

FINITE ELEMENT METHOD WITH GRÜNWARD-LETNIKOV TYPE APPROXIMATION IN TIME FOR A CONSTANT TIME DELAY SUBDIFFUSION EQUATION*

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Abstract

In this work, a subdiffusion equation with constant time delay τ is considered. First, the regularity of the solution to the considered problem is investigated, finding that its first-order time derivative exhibits singularity at $t = 0^+$ and its second-order time derivative shows singularity at both $t = 0^+$ and τ^+ , while the solution can be decomposed into its singular and regular components. Then, we derive a fully discrete finite element scheme to solve the considered problem based on the standard Galerkin finite element method in space and the Grünwald-Letnikov type approximation in time. The analysis shows that the developed numerical scheme is stable. In order to discuss the error estimate, a new discrete Grönwall inequality is established. Under the above decomposition of the solution, we obtain a local error estimate in time for the developed numerical scheme. Finally, some numerical tests are provided to support our theoretical analysis.

Mathematics subject classification: 65M06, 65M12, 65M60.

Key words: Subdiffusion equation with constant time delay, Finite element method, Grünwald-Letnikov type approximation, Discrete Grönwall inequality, Error estimate.

1. Introduction

It is well known that differential equation is a powerful tool for understanding and describing some complex phenomena in the natural world. In view of the fact that there exist numerous real-world systems which depend on their historical states, this makes the delay differential equation show wide application potential in various fields such as physics, biology and medicine [12, 33]. The emergence of this trend has aroused great interest of mathematicians in delay differential equations [2, 4, 11, 13]. Compared with integer-order model, fractional differential equation often exhibits high reliability in capturing memory effect, hereditary characteristic and non-locality of complex systems. Therefore, the fractional models often are used to describe these practical problems [6, 15, 16, 34]. It is noteworthy that some systems, such as biological

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systems, inherently possess time delay and exhibit long-range memory. Traditional models, including integer-order equations with delay and fractional equations without delay, are difficult to comprehensively capture the complex interplay between memory and delay. Since models with fractional operator and time delay allow greater degrees of freedom in modeling and in describing systems with long-time memory, they provide a powerful tool for disclosing the underlying laws governing these systems [3,10,30,31,40]. In this paper, we consider the following time-fractional diffusion equation with constant time delay:

$$\partial_t u(x, t) = \partial_t^{1-\alpha} (p\Delta u(x, t) + au(x, t)) + bu(x, t - \tau) + f(x, t), \quad (x, t) \in \Omega \times (0, K\tau), \quad (1.1)$$

$$u(x, t) = \varphi(x, t), \quad (x, t) \in \bar{\Omega} \times [-\tau, 0], \quad (1.2)$$

$$u(x, t)|_{\partial\Omega} = 0, \quad t \in [-\tau, K\tau], \quad (1.3)$$

where $\Omega = (0, L)$, L, p, a, b are some constants satisfying $L > 0, p > 0, a \leq 0, b \neq 0$, $f(x, t)$ is a continuous function, $\tau > 0$ is the time delay parameter, K is a given positive integer, $\varphi(x, t)$ is a continuous function in $\bar{\Omega} \times [-\tau, 0]$, and $\partial_t^l u$ denote as the l -th order derivative with respect to variable t and $\partial_t(\cdot) := \partial_t^1(\cdot)$. The Riemann-Liouville fractional derivative $\partial_t^{1-\alpha} u(x, t)$ is defined by [5]

$$\partial_t^{1-\alpha} u(x, t) = \frac{\partial}{\partial t} \int_0^t \omega_\alpha(t-s) u(x, s) ds, \quad 0 < \alpha < 1,$$

where $\omega_\beta(t) := t^{\beta-1}/\Gamma(\beta)$ for $\beta, t > 0$ is the convolution kernel function.

At present, there have been some advancements in the theoretical studies of fractional diffusion equations with delay. Prakash *et al.* [29] proposed the invariant subspace approach to find the exact solution of time-fractional reaction-diffusion equation with time delay. Zhu *et al.* [44] investigated the local and nonlocal existence of mild solution for a nonlinear time delay fractional reaction-diffusion equation. By using the semigroup theory of operators and the monotone iterative technique, reference [19] obtained the existence and uniqueness of mild solutions for a time-space diffusion equation involving delay. Yao and Yang [37] investigated the asymptotic stability and long-time decay rates for a fractional diffusion-wave equation with time delay. Meanwhile, some researchers proposed a lot of effective numerical methods to solve the time fractional diffusion equation with delay. For example, in [18, 43], researchers discussed finite difference methods for constant time delay subdiffusion equation. In [1, 20, 41], some finite difference schemes were devised to solve the time fractional diffusion-wave equations with delay. We also noted that some studiers use finite difference method to solve distributed order fractional diffusion equation [28]. In [26, 27], the authors considered the finite element methods for the time delay subdiffusion equation and investigated their convergence and superconvergence. Based on L1 formula in time and Galerkin spectral method in space, Zaky *et al.* [38] developed an effective numerical method for variable-order time fractional reaction-diffusion equation with delay. In order to analyze the stability and convergence of finite difference scheme of time fractional multi-delayed diffusion equation, Hendy and Macías-Díaz [14] proposed a novel discrete Grönwall inequality.

Recently, Tan *et al.* [35] discuss a constant time delay fractional diffusion equation from both theoretical and numerical perspectives. They not only derive its exact solution, but also reveal the multiple singularity phenomena of the solution in time. Furthermore, Cen and Vong [8] obtain a sharp multi-singularity result of a class of delay fractional ordinary differential equation, and two corrected L-type schemes are constructed in [7] to overcome this multi-singularity. Subsequently, by using variable-step L1 method, references [5, 25] obtain effective numerical

schemes with satisfactory temporal convergence accuracy when the exact solution has multiple singularities in time. It is worth pointing out that the regularity results of references [5, 8, 25] show that the first derivative of the solution with respect to variable t satisfies

$$|\partial_t u| \leq C \left(1 + (t - (k-1)\tau)^{k\alpha-1}\right), \quad t \in ((k-1)\tau, k\tau], \quad (1.4)$$

where $k = 1, 2, \dots$. However, compared with previous works, there are some different regularity results for the present problem (1.1)-(1.3). Therefore, it motivates us to develop new numerical method to handle the different smoothness. In this paper, the main contributions of this work are threefold:

- We investigate the problem (1.1)-(1.3) and discover that the first derivative of the solution with respect to variable t has

$$|\partial_t u| \leq C(1 + t^{\alpha-1}), \quad t \in ((k-1)\tau, k\tau], \quad k = 1, 2, \dots, K, \quad (1.5)$$

the second derivative of the solution with respect to variable t satisfies

$$\begin{aligned} |\partial_t^2 u| &\leq C(1 + t^{\alpha-2}), & t \in (0, \tau], \\ |\partial_t^2 u| &\leq C(1 + (t - \tau)^{\alpha-1}), & t \in ((k-1)\tau, k\tau] \end{aligned} \quad (1.6)$$

for $k = 2, 3, \dots, K$, and the solution can be decomposed into its singular and regular components. The regularity results (1.5) and (1.6) indicate that $\partial_t u$ will blow up only at $t = 0$, and $\partial_t^2 u$ will blow up only at $t = 0$ and $t = \tau$. It implies that the problem (1.1)-(1.3) has better regularity than the problem considered in [5, 8, 25].

- In view of the good regularity of exact solution and in order to avoid the round-off error which may be caused by dense graded mesh, we use the Grünwald-Letnikov type approximation to discretize the time fractional operators, propose a fully discrete finite element scheme for (1.1)-(1.3), and discuss its stability.

- We establish a new discrete Grönwall inequality, and utilize it to investigate the convergence based on the regularity results (1.5)-(1.6). It is worth noting that the obtained results are pointwise-in-time error estimates.

