

# ANALYSIS OF ARBITRARY HIGH ORDER SPECTRAL VOLUME METHOD FOR HYPERBOLIC CONSERVATION LAWS OVER RECTANGULAR MESHES\*

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## Abstract

This paper investigates two spectral volume (SV) methods applied to 2D linear hyperbolic conservation laws on rectangular meshes. These methods utilize upwind fluxes and define control volumes using Gauss-Legendre (LSV) and right-Radau (RRSV) points within mesh elements. Within the framework of Petrov-Galerkin method, a unified proof is established to show that the proposed LSV and RRSV schemes are energy stable and have optimal error estimates in the  $L^2$  norm. Additionally, we demonstrate superconvergence properties of the SV method at specific points and analyze the error in cell averages under appropriate initial and boundary discretizations. As a result, we show that the RRSV method coincides with the standard upwind discontinuous Galerkin method for hyperbolic problems with constant coefficients. Numerical experiments are conducted to validate all theoretical findings.

*Mathematics subject classification:* 65M15, 65M60, 65N30.

*Key words:* Spectral volume methods, Energy stable, Superconvergence, Hyperbolic conservation laws.

## 1. Introduction

The spectral volume method is a significant high-order numerical technique for hyperbolic equations, extending the classical Godunov finite volume method [20]. Similar to other high-order methods such as the k-exact finite volume method [4, 19], the essentially nonoscillatory (ENO) method [11, 22], the weighted ENO (WENO) method [25, 27], and the discontinuous Galerkin (DG) method [15–17, 23], the SV method offers numerous advantages including high-order accuracy, compact stencils, and geometric flexibility (suitable for unstructured grids). Notably, the SV method preserves conservation laws on finer meshes, potentially offering higher resolution for discontinuities compared to other high-order methods (see [28]). Since its intro-

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duction in 2002 [36] and subsequent developments by Wang *et al.* [34, 37–39], the SV method has found widespread application in solving various PDEs such as the shallow water equation [14, 18], Navier-Stokes equation [21, 26, 29, 30], and electromagnetic field equations [26].

Despite the widespread application of SV methods, the exploration of their mathematical theory remains nascent, with limited available information on their convergence and superconvergence properties. To our knowledge, considerable attention has focused on the stability of lower-order SV schemes. For instance, Wang *et al.* [34, 38, 39] introduced the “Lebesgue constant” to assess SV methods in their early work. Van den Abeele *et al.* [31–33] investigated the stability of various low-order SV schemes: They found that 1D third and fourth-order SV schemes based on Gauss-Lobatto distributions are weakly unstable, whereas a 2D second-order SV scheme is stable, and a two-parameter family of third-order SV schemes exhibits weak instability. Zhang and Shu [35] employed Fourier-type analysis to study the stability of  $p$ -th order ( $p \leq 3$ ) SV schemes over uniform meshes for 1D hyperbolic equations. In summary, there appears to be a lack of systematic studies on the mathematical theory encompassing stability, optimal-order convergence, and superconvergence.

A primary contribution of this study is to present a unified analysis of the stability and optimal-order convergence properties of SV schemes of arbitrary order for two-dimensional linear hyperbolic problems on rectangular meshes. To achieve this goal, we initially develop two classes of high-order SV schemes using control volumes defined by Gauss-Legendre points or right-Radau points within subintervals of the mesh. Ensuring stability, we adopt upwind numerical fluxes and reformulate the SV scheme in Petrov-Galerkin form. By introducing a specific mapping from the trial to the test space, we recast the SV method as a specialized Galerkin method. Leveraging the Galerkin framework and established numerical quadrature techniques, we establish the energy stability and optimal convergence rates of the LSV and RRSV schemes. As a notable outcome, we demonstrate that the proposed SV method can be interpreted as a discrete variant involving numerical quadratures of the well-known discontinuous Galerkin method. Particularly, the RRSV method exhibits identical behavior to the DG method when applied to hyperbolic equations with constant coefficients.

Another significant contribution of this work is the discovery and proof of notable superconvergence properties inherent in the SV method. Research into superconvergence behaviors of numerical methods has been extensive over the past decades (see, e.g. [1, 3, 5–9, 12, 13, 24]). Among these studies, a well-known result is the  $2k$ -conjecture, which initially conjectured (based on substantial numerical evidence) that bi- $k$  finite element approximations on rectangular meshes for the Poisson equation converge at a rate of  $h^{2k}$  at vertices of the mesh. This conjecture was proven by Chen *et al.* [12] after nearly 40 years. Subsequently, Cao *et al.* [9] demonstrated that the  $2k$ -conjecture holds for finite volume methods (FVM) and further extended it to the  $(2k + 1)$ -conjecture for DG methods in [6] for 2D hyperbolic equations.

However, very little is known about the superconvergence properties of the SV method. To our knowledge, prior to our recent work on SV methods for 1D linear hyperbolic problems [10], no superconvergence analysis of the SV method had been reported. Motivated by the successful application of correction functions in DG and FVM methods (see, e.g. [6, 9]), we introduce a novel method to investigate the superconvergence properties of the proposed SV schemes. We prove that the  $2k$ -conjecture holds true for LSV, while the  $(2k + 1)$ -conjecture is valid for RRSV. Specifically, the bi- $k$  LSV approximation and RRSV approximation over rectangular meshes converge at rates of  $h^{2k}$  and  $h^{2k+1}$ , respectively, at downwind nodal points. Additionally, we derive several other superconvergence results for the SV solution at specific points or lines: