Journal of Computational Mathematics
Vol.44, No.3, 2026, 819–842.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2505-m2025-0062

SUPERLINEARLY CONVERGENT ALGORITHMS FOR STOCHASTIC TIME-FRACTIONAL EQUATIONS DRIVEN BY WHITE NOISE*

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Abstract

The numerical analysis of stochastic time-fractional equations exhibits a significantly low-order convergence rate since the limited regularity of model caused by the nonlocal operator and the presence of noise. In this work, we consider stochastic time-fractional equations driven by integrated white noise, where ${}^C D_t^\alpha \psi(x,t), 0<\alpha<2$ and $I_t^\gamma \dot{W}(x,t)$, $0<\gamma<1$. We first establish the regularity of the mild solution. Then superlinear convergence rate

$$\left(\mathbb{E}\left\|\psi(\cdot,t_n)-\psi^n\right\|^2\right)^{\frac{1}{2}} = O\left(\tau^{\alpha+\gamma-\frac{\alpha d}{4}-\frac{1}{2}-\varepsilon}\right)$$

with sufficiently small ε term in the exponent is established based on the modified two-step backward difference formula methods. Here d represents the spatial dimension, ψ^n denotes the approximate solution at the n-th time step, and $\mathbb{E}$ is the expectation operator. Numerical experiments are performed to verify the theoretical results. To the best of our knowledge, this is the first topic on the superlinear convergence analysis for the stochastic time-fractional equations with integrated white noise.

Mathematics subject classification: 60H35, 34A08.
Key words: Stochastic fractional evolution equation, Integrated white noise, Superlinear convergence analysis.

1. Introduction

We are interested in the error estimates of modified two-step backward difference formula (BDF2) methods for solving the stochastic time-fractional evolution equation driven by integrated white noise [10, 15, 16], with $\alpha \in (1,2)$ and $\gamma \in (0,1)$,

$$\begin{cases} {}^C D_t^\alpha \psi(x,t) - A\psi(x,t) = f(x,t) + I_t^\gamma \dot{W}(x,t), & (x,t) \in \mathcal{O}\times\mathbb{R}_+, \\ \psi(x,0) = \upsilon(x), \quad \partial_t \psi(x,0) = b(x), & x \in \mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O} \subset \mathbb{R}^d$, $d=1,2,3$ is a bounded domain with Lipschitz boundary $\partial\mathcal{O}$ and d denotes the spatial dimension. The operator A denotes the Laplacian Δ on a convex polyhedral domain $\mathcal{O}$ with $\mathcal{D}(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$.

Here $W(x,t)$ is a cylindrical Wiener process with a covariance operator $Q=I$ on $L^2(\mathcal{O})$ with respect to a filtration $\{\mathscr{F}_t\}_{t\ge 0}$ on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ in [27]. And white noise

* Received February 21, 2025 / Revised version received May 10, 2025 / Accepted May 12, 2025 / Published online September 1, 2025 /
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$\dot{W}(x,t)$ is the time derivative of $W(x,t)$ with

$$W(x,t)=\sum_{j=1}^{\infty}\beta_j(t)\varphi_j(x), \tag{1.2}$$

where $\beta_j(t)$ are the independently and identically distributed Brownian motions and $\varphi_j(x)$, $j=1,2,\ldots$, are the L^2-norm normalized eigenfunctions of the operator $-\Delta$ corresponding to the eigenvalues $\lambda_j,\ j=1,2,\ldots$, arranged in nondecreasing order.

Note that the additive noise $\dot{W}(x,t)$ is expressed by [8]

$$\dot{W}(x,t)=\frac{dW(x,t)}{dt}=\sum_{j=1}^{\infty}\sigma_j(t)\dot{\beta}_j(t)\varphi_j(x), \tag{1.3}$$

where $\sigma_j(t)$ is the rapidly decay function as j increases with $\sum_{j=1}^{\infty}\sigma_j^2<\infty$.

The deterministic problems associated with model (1.1) arise in many areas of the applied sciences, such as the dynamics of viscoelastic materials, through water around rocks, and the transport of chemical contaminants [11, 17, 22]. The fractionally integrated noise $I_t^{\gamma}\dot{W}(x,t)$ characterizes random effects on particle motion in medium with memory or particles subject to sticking and trapping [1, 5, 12, 15, 16].

The solution of model (1.1) may be decomposed into the solution of the stochastic problem

$$\begin{cases} {}^{C}D_t^{\alpha}u(x,t)-Au(x,t)=I_t^{\gamma}\dot{W}(x,t), & (x,t)\in\mathcal{O}\times\mathbb{R}_+,\\ u(x,0)=\upsilon(x),\quad \partial_t u(x,0)=b(x), & x\in\mathcal{O}, \end{cases} \tag{1.4}$$

plus the solution of the deterministic problem

$$\begin{cases} {}^{C}D_t^{\alpha}v(x,t)-Av(x,t)=f(x,t), & (x,t)\in\mathcal{O}\times\mathbb{R}_+,\\ v(x,0)=0,\quad \partial_t v(x,0)=0, & x\in\mathcal{O}. \end{cases} \tag{1.5}$$

The operators ${}^{C}D_t^{\alpha}$ and I_t^{γ} denote the Caputo fractional derivative of order $\alpha\in(1,2)$ and Riemann-Liouville integral of order $\gamma\in(0,1)$, respectively, defined by

$$\begin{aligned} {}^{C}D_t^{\alpha}\psi(t)&=\frac{1}{\Gamma(2-\alpha)}\int_0^t(t-s)^{1-\alpha}\frac{d^2}{ds^2}\psi(s)ds,\\ I_t^{\gamma}\psi(t)&=\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}\psi(s)ds=\frac{1}{\Gamma(\gamma)}t^{\gamma-1}*\psi(t), \end{aligned}$$

where $*$ denotes the convolution integral operators

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau.$$

Numerical methods of (1.5) have been widely investigated by various authors. If $f(x,t)$ is smooth in time, one approach involves employing variable time-stepping schemes, such as geometric meshes or graded meshes [23,24]. These schemes are particularly effective in capturing the singularities of the solution at $t=0$. Another method is the utilization of convolution quadrature, which can be generated using BDFk or Lagrange interpolation of degree k, as discussed in [7, 13, 14, 21].