The structure of this work is as follows. In Section 2, the exact solution of (1.1)-(1.3) is obtained and its regularity is discussed. In Section 3, we establish a fully discrete finite element scheme for the problem (1.1)-(1.3), and investigate the stability. Based on the regularity results, the convergence is discussed in Section 4. Finally, numerical tests are presented to verify the theoretical results in Section 5 and the conclusion is given in Section 6.

Notation. In this paper, C is a general constant, which can be different in different situations. Denote (\cdot, \cdot) and $\|\cdot\|_0$ as the inner product and norm of $L^2(0, L)$. Define the solution $u(x, t)$, $t \in ((k-1)\tau, k\tau]$ of (1.1)-(1.3) by $u_{k\tau}(x, t)$, and $u_\tau(x, t) := u_{1\tau}(x, t)$.

2. The Regularity of the Solution

In this section, the solution to (1.1)-(1.3) is obtained by the variable separation method and the Laplace transform method, and we discuss its regularity. First, when $(x, t) \in [0, L] \times (0, \tau]$,

the problem (1.1)-(1.3) can be written as

$$\begin{cases} \partial_t u(x, t) = \partial_t^{1-\alpha}(-G(u)) + F_\tau(x, t), & 0 < x < L, \quad 0 < t \leq \tau, \\ u(x, t) = \varphi(x, t), & 0 \leq x \leq L, \quad -\tau \leq t \leq 0, \\ u(0, t) = 0, \quad u(L, t) = 0, & -\tau \leq t \leq \tau, \end{cases} \quad (2.1)$$

where

$$G(u) := -p\Delta u(x, t) - au(x, t), \quad F_\tau(x, t) := bu(x, t - \tau) + f(x, t).$$

Similar to [5], it follows from the variable separation method that

$$u_\tau(x, t) = \sum_{i=1}^{\infty} T_i(t) X_i(x), \quad T_i(t) = (u_\tau(x, t), X_i), \quad (2.2)$$

where X_i is the eigenfunction corresponding to the eigenvalue $\lambda_i > 0$ satisfying $\|X_i\|_0 = 1$ and

$$G(X_i) = \lambda_i X_i, \quad X_i(0) = X_i(L) = 0, \quad (2.3)$$

and

$$\varphi(x, t - \tau) = \sum_{i=1}^{\infty} \varphi_i(t - \tau) X_i(x), \quad F_\tau(x, t) = \sum_{i=1}^{\infty} F_{\tau,i}(t) X_i(x), \quad (2.4)$$

where

$$\varphi_i(t - \tau) = (\varphi(x, t - \tau), X_i), \quad F_{\tau,i}(t) = (F_\tau(x, t), X_i), \quad t \in (0, \tau].$$

Combining (2.1), (2.2) and (2.4), then we have

$$\partial_t T_i(t) = \partial_t^{1-\alpha}(-\lambda_i T_i(t)) + F_{\tau,i}(t).$$

Taking the Laplace transform of the above equation yields

$$s\tilde{T}_i(s) - \varphi_i(0) = -s^{1-\alpha}\lambda_i\tilde{T}_i(s) + \tilde{F}_{\tau,i}(s),$$

i.e.,

$$\tilde{T}_i(s) = \frac{\varphi_i(0)}{s + \lambda_i s^{1-\alpha}} + \frac{\tilde{F}_{\tau,i}(s)}{s + \lambda_i s^{1-\alpha}}.$$

Furthermore, from the inverse Laplace transform, one has

$$T_i(t) = \varphi_i(0)E_\alpha(-\lambda_i t^\alpha) + \int_0^t E_\alpha(-\lambda_i s^\alpha) F_{\tau,i}(t-s) ds, \quad (2.5)$$

where

$$E_{\mu,\nu}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\mu + \nu)}$$

is the Mittag-Leffler function. Therefore, we can obtain the solution of (2.1) by substituting the above equation into (2.2) as follows:

$$u_\tau(x, t) = \sum_{i=1}^{\infty} \left(\varphi_i(0)E_\alpha(-\lambda_i t^\alpha) + \int_0^t E_\alpha(-\lambda_i s^\alpha) F_{\tau,i}(t-s) ds \right) X_i(x). \quad (2.6)$$

In fact, (2.6) implies that the problem (1.1)-(1.3) exists a unique solution in the domain $[0, L] \times (0, \tau]$. Moreover, by performing similar manipulation to $[0, L] \times ((k-1)\tau, k\tau]$, $k = 2, 3, \dots, K$, it is clear that the problem (1.1)-(1.3) has a unique solution in $[0, L] \times ((k-1)\tau, k\tau]$, i.e.,

$$u_{k\tau}(x, t) = \sum_{i=1}^{\infty} \left(\varphi_i(0) E_{\alpha}(-\lambda_i t^{\alpha}) + \int_0^t E_{\alpha}(-\lambda_i s^{\alpha}) F_{k\tau, i}(t-s) ds \right) X_i(x), \quad (2.7)$$

where $\varphi_i(0) = (\varphi(x, 0), X_i)$, $F_{k\tau, i}(t) = (F_{k\tau}(x, t), X_i(x))$, and

$$F_{k\tau}(x, t) = \begin{cases} b\varphi(x, t-\tau) + f(x, t), & 0 \leq x \leq L, \quad 0 < t \leq \tau, \\ bu_{\tau}(x, t-\tau) + f(x, t), & 0 \leq x \leq L, \quad \tau < t \leq 2\tau, \\ \vdots \\ bu_{(k-1)\tau}(x, t-\tau) + f(x, t), & 0 \leq x \leq L, \quad (k-1)\tau < t \leq k\tau. \end{cases} \quad (2.8)$$

In order to investigate the regularity of the solution, we give two auxiliary lemmas firstly.

Lemma 2.1 ([32]). *If $\alpha > 0, \lambda > 0, t > 0$ and positive integer $m \in \mathbb{N}$, then*

$$\frac{d^m}{dt^m} E_{\alpha, 1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-m} E_{\alpha, \alpha-m+1}(-\lambda t^{\alpha}).$$

Moreover, if $0 < \alpha < 1$ and $\eta \geq 0$, then $E_{\alpha, \alpha}(-\eta) \geq 0$.

From Lemma 2.1, it is easy to know that

$$\int_0^t |\lambda_i s^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i s^{\alpha})| ds = 1 - E_{\alpha, 1}(-\lambda_i t^{\alpha}) \leq C. \quad (2.9)$$

Let $\gamma > 0$. Define the space $D(G^{\gamma})$ and its norm as follows [5]:

$$D(G^{\gamma}) = \left\{ g \in L^2(0, L) : \sum_{i=1}^{\infty} \lambda_i^{2\gamma} |(X_i, g)|^2 \leq \infty \right\},$$

$$\|g\|_{G^{\gamma}} = \left(\sum_{i=1}^{\infty} \lambda_i^{2\gamma} |(X_i, g)|^2 \right)^{\frac{1}{2}}.$$

Lemma 2.2. *Suppose that*

- (I) $\|\varphi(x, t-\tau)\|_{G^{3/2}} \leq C$ and $\|\partial_t \varphi(x, t-\tau)\|_{G^{1/2}} \leq C, t \in [0, \tau]$;
- (II) $\|f(x, t)\|_{G^{3/2}} \leq C, t \in [0, K\tau]$, and $\|\partial_t f(x, t)\|_{G^{1/2}} \leq C(1+t^{\alpha-1}), t \in (0, K\tau]$.

