Journal of Computational Mathematics
Vol.44, No.3, 2026, 871–890.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2510-m2024-0087

AN ANALYSIS OF THE SATURATION ASSUMPTION FOR POLYNOMIAL INTERPOLATION WITH APPLICATION TO ADAPTIVE FINITE ELEMENT METHODS*

Randolph E. Bank[1)]
Department of Mathematics, University of California San Diego, La Jolla, California 92093-0112, USA
Email: rbank@ucsd.edu
Jinchao Xu
Applied Mathematics and Computational Science, Computer, Electrical and Mathematical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal 23955, Saudi Arabia
Email: jinchao.xu@kaust.edu.sa
Harry Yserentant
Institut für Mathematik, Technische Universität Berlin, 10623 Berlin, Germany
Email: yserentant@math.tu-berlin.de

Abstract

The saturation assumption plays a central role in much of the analysis of a posteriori error estimates and refinement algorithms for adaptive finite element methods. In this work we provide an analysis of this assumption in the simple setting of interpolation. We have proved elsewhere [Bank and Yserentant, Numer. Math., 131:1 (2015)] that interpolation error is both reliable and efficient as an a posteriori error estimate. Thus behavior of interpolation error is indicative of the behavior of the error in the exact finite element solution of a PDE as well as any practical a posteriori error estimate that is also reliable and efficient.

Mathematics subject classification: 65N30, 65N15, 65N50.
Key words: Saturation assumption, Adaptive feedback loop, A posteriori error estimates.

1. Introduction

Consider a single shape regular simplicial finite element t in $\mathcal{R}^d$. Let h denote the diameter of t. On element t, we define a family of polynomial spaces $\mathcal{S}_p(t)$, consisting of polynomials of degree less than or equal to p. Let $H^k(t)$ denote the Sobolev space for appropriately chosen k and let $\mathcal{I}_p : H^k(t) \to \mathcal{S}_p(t)$ denote the usual Lagrange interpolation operator associated with the space $\mathcal{S}_p(t)$. We assume that k is sufficiently large that the usual a priori estimates of the form

$$\|(1-\mathcal{I}_p)u\|_{H^r(t)} \le C(q,r)h^{q-r}|u|_{H^q(t)} \tag{1.1}$$

for $0 \le r < q \le p+1$ hold. In this work, we focus mainly on the $H^1(t)$-seminorm. $|u|_{H^1(t)} \equiv \|\nabla u\|_{L_2(t)}$, although much of the analysis can be generalized to other Sobolev norms. For simplicity in notation, we will write $|\cdot|_1$ or $|\cdot|_{1,t}$ where appropriate.

* Received April 22, 2024 / Revised version received May 9, 2025 / Accepted October 14, 2025 / Published online March 2, 2026 /
1) Corresponding author

Next we assume some refinement of element t. This could be h-refinement (division of t into several smaller elements with the same polynomial degree as t) or p-refinement (increasing the degree of the polynomial space by one). Let $\hat{\mathcal{I}}_p$ denote the canonical interpolation operator applied to the refinement of t. Thus in the case of p-refinement we have $\hat{\mathcal{I}}_p \equiv \mathcal{I}_{p+1}$, or in the case of h-refinement, $\hat{\mathcal{I}}_p$ represents $\mathcal{I}_p$ applied to each of the newly refined child elements of t.

The local saturation assumption reads

$$|u - \hat{\mathcal{I}}_p u|_{1,t} \le \beta |u - \mathcal{I}_p u|_{1,t} \tag{1.2}$$

for some $\beta = \beta(u) < 1$. Informally, the saturation assumption asserts that the refinement of element t will result in a reduction of the interpolation error.

Our main interest in (1.2) is its use in the study of adaptive finite element methods for solving partial differential equations. Here one makes an a posteriori error estimate using the finite element solution u_h, and then employs this estimate to guide refinement of the finite element subspace in an adaptive feedback loop. Success of such an adaptive procedure depends on both the quality of the a posteriori error estimates, and that the adaptive feedback loop results in a significant reduction of the error. Saturation assumptions can play an important role in analyzing both of these aspects of adaptive procedures. The books of Babuška and Strouboulis [1], Babuška *et al.* [2], Deuflhard and Weiser [10], Verfürth [15], and the references cited therein, together provide a rather complete overview of these topics and the role played by the saturation assumption. Several authors have studied the validation of this assumption, as well as possible alternative approaches. See for example, Nochetto [13], Dörfler and Nochetto [11], Carstensen *et al.* [9], Bulle *et al.* [8], Praetorius *et al.* [14], and Bank *et al.* [4].

What these and other works have in common is that they consider the application of a global saturation assumption to the finite element solution u_h of the partial differential equation, often restricted to the case of p-refinement. In (1.2), we consider its application to interpolation, and allow both h- and p-refinement. Since interpolation is also quite local, this setting is simple in comparison. On the other hand, (1.2) still has important implications for adaptive finite element methods. Bank and Yserentant [5] show that interpolation error is a local lower bound on the error for any finite element approximation of a given function u, in particular the finite element solution u_h. Estimate (1.2) is the key assumption in that analysis. Coupled with standard a priori estimates, this shows that interpolation error is both reliable and efficient as an a posteriori error estimator. As a practical matter, of course it could not be used as such, since the interpolant is generally not available in such situations.[1)] However, if we have a practical a posteriori error estimate e_h that is both reliable and efficient, we have the estimates

$$c_1 |u - \mathcal{I}_p u|_{1,\Omega} \le |u - u_h|_{1,\Omega} \le c_2 |u - \mathcal{I}_p u|_{1,\Omega},$$
$$C_1 |e_h|_{1,\Omega} \le |u - u_h|_{1,\Omega} \le C_2 |e_h|_{1,\Omega},$$

where Ω is the domain of the given partial differential equation. It follows that

$$\frac{c_1}{C_2} |u - \mathcal{I}_p u|_{1,\Omega} \le |e_h|_{1,\Omega} \le \frac{C_1}{c_2} |u - \mathcal{I}_p u|_{1,\Omega},$$

that suggests the true finite element error, the interpolation error, and the computed a posteriori error estimate all behave in the same way. See also [12].

[1)] The PLTMG software package [3] does use local interpolation formulas as its local error indicators and its a posteriori estimate. Approximations of the required derivatives are obtained using a superconvergence algorithm.