Then for $k \geq 1$, $\|u_{k\tau}(x, t)\|_{G^{3/2}}, \|F_{k\tau}(x, t)\|_{G^{3/2}} \in L^{\infty}(((k-1)\tau, k\tau])$,

$$\|\partial_t u_{k\tau}(x, t)\|_{G^{1/2}}^2 \leq C(1+t^{2(\alpha-1)}), \quad \|\partial_t F_{k\tau}(x, t)\|_{G^{1/2}} \leq C(1+t^{\alpha-1}),$$

and for $k \geq 2, t \in (j\tau, (j+1)\tau]$ with $0 \leq j \leq k-1$,

$$\|\partial_t F_{k\tau}(x, t)\|_{G^{1/2}} \leq C(1+(t-\tau)^{\alpha-1}).$$

Proof. When $k = 1$, it follows from (2.8), the conditions (I) and (II) that

$$\begin{aligned} \|F_\tau(x, t)\|_{G^{3/2}}^2 &= \|b\varphi(x, t - \tau) + f(x, t)\|_{G^{3/2}}^2 \\ &\leq C \left[\|\varphi(x, t - \tau)\|_{G^{3/2}}^2 + \|f(x, t)\|_{G^{3/2}}^2 \right]. \end{aligned} \quad (2.10)$$

Therefore, (2.10) implies that $\|F_\tau(x, t)\|_{G^{3/2}} \in L^\infty([0, \tau])$. By (2.5) and the boundedness of Mittag-Leffler function, we know that

$$|T_i(t)|^2 \leq \left(|\varphi_i(0)| + \int_0^t |F_{\tau,i}(t-s)| ds \right)^2 \leq C \left[|\varphi_i(0)|^2 + \int_0^t |F_{\tau,i}(t-s)|^2 ds \right]. \quad (2.11)$$

The combination of (2.5), (2.6) and (2.11) yields

$$\|u_\tau(x, t)\|_{G^{3/2}}^2 = \sum_{i=1}^{\infty} \lambda_i^3 |T_i(t)|^2 \leq C \left[\|\varphi(x, 0)\|_{G^{3/2}}^2 + \int_0^\tau \|F_\tau(x, t-s)\|_{G^{3/2}}^2 ds \right] \leq C. \quad (2.12)$$

Hence, (2.12) indicates that $\|u_\tau(x, t)\|_{G^{3/2}} \in L^\infty((0, \tau])$. Now, we suppose that $\|u_{k\tau}(x, t)\|_{G^{3/2}}$, $\|F_{k\tau}(x, t)\|_{G^{3/2}} \in L^\infty(((k-1)\tau, k\tau])$ are true for $k \leq n$. When $k = n+1$, (2.8) gives

$$\begin{aligned} \|F_{(n+1)\tau}(x, t)\|_{G^{3/2}}^2 &= \|bu_{n\tau}(x, t - \tau) + f(x, t)\|_{G^{3/2}}^2 \\ &\leq C \left[\|u_{n\tau}(x, t - \tau)\|_{G^{3/2}}^2 + \|f(x, t)\|_{G^{3/2}}^2 \right] \leq C. \end{aligned} \quad (2.13)$$

Besides, from (2.7), one has

$$|u_{(n+1)\tau,i}(t)|^2 \leq C \left[|\varphi_i(0)|^2 + \int_0^{(n+1)\tau} |F_{(n+1)\tau,i}(t-s)|^2 ds \right].$$

The above inequality leads to

$$\|u_{(n+1)\tau}(x, t)\|_{G^{3/2}}^2 \leq C \left[\|\varphi(x, 0)\|_{G^{3/2}}^2 + \int_0^{(n+1)\tau} \|F_{(n+1)\tau}(x, s)\|_{G^{3/2}}^2 ds \right] \leq C. \quad (2.14)$$

According to (2.13), (2.14) and mathematical induction, we can conclude that $\|u_{k\tau}(x, t)\|_{G^{3/2}}$, $\|F_{k\tau}(x, t)\|_{G^{3/2}} \in L^\infty(((k-1)\tau, k\tau])$ hold for $k = 1, 2, \dots, K$.

Next, we discuss $\|\partial_t u_{k\tau}(t)\|_{G^{1/2}}$ and $\|\partial_t F_{k\tau}(t)\|_{G^{1/2}}$. First, it is easy to know that the above results imply

$$\sum_{i=1}^{\infty} |F_{k\tau,i}(t)| \leq \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} \lambda_i |F_{k\tau,i}(t)|^2 \right)^{\frac{1}{2}} \leq C, \quad (2.15)$$

$$\sum_{i=1}^{\infty} \lambda_i |F_{k\tau,i}(t)| \leq \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} \lambda_i^3 |F_{k\tau,i}(t)|^2 \right)^{\frac{1}{2}} \leq C, \quad (2.16)$$

where the well known fact $\lambda_i \approx i^2$ is used. Taking the first derivative of $u_{k\tau}(x, t)$, $k = 1, 2, \dots, K$ with respect to t and applying Lemma 2.1, it yields

$$\begin{aligned} \partial_t u_{k\tau}(x, t) &= \sum_{i=1}^{\infty} \left\{ -\lambda_i t^{\alpha-1} \varphi_i(0) E_{\alpha, \alpha}(-\lambda_i t^\alpha) + E_\alpha(-\lambda_i t^\alpha) F_{k\tau,i}(0) \right. \\ &\quad \left. + \int_0^t E_\alpha(-\lambda_i s^\alpha) \frac{d}{dt} F_{k\tau,i}(t-s) ds \right\} X_i(x) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty} \left\{ -\lambda_i t^{\alpha-1} \varphi_i(0) E_{\alpha,\alpha}(-\lambda_i t^\alpha) + E_\alpha(0) F_{k\tau,i}(t) \right. \\
&\quad \left. + \int_0^t (-\lambda_i s^{\alpha-1}) E_{\alpha,\alpha}(-\lambda_i s^\alpha) F_{k\tau,i}(t-s) ds \right\} X_i(x), \quad (2.17)
\end{aligned}$$

where the integration by parts is used. Therefore, one has

$$\begin{aligned}
|\partial_t u_{k\tau,i}(t)|^2 &\leq C \left[t^{2(\alpha-1)} \lambda_i^2 |\varphi_i(0)|^2 + |F_{k\tau,i}(t)|^2 \right. \\
&\quad \left. + \left(\int_0^t \lambda_i (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) F_{k\tau,i}(s) ds \right)^2 \right]. \quad (2.18)
\end{aligned}$$

Applying the Young's inequality for convolution, the Cauchy-Schwarz inequality and (2.9), it holds

$$\begin{aligned}
|\partial_t u_{k\tau,i}(t)|^2 &\leq C \left[t^{2(\alpha-1)} \lambda_i^2 |\varphi_i(0)|^2 + |F_{k\tau,i}(t)|^2 \right. \\
&\quad \left. + \left(\int_0^t |\lambda_i s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i s^\alpha)| ds \right)^2 \left(\int_0^t |F_{k\tau,i}(s)|^2 ds \right) \right] \\
&\leq C \left[t^{2(\alpha-1)} \lambda_i^2 |\varphi_i(0)|^2 + |F_{k\tau,i}(t)|^2 + \int_0^t |F_{k\tau,i}(s)|^2 ds \right].
\end{aligned}$$

The above inequality demonstrates

$$\begin{aligned}
\|\partial_t u_{k\tau}(x, t)\|_{G^{1/2}}^2 &\leq C \left[t^{2(\alpha-1)} \sum_{i=1}^{\infty} \lambda_i^3 |\varphi_i(0)|^2 + \sum_{i=1}^{\infty} \lambda_i |F_{k\tau,i}(t)|^2 + \int_0^t \sum_{i=1}^{\infty} \lambda_i |F_{k\tau,i}(s)|^2 ds \right] \\
&\leq C(1 + t^{2(\alpha-1)}). \quad (2.19)
\end{aligned}$$

Since

$$\begin{aligned}
\partial_t F_{k\tau}(x, t) &= b \partial_t \varphi(x, t - \tau) + \partial_t f(x, t) \quad \text{for } k = 1, \\
\partial_t F_{k\tau}(x, t) &= b \partial_t u_{j\tau}(x, t - \tau) + \partial_t f(x, t), \\
j\tau < t \leq (j+1)\tau, \quad 1 \leq j \leq k-1 \quad &\text{for } k = 2, 3, \dots, K,
\end{aligned}$$

it is clear that the conditions (I) and (II) give

$$\begin{aligned}
\|\partial_t F_{k\tau}(x, t)\|_{G^{1/2}}^2 &\leq C [\|\partial_t \varphi(x, t - \tau)\|_{G^{1/2}}^2 + \|\partial_t f(x, t)\|_{G^{1/2}}^2] \\
&\leq C(1 + t^{2(\alpha-1)}) \quad \text{for } k = 1, \quad (2.20)
\end{aligned}$$

and for $k \geq 2$, $1 \leq j \leq k-1$, $t \in (j\tau, (j+1)\tau]$, one has

$$\begin{aligned}
\|\partial_t F_{k\tau}(x, t)\|_{G^{1/2}}^2 &\leq C [\|\partial_t u_{j\tau}(x, t - \tau)\|_{G^{1/2}}^2 + \|\partial_t f(x, t)\|_{G^{1/2}}^2] \\
&\leq C(1 + (t - \tau)^{2(\alpha-1)}). \quad (2.21)
\end{aligned}$$

This complete the proof of Lemma 2.2. \square

Now we state the regularity result.

Theorem 2.1. *Suppose that the assumptions (I) and (II) of Lemma 2.2 hold. Then the solution of (1.1)-(1.3) which is expressed by (2.6) and (2.7) satisfies*

$$\begin{aligned} |\partial_t^l u_{k\tau}(x, t)| &\leq C(1 + t^{\alpha-l}), & k = 1, 2, \dots, K, \quad l = 0, 1, \\ |\partial_t^2 u_\tau(x, t)| &\leq C(1 + t^{\alpha-2}), \\ |\partial_t^2 u_{k\tau}(x, t)| &\leq C(1 + (t - \tau)^{\alpha-1}), & k = 2, 3, \dots, K, \\ |\partial_x^n u_{k\tau}(x, t)| &\leq C, & k = 1, 2, \dots, K, \quad n = 0, 1, 2. \end{aligned}$$

Furthermore, under the conditions (I) and (II) of Lemma 2.2 and $|G^{m+1}(\varphi(x, 0))| \leq C$, the solution of (1.1)-(1.3) can be decomposed as

$$u(x, t) = \sum_{j=0}^m \gamma_j t^{j\alpha} + \tilde{\gamma}t + Y(x, t), \quad (x, t) \in [0, L] \times [0, K\tau], \quad (2.22)$$

where m is the smallest positive integer to ensure that $(m+1)\alpha > 1$, γ_j and $\tilde{\gamma}$ are bounded functions with respect to the variable x , and $Y(\cdot, t)$ satisfies

$$\begin{aligned} Y(\cdot, 0) = \partial_t Y(\cdot, t)|_{t=0} &= 0, & Y(\cdot, t) &\in C^1[0, K\tau], \\ |\partial_t Y(\cdot, t)| &\leq C(1 + t^{(m+1)\alpha-1}), & \partial_t^2 Y(\cdot, t) &\in L^1[0, K\tau]. \end{aligned} \quad (2.23)$$

Proof. First, according to (2.6), (2.7) and (2.15), it is obvious that

$$\begin{aligned} |u_{k\tau}(x, t)| &= \left| \sum_{i=1}^{\infty} \left[\varphi_i(0) E_\alpha(-\lambda_i t^\alpha) + \int_0^t E_\alpha(-\lambda_i s^\alpha) F_{k\tau, i}(t-s) ds \right] X_i(x) \right| \\ &\leq C \left[\sum_{i=1}^{\infty} |\varphi_i(0)| + \int_0^t \sum_{i=1}^{\infty} |F_{k\tau, i}(t-s)| ds \right] \leq C(1 + t^\alpha). \end{aligned} \quad (2.24)$$

Moreover, it follows from the condition (I) and (2.15)-(2.17) that

$$\begin{aligned} |\partial_t u_{k\tau}(x, t)| &\leq C \left[t^{\alpha-1} \sum_{i=1}^{\infty} \lambda_i |\varphi_i(0)| + \sum_{i=1}^{\infty} |F_{k\tau, i}(t)| + \int_0^t (t-s)^{\alpha-1} \sum_{i=1}^{\infty} \lambda_i |F_{k\tau, i}(s)| ds \right] \\ &\leq C(1 + t^{\alpha-1}). \end{aligned} \quad (2.25)$$

Now we discuss $|\partial_t^2 u_{k\tau}(x, t)|$, $k = 1, 2, \dots, K$. Taking the first time derivative on both sides of (2.17) yields

$$\begin{aligned} \partial_t^2 u_{k\tau}(x, t) &= \sum_{i=1}^{\infty} \left\{ -\lambda_i t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_i t^\alpha) \varphi_i(0) + E_\alpha(0) \partial_t F_{k\tau, i}(t) \right. \\ &\quad + (-\lambda_i t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i t^\alpha)) F_{k\tau, i}(0) \\ &\quad \left. + \int_0^t -\lambda_i s^{\alpha-1} E_{\alpha, \alpha}(-\lambda_i s^\alpha) \partial_t F_{k\tau, i}(t-s) ds \right\} X_i(x). \end{aligned} \quad (2.26)$$

Furthermore, we have

$$|\partial_t^2 u_{k\tau}(x, t)| \leq C \left[t^{\alpha-2} \sum_{i=1}^{\infty} \lambda_i |\varphi_i(0)| + \sum_{i=1}^{\infty} |\partial_t F_{k\tau, i}(t)| + t^{\alpha-1} \sum_{i=1}^{\infty} \lambda_i |F_{k\tau, i}(0)| \right]$$

$$\begin{aligned}
& + \sum_{i=1}^{\infty} \left| \int_0^t -\lambda_i s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i s^\alpha) \partial_t F_{k\tau,i}(t-s) ds \right| \\
& \leq C \left[t^{\alpha-2} \|\varphi(x, 0)\|_{G^{3/2}} + \|\partial_t F_{k\tau}(x, t)\|_{G^{1/2}} \right. \\
& \quad \left. + t^{\alpha-1} \|F_{k\tau}(x, 0)\|_{G^{3/2}} + \int_0^t \|\partial_s F_{k\tau}(x, s)\|_{G^{1/2}} ds \right]. \tag{2.27}
\end{aligned}$$

According to Lemma 2.2, when $k = 1$ it yields

$$|\partial_t^2 u_\tau(x, t)| \leq C(1 + t^{\alpha-2}), \tag{2.28}$$

and when $k \geq 2$ it gives

$$\begin{aligned}
|\partial_t^2 u_{k\tau}(x, t)| & \leq C \left[1 + (t - \tau)^{\alpha-1} + \int_0^\tau \|\partial_s F_{k\tau}(x, s)\|_{G^{1/2}} ds + \int_\tau^t \|\partial_s F_{k\tau}(x, s)\|_{G^{1/2}} ds \right] \\
& \leq C[1 + (t - \tau)^{\alpha-1}], \tag{2.29}
\end{aligned}$$

where the fact that $k \geq 2$ means $t > \tau$ is used.

Next we discuss the boundedness of $\partial_x^n u_{k\tau}(x, t)$, $n = 1, 2$. By [24, pp. 151–152], it is easy to know that

$$\left| \sum_{i=1}^{\infty} \varphi_i(0) E_\alpha(-\lambda_i t^\alpha) X_i^{(n)}(x) \right| \leq C, \quad n = 1, 2. \tag{2.30}$$

In addition, by performing analogous manipulation of [24, pp. 151–152], it is obvious that

$$\sum_{i=1}^{\infty} \left| F_{k\tau,i}(t-s) E_\alpha(-\lambda_i s^\alpha) X_i^{(n)}(x) \right| \leq C, \quad n = 1, 2. \tag{2.31}$$

Since

$$\begin{aligned}
|\partial_x^n u_{k\tau}(x, t)| & = \left| \sum_{i=1}^{\infty} \varphi_i(0) E_\alpha(-\lambda_i t^\alpha) X_i^{(n)}(x) \right. \\
& \quad \left. + \int_0^t \sum_{i=1}^{\infty} F_{k\tau,i}(t-s) E_\alpha(-\lambda_i s^\alpha) X_i^{(n)}(x) ds \right| \\
& \leq \left| \sum_{i=1}^{\infty} \varphi_i(0) E_\alpha(-\lambda_i t^\alpha) X_i^{(n)}(x) \right| \\
& \quad + \int_0^t \left| \sum_{i=1}^{\infty} F_{k\tau,i}(t-s) E_\alpha(-\lambda_i s^\alpha) X_i^{(n)}(x) \right| ds, \tag{2.32}
\end{aligned}$$

we can immediately obtain $|\partial_x^n u_{k\tau}(x, t)| \leq C$, $n = 1, 2$ from (2.30) and (2.31).

Now we discuss (2.22) and (2.23). For $t \in ((k-1)\tau, k\tau]$, let

$$\begin{aligned}
Y_1(x, t) & = \sum_{i=1}^{\infty} \int_0^t E_\alpha(-\lambda_i s^\alpha) F_{k\tau,i}(t-s) ds \cdot X_i - E_\alpha(0) F_\tau(x, 0) t, \\
Y_2(x, t) & = \sum_{i=1}^{\infty} \varphi_i(0) \sum_{j=m+1}^{\infty} \frac{(-\lambda_i t^\alpha)^j}{\Gamma(j\alpha + 1)} \cdot X_i.
\end{aligned}$$

Then it follows from (2.6) and (2.7) that the solution of (1.1)-(1.3) can be written as

$$\begin{aligned} u_{k\tau}(x, t) &= \sum_{j=0}^m \frac{(-1)^j t^{j\alpha}}{\Gamma(j\alpha + 1)} G^j(\varphi(x, 0)) + E_\alpha(0)F_\tau(x, 0)t + Y(x, t), \\ &:= \sum_{j=0}^m \gamma_j t^{j\alpha} + \tilde{\gamma}t + Y(x, t), \quad t \in ((k-1)\tau, k\tau], \end{aligned} \quad (2.33)$$

where $Y(x, t) = Y_1(x, t) + Y_2(x, t)$. First, it is obvious that γ_j is bounded, and the boundedness of Mittag-Leffler function and the continuity of φ, f can ensure the boundedness of $\tilde{\gamma}$. According to the definition of $Y_1(x, t)$ and $Y_2(x, t)$, it is easy to check $Y(\cdot, 0) = 0$. Furthermore, it follows from (2.6) and (2.7) that

$$Y_1(x, t) = u_{k\tau}(x, t) - \sum_{i=1}^{\infty} \varphi_i(0)E_\alpha(-\lambda_i t^\alpha)X_i(x) - E_\alpha(0)F_\tau(x, 0)t, \quad (x, t) \in [0, L] \times ((k-1)\tau, k\tau].$$

Taking the first derivative on both sides of the above equation on t and according to (2.17), it yields

$$\begin{aligned} \partial_t Y_1(x, t) &= \sum_{i=1}^{\infty} \left\{ E_\alpha(0)F_{k\tau, i}(t) + \int_0^t (-\lambda_i s^{\alpha-1})E_{\alpha, \alpha}(-\lambda_i s^\alpha)F_{k\tau, i}(t-s)ds \right\} X_i(x) \\ &\quad - E_\alpha(0)F_\tau(x, 0), \quad (x, t) \in [0, L] \times ((k-1)\tau, k\tau]. \end{aligned}$$

Besides, it follows from (2.6)-(2.8) that $F_{k\tau}$ is obviously a continuous function on t . Therefore

$$\partial_t Y_1(\cdot, t)|_{t=0} = 0, \quad Y_1(\cdot, t) \in C^1[0, K\tau], \quad |\partial_t Y_1(\cdot, t)| \leq C.$$

Note that

$$\begin{aligned} \partial_t^2 Y_1(x, t) &= \sum_{i=1}^{\infty} \left\{ E_\alpha(0)\partial_t F_{k\tau, i}(t) + (-\lambda_i t^{\alpha-1})E_{\alpha, \alpha}(-\lambda_i t^\alpha)F_{k\tau, i}(0) \right. \\ &\quad \left. + \int_0^t -\lambda_i s^{\alpha-1}E_{\alpha, \alpha}(-\lambda_i s^\alpha)\partial_t F_{k\tau, i}(t-s)ds \right\} X_i(x) \\ &\leq C \left[\|\partial_t F_{k\tau}(x, t)\|_{G^{1/2}} + \|F_{k\tau}(x, 0)\|_{G^{3/2}} t^{\alpha-1} + \int_0^t \|\partial_t F_{k\tau}(x, t-s)\|_{G^{1/2}} ds \right]. \end{aligned} \quad (2.34)$$

Hence Lemma 2.2 implies that

$$\begin{aligned} \partial_t^2 Y_1(x, t) &\leq C(1 + t^{\alpha-1}), \quad t \in (0, \tau], \\ \partial_t^2 Y_1(x, t) &\leq C(1 + (t - \tau)^{\alpha-1}), \quad t \in (\tau, K\tau]. \end{aligned}$$

For $\partial_t Y_2(x, t)$, it is clear that

$$\partial_t Y_2(x, t) = \sum_{i=1}^{\infty} \varphi_i(x, 0)(-\lambda_i)^{m+1} t^{(m+1)\alpha-1} X_i \cdot \left(\sum_{j=0}^{\infty} \frac{(-\lambda_i)^j t^{j\alpha}}{\Gamma(j\alpha + (m+1)\alpha)} \right).$$

The boundedness of the Mittag-Leffler function gives

$$|\partial_t Y_2(x, t)| \leq C |G^{m+1}(\varphi(x, 0))| t^{(m+1)\alpha-1}.$$

Similarly, we can also obtain

$$|\partial_t^2 Y_2(x, t)| \leq C |G^{m+1}(\varphi(x, 0))| t^{(m+1)\alpha-2}.$$

Therefore, when $(m+1)\alpha > 1$, one has

$$\partial_t Y_2(\cdot, t)|_{t=0} = 0, \quad Y_2(\cdot, t) \in C^1[0, K\tau], \quad |\partial_t^2 Y_2(\cdot, t)| \leq C t^{(m+1)\alpha-2}.$$

The above discussion means that (2.22) and (2.23) hold. \square

3. Fully Discrete Scheme and Its Stability

Similar to [5, Eq. (3.1)] to take the Riemann-Liouville integral ${}_0I_t^{1-\alpha}$ to the both sides of (1.1), then by [17, Property 2.4] the problem (1.1)-(1.3) can be transformed into

$${}_0^C D_t^\alpha u(x, t) = p\Delta u(x, t) + au(x, t) + b_0 I_t^{1-\alpha} u(x, t - \tau) + \tilde{f}(x, t), \quad (x, t) \in \Omega \times (0, K\tau], \quad (3.1)$$

$$u(x, t) = \varphi(x, t), \quad (x, t) \in \bar{\Omega} \times [-\tau, 0], \quad (3.2)$$

$$u(x, t)|_{\partial\Omega} = 0, \quad t \in [-\tau, K\tau], \quad (3.3)$$

where

$${}_0I_t^\beta u(x, t) := \int_0^t \omega_\beta(t-s) u(x, s) ds, \quad \tilde{f}(x, t) = {}_0I_t^{1-\alpha} f(x, t)$$

and ${}_0^C D_t^\alpha u(x, t)$ is the Caputo fractional derivative defined by

$${}_0^C D_t^\alpha u(x, t) = \int_0^t \omega_{1-\alpha}(t-s) \partial_s u(x, s) ds.$$

In order to establish a suitable numerical scheme to facilitate the discussion of error estimates, when $t > \tau$, we rewrite the third term of the right-hand side of (3.1) as

$$\begin{aligned} {}_0I_t^{1-\alpha} u(x, t - \tau) &= \int_\tau^t \omega_{1-\alpha}(t-s) u(x, s - \tau) ds + \int_0^\tau \omega_{1-\alpha}(t-s) \varphi(x, s - \tau) ds \\ &= \int_0^{t-\tau} \omega_{1-\alpha}(t-\tau-s) u(x, s) ds + \int_0^t \omega_{1-\alpha}(t-s) \tilde{\varphi}(x, s - \tau) ds \\ &\quad - \partial_t \varphi(x, 0) \int_\tau^t \omega_{1-\alpha}(t-s)(s-\tau) ds - \varphi(x, 0) \int_\tau^t \omega_{1-\alpha}(t-s) ds \\ &= \int_0^{t-\tau} \omega_{1-\alpha}(t-\tau-s) (u(x, s) - \varphi(x, 0)) ds \\ &\quad + \int_0^t \omega_{1-\alpha}(t-s) (\tilde{\varphi}(x, s - \tau) - \varphi(x, -\tau)) ds \\ &\quad - \left[\frac{(t-\tau)^{2-\alpha}}{\Gamma(3-\alpha)} \partial_t \varphi(x, 0) - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \varphi(x, -\tau) \right], \end{aligned} \quad (3.4)$$

where

$$\tilde{\varphi}(x, t - \tau) = \begin{cases} \varphi(x, t - \tau), & t \leq \tau, \\ \partial_t \varphi(x, 0)(t - \tau) + \varphi(x, 0), & t > \tau. \end{cases} \quad (3.5)$$

The above equality implies that

$${}_0I_t^{1-\alpha}u(x, t-\tau) = \begin{cases} {}_0I_t^{1-\alpha}(\varphi(x, t-\tau) - \varphi(x, -\tau)) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\varphi(x, -\tau), & t \leq \tau, \\ {}_0I_{t-\tau}^{1-\alpha}(u(x, t) - \varphi(x, 0)) + {}_0I_t^{1-\alpha}(\tilde{\varphi}(x, t-\tau) - \varphi(x, -\tau)) \\ \quad - \left[\frac{(t-\tau)^{2-\alpha}}{\Gamma(3-\alpha)}\partial_t\varphi(x, 0) - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\varphi(x, -\tau) \right], & t > \tau. \end{cases} \quad (3.6)$$

Therefore, for $t = t_n$, (3.1) can be rewritten as

$${}^C D_t^\alpha u(x, t_n) = p\Delta u(x, t_n) + au(x, t_n) + b{}_0\bar{I}_t^{1-\alpha}u(x, t_n - \tau) + F^n(x), \quad (3.7)$$

where

$${}_0\bar{I}_t^{1-\alpha}u(x, t_n - \tau) = \begin{cases} {}_0I_t^{1-\alpha}(\varphi(x, t_n - \tau) - \varphi(x, -\tau)), & t_n \leq \tau, \\ {}_0I_t^{1-\alpha}(\tilde{\varphi}(x, t_n - \tau) - \varphi(x, -\tau)) + {}_0I_{t-\tau}^{1-\alpha}(u(x, t_n) - \varphi(x, 0)), & t_n > \tau, \end{cases}$$

and

$$F^n(x) = \begin{cases} \tilde{f}^n(x) + b\frac{t_n^{1-\alpha}}{\Gamma(2-\alpha)}\varphi(x, -\tau), & t_n \leq \tau, \\ \tilde{f}^n(x) - b\left[\frac{(t_n - \tau)^{2-\alpha}}{\Gamma(3-\alpha)}\partial_t\varphi(x, 0) - \frac{t_n^{1-\alpha}}{\Gamma(2-\alpha)}\varphi(x, -\tau) \right], & t_n > \tau. \end{cases}$$

Now we consider the discretization of fractional operators. For a given integer $N > 0$, let $\rho = \tau/N$, $t_n = n\rho$, $-N \leq n \leq KN$, $u^k = u(t_k)$ and $\nabla_t u^k = u^k - u^{k-1}$, $1 \leq k \leq KN$, where $u(t_k)$ denotes the value of u at $t = t_k$. According to [16, Remark 2.13] and Theorem 2.1, we notice that

$${}^C D_t^\alpha u(t) = \partial_t^\alpha(u(t) - u(0)). \quad (3.8)$$

Therefore, the Grünwald-Letnikov type approximation [9, Eq. (7)] gives

$${}^C D_t^\alpha u^n = \rho^{-\alpha} \sum_{k=0}^n g_k^{(\alpha)}(u^{n-k} - u^0) + r_1^n, \quad (3.9)$$

where

$$g_k^{(\alpha)} := (-1)^k \binom{\alpha}{k},$$

and r_1^n is the corresponding local truncation error. Furthermore, we generalize the above Grünwald-Letnikov type approximation to discretize the fractional integral operators. If $t_n \leq \tau$, then

$${}_0I_t^{1-\alpha}(\varphi(t_n - \tau) - \varphi(-\tau)) = \rho^{1-\alpha} \sum_{k=0}^n g_{n-k}^{(\alpha-1)}(\varphi(t_k - \tau) - \varphi(-\tau)) + r_2^{n-N}, \quad (3.10)$$

and if $t_n > \tau$, then

$$\begin{aligned}
& {}_0I_t^{1-\alpha}(\tilde{\varphi}(t_n - \tau) - \varphi(-\tau)) + {}_0I_{t-\tau}^{1-\alpha}(u(t_n) - \varphi(0)) \\
&= \rho^{1-\alpha} \sum_{k=0}^n g_{n-k}^{(\alpha-1)}(\tilde{\varphi}(t_k - \tau) - \varphi(-\tau)) + \rho^{1-\alpha} \sum_{k=0}^{n-N} g_{n-N-k}^{(\alpha-1)}(u(t_k) - \varphi(0)) + r_2^{n-N} \\
&= \rho^{1-\alpha} \sum_{k=0}^{N-1} g_{n-k}^{(\alpha-1)}\tilde{\varphi}(t_k - \tau) + \rho^{1-\alpha} \sum_{k=0}^{n-N} g_{n-N-k}^{(\alpha-1)}u(t_k) + \rho^{1-\alpha} \sum_{k=N}^n g_{n-k}^{(\alpha-1)}\tilde{\varphi}(t_k - \tau) \\
&\quad - \varphi(-\tau) \cdot \rho^{1-\alpha} \sum_{k=0}^n g_{n-k}^{(\alpha-1)} - \varphi(0) \cdot \rho^{1-\alpha} \sum_{k=0}^{n-N} g_{n-N-k}^{(\alpha-1)} + r_2^{n-N} \\
&= \rho^{1-\alpha} \sum_{k=0}^n g_{n-k}^{(\alpha-1)}u^{k-N} + \partial_t \varphi(0) \cdot \rho^{1-\alpha} \sum_{k=N}^n g_{n-k}^{(\alpha-1)}(t_k - \tau) \\
&\quad - \varphi(-\tau) \cdot \rho^{1-\alpha} \sum_{k=0}^n g_{n-k}^{(\alpha-1)} + r_2^{n-N}. \tag{3.11}
\end{aligned}$$

It should be pointed out that the above discretization of fractional integrals can be considered as a special case of Lubich's convolution quadrature method [22, 23], and (3.10) and (3.11) can be uniformly written as

$${}_0\bar{I}_t^{1-\alpha}u^{n-N} = J^{1-\alpha}u^{n-N} + H^n + r_2^{n-N}, \tag{3.12}$$

where

$$\begin{aligned}
J^{1-\alpha}u^{n-N} &= \rho^{1-\alpha} \sum_{k=0}^n g_{n-k}^{(\alpha-1)}u^{k-N}, \\
H^n &= \partial_t \varphi(0) \cdot \rho^{1-\alpha} \sum_{k=N}^n g_{n-k}^{(\alpha-1)}(t_k - \tau) - \varphi(-\tau) \cdot \rho^{1-\alpha} \sum_{k=0}^n g_{n-k}^{(\alpha-1)},
\end{aligned}$$

and the local truncation error

$$r_2^{n-N} = {}_0\bar{I}_t^{1-\alpha}u^{n-N} - J^{1-\alpha}u^{n-N} - H^n.$$

In addition, according to [9], one has

$$g_k^{(\alpha)} = \rho^\alpha (A_k - A_{k-1})$$

with

$$A_{k-1} = \rho^{-\alpha} \frac{\Gamma(k - \alpha)}{\Gamma(1 - \alpha)\Gamma(k)}, \quad k \geq 1.$$

Hence (3.9) can be further rewritten as

$${}_0^C D_t^\alpha u(t_n) = \sum_{k=1}^n A_{n-k} \nabla_t u^k + r_1^n := \bar{\partial}_\rho^\alpha u^n + r_1^n. \tag{3.13}$$

We also note that

$$g_k^{(\alpha-1)} = \frac{\Gamma(k + 1 - \alpha)}{\Gamma(1 - \alpha)\Gamma(k + 1)}.$$

It means that

$$J^{1-\alpha} u^{n-N} = \rho \sum_{k=0}^n A_k u^{n-N-k}. \quad (3.14)$$

Let X_h be a continuous piecewise linear finite element space of $H_0^1(\Omega)$ under the quasi-uniform partition on Ω with the maximum diameter h . By (3.12) and (3.13), we propose the fully discrete finite element scheme of problem (3.1)-(3.3): Find $u_h^n \in X_h, n = 1, 2, \dots, KN$ such that

$$\begin{aligned} (\bar{\partial}_\rho^\alpha u_h^n, v_h) &= -p(\nabla u_h^n, \nabla v_h) + a(u_h^n, v_h) + b(J^{1-\alpha} u_h^{n-N}, v_h) \\ &\quad + b(H^n, v_h) + (F^n, v_h), \quad \forall v_h \in X_h, \end{aligned} \quad (3.15)$$

where $u_h^n \in X_h, -N \leq n \leq 0$ is a suitable approximation of the initial value function $\varphi(t_n)$.

In order to discuss the stability of the fully discrete finite element scheme (3.15), we introduce three useful lemmas firstly.

Lemma 3.1. *For the coefficients $A_n, 1 \leq n \leq KN$, one has*

$$A_n > 0, \quad \rho \sum_{k=0}^n A_k < \rho^{1-\alpha} + \frac{t_n^{1-\alpha}}{\Gamma(2-\alpha)}.$$

Proof. According to the definition of A_n , it is obvious that $A_n > 0$ and

$$\rho \sum_{k=0}^n A_k = \rho^{1-\alpha} \sum_{k=1}^{n+1} \frac{\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k)}. \quad (3.16)$$

From [9, Eqs. (3) and (4)], we know that

$$\frac{\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k)} < \frac{(k-1)^{-\alpha}}{\Gamma(1-\alpha)}, \quad k \geq 2. \quad (3.17)$$

Therefore

$$\begin{aligned} \rho \sum_{k=0}^n A_k &< \rho^{1-\alpha} + \rho^{1-\alpha} \sum_{k=2}^{n+1} \frac{(k-1)^{-\alpha}}{\Gamma(1-\alpha)} \\ &< \rho^{1-\alpha} + \frac{\rho^{1-\alpha}}{\Gamma(1-\alpha)} \sum_{k=2}^{n+1} \int_{k-2}^{k-1} s^{-\alpha} ds \\ &< \rho^{1-\alpha} + \frac{t_n^{1-\alpha}}{\Gamma(2-\alpha)}. \end{aligned} \quad (3.18)$$

The proof is complete. \square

Define

$$P_0 = \frac{1}{A_0}, \quad P_n = \frac{1}{A_0} \sum_{j=0}^{n-1} P_j (A_{n-j-1} - A_{n-j}), \quad (3.19)$$

and

$$K_{\beta,n} = \begin{cases} 1 + \frac{1-n^{1-\beta}}{\beta-1}, & \beta \neq 1, \\ 1 + \ln n, & \beta = 1. \end{cases} \quad (3.20)$$

Then the following inequality and properties hold.

Lemma 3.2 ([9, 21]). Let $\{g^j\}_{j=1}^{KN}$ and $\{\mu_j\}_{j=0}^{KN}$ be given nonnegative sequences. Suppose that there exists a constant Λ such that $\Lambda \geq \sum_{j=0}^{KN} \mu_j$ and the step size satisfies $\rho \cdot \sqrt[3]{2\pi_A \Gamma(2-\alpha)\Lambda} \leq 1$. Then, for any nonnegative sequence $\{v^j\}_{j=0}^{KN}$ such that

$$\sum_{k=1}^n A_{n-k} \nabla_t v^k \leq \sum_{k=1}^n \mu_{n-k} v^k + g^n, \quad 1 \leq n \leq KN, \quad (3.21)$$

it holds that

$$v^n \leq 2E_\alpha(2\pi_A \Lambda t_n^\alpha) \left(v^0 + \max_{1 \leq k \leq n} \sum_{j=1}^k P_{k-j} g^j \right), \quad 1 \leq n \leq KN, \quad (3.22)$$

where the value of π_A , i.e. $\pi_A = 2^\alpha$ has been discussed in [9].

Lemma 3.3 ([9]). For the discrete coefficient P_j , it has the following properties:

$$\begin{aligned} (i) \quad & \sum_{j=1}^n P_{n-j} \leq \frac{2^\alpha t_n^\alpha}{\Gamma(1+\alpha)}, \\ (ii) \quad & \sum_{j=k}^n P_{n-j} A_{j-k} = 1, \quad 1 \leq k \leq n, \\ (iii) \quad & \sum_{j=1}^n j^{-\beta} P_{n-j} \leq \rho^\alpha n^{-\beta} + \frac{\rho^\alpha}{\Gamma(\alpha)} \left[K_{\beta,n} \left(\frac{n}{2}\right)^{\alpha-1} + \frac{1}{\alpha} \left(\frac{n}{2}\right)^{\alpha-\beta} \right], \end{aligned}$$

where $\beta \geq 0$ is a constant and $n = 1, 2, \dots, KN$.

Now we state the stability of the fully discrete finite element scheme (3.15).

Theorem 3.1. Let u_h^n be the solution to (3.15), and denote $c = \sup_{0 \leq k \leq N} \|u_h^{k-N}\|_0$. Then for $1 \leq n \leq KN$, it holds

$$\|u_h^n\|_0 \leq 2E_\alpha(2^{\alpha+1} \Lambda t_n^\alpha) \left(\|u_h^0\|_0 + \frac{2^\alpha t_n^\alpha}{\Gamma(1+\alpha)} \max_{1 \leq k \leq n} y^k \right), \quad (3.23)$$

where if $1 \leq n \leq N$, then

$$\Lambda = 0, \quad y^k = |b| \left(\rho^{1-\alpha} + \frac{t_k^{1-\alpha}}{\Gamma(2-\alpha)} \right) (c + \|\varphi(-\tau)\|_0) + \|F^k\|_0,$$

and if $N+1 \leq n \leq KN$, then

$$\Lambda = |b| \left(\rho^{1-\alpha} + \frac{(K\tau)^{1-\alpha}}{\Gamma(2-\alpha)} \right), \quad y^k = c|b|\rho \sum_{j=0}^N A_{k-j} + |b| \|H^k\|_0 + \|F^k\|_0.$$

Proof. Taking $v_h = u_h^n$ in (3.15), we have

$$\begin{aligned} (\bar{\partial}_\rho^\alpha u_h^n, u_h^n) &= -p(\nabla u_h^n, \nabla u_h^n) + a(u_h^n, u_h^n) + b(J^{1-\alpha} u_h^{n-N}, u_h^n) \\ &\quad + b(H^n, u_h^n) + (F^n, u_h^n), \end{aligned} \quad (3.24)$$

i.e.

$$\begin{aligned} A_0 \|u_h^n\|_0^2 &\leq \sum_{k=1}^{n-1} (A_{k-1} - A_k) \|u_h^{n-k}\|_0 \|u_h^n\|_0 + A_{n-1} \|u_h^0\|_0 \|u_h^n\|_0 - p \|\nabla u_h^n\|_0^2 + a \|u_h^n\|_0^2 \\ &\quad + |b| \|J^{1-\alpha} u_h^{n-N}\|_0 \|u_h^n\|_0 + |b| \|H^n\|_0 \|u_h^n\|_0 + \|F^n\|_0 \|u_h^n\|_0. \end{aligned} \quad (3.25)$$

Using the fact $p > 0, a \leq 0$ and $A_{k-1} > A_k$ mentioned in [9], (3.25) leads to

$$\sum_{k=1}^n A_{n-k} \nabla_t \|u_h^k\|_0 \leq |b| \|J^{1-\alpha} u_h^{n-N}\|_0 + |b| \|H^n\|_0 + \|F^n\|_0. \quad (3.26)$$

It follows from Lemma 3.1 and (3.26) that if $1 \leq n \leq N$, then

$$\begin{aligned} \sum_{k=1}^n A_{n-k} \nabla_t \|u_h^k\|_0 &\leq |b| \rho \sum_{k=0}^n A_k \|u_h^{n-N-k}\|_0 + |b| \rho \sum_{k=0}^n A_k \|\varphi(-\tau)\|_0 + \|F^n\|_0 \\ &\leq |b| \left(\rho^{1-\alpha} + \frac{t_n^{1-\alpha}}{\Gamma(2-\alpha)} \right) (c + \|\varphi(-\tau)\|_0) + \|F^n\|_0, \end{aligned} \quad (3.27)$$

and if $N+1 \leq n \leq KN$, then

$$\begin{aligned} \sum_{k=1}^n A_{n-k} \nabla_t \|u_h^k\|_0 &\leq |b| \rho \sum_{k=1}^N A_{n-k} \|u_h^{k-N}\|_0 + |b| \rho \sum_{k=N+1}^n A_{n-k} \|u_h^{k-N}\|_0 + |b| \|H^n\|_0 + \|F^n\|_0 \\ &\leq |b| \rho \sum_{k=1}^{n-N} A_{n-N-k} \|u_h^k\|_0 + \left(c |b| \rho \sum_{k=0}^N A_{n-k} + |b| \|H^n\|_0 + \|F^n\|_0 \right). \end{aligned} \quad (3.28)$$

Applying Lemma 3.2 to (3.27) and (3.28), and noting that inequalities mentioned in Lemmas 3.1 and 3.3, it yields (3.23) immediately. \square

4. Convergence Analysis

In this section, we consider the convergence of the fully discrete scheme (3.15). Now we give three useful lemmas.

Lemma 4.1 ([39, Lemma 2.1]). *Let $w(t) = t^\sigma$, $\sigma > 0$ and β is a real number. Then*

$$\left| {}_{RL}D_{0,t}^\beta w^n - \rho^{-\beta} \sum_{j=0}^n g_j^{(\beta)} w^{n-j} \right| \leq C \left(\frac{\beta}{2} \left| \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\beta)} \right| \rho t_n^{\sigma-1-\beta} + \rho^2 t_n^{\sigma-2-\beta} \right), \quad (4.1)$$

where ${}_{RL}D_{0,t}^\beta w$ represents the Riemann-Liouville fractional derivative $\partial_t^\beta w$ for $\beta > 0$, and ${}_{RL}D_{0,t}^\beta w$ is the Riemann-Liouville fractional integral ${}_0I_t^{-\beta} w$ for $\beta < 0$.

Lemma 4.2 ([42, Theorem 1]). *Assume that $w(t) \in C^1[0, K\tau]$, $\partial_t^2 w(t) \in L^1[0, K\tau]$ and $w(0) = \partial_t w(0) = 0$. Then*

$$|{}^C D_t^\alpha w^n - \bar{\partial}_\rho^\alpha w^n| \leq C\rho. \quad (4.2)$$

Combining (2.22), (2.23), (3.8), Lemmas 4.1 and 4.2, it is clear that the local truncation error r_1^n in (3.13) can be bounded by

$$\begin{aligned} |r_1^n| &\leq C \left[\sum_{j=1}^m \left(\left| \frac{\Gamma(j\alpha+1)}{\Gamma((j-1)\alpha)} \right| \rho t_n^{(j-1)\alpha-1} + \rho^2 t_n^{(j-1)\alpha-2} \right) + \rho t_n^{-\alpha} + \rho^2 t_n^{-\alpha-1} + \rho \right] \\ &\leq C (\rho t_n^{-1} + \rho^2 t_n^{-2} + \rho t_n^{-\alpha}) \leq C \rho t_n^{-1}. \end{aligned} \quad (4.3)$$

Lemma 4.3 ([23, Theorem 3.1]). *Assume that $w(t) \in C^1[0, K\tau]$. Then*

$$\left| {}_0I_t^{1-\alpha} w^n - \rho^{1-\alpha} \sum_{j=0}^n g_{n-j}^{(\alpha-1)} w^j \right| \leq C t_n^{-\alpha} \rho \left(|w^0| + t_n \max_{0 \leq s \leq t_n} |\partial_t w(s)| \right). \quad (4.4)$$

Suppose that

$$\varphi(t) \in C^1([-\tau, 0]), \quad |\partial_t \varphi(t)| \leq C \quad (4.5)$$

for all $x \in [0, L]$. Then it follows from Lemmas 4.1 and 4.3 that

$$|r_2^{n-N}| \leq C \rho, \quad n \geq 1, \quad (4.6)$$

where the splitting

$$u = \left(\sum_{j=1}^m \gamma_j t^{j\alpha} + \tilde{\gamma} t \right) + (\varphi(0) + Y)$$

mentioned in (2.22) is used. Furthermore, for P_j defined in (3.19), the combination of Lemma 3.3(i) and (4.6) gives

$$\sum_{j=1}^n P_{n-j} |r_2^{j-N}| \leq C \rho. \quad (4.7)$$

In order to obtain a new Grönwall inequality to discuss the convergence of the numerical scheme (3.15), we give the following lemma.

Lemma 4.4. *Let*

$$\begin{aligned} Z_1 &= p_1 (1, 1, \dots, 1)^\top \in \mathbb{R}^n, \\ Z_2 &= p_2 \rho \left((1 + \ln n) t_n^{\alpha-1}, (1 + \ln(n-1)) t_{n-1}^{\alpha-1}, \dots, t_1^{\alpha-1} \right)^\top, \end{aligned}$$

and

$$M = p_3 \rho \underbrace{\begin{pmatrix} 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}}_N, \quad (4.8)$$

where p_1, p_2, p_3 are some nonnegative constants and $n \in [(k-1)N + 1, kN], k \geq 1$. Then we have

(i) $M^j = \mathbb{O}$, $j \geq k$.

(ii) For $M^q Z_1$, $q = 0, 1, \dots, k-1$, the following inequalities hold:

$$M^q Z_1 \leq Cp_1 \left(1, 1, \dots, 1, \underbrace{0, \dots, 0}_{qN} \right)^\top, \quad (4.9)$$

$$\sum_{j=0}^{k-1} M^j Z_1 \leq Cp_1 \left(\underbrace{k, \dots, k}_{n-(k-1)N}, \underbrace{k-1, \dots, k-1}_N, \dots, \underbrace{1, \dots, 1}_N \right)^\top. \quad (4.10)$$

(iii) For $M^q Z_2$, $q = 1, 2, \dots, k-1$, the following inequalities hold:

$$M^q Z_2 \leq C\rho \left(1 + \ln(n - qN), 1 + \ln(n - qN - 1), \dots, 1, \underbrace{0, \dots, 0}_{qN} \right)^\top, \quad (4.11)$$

and

$$\begin{aligned} \sum_{j=1}^{k-1} M^j Z_2 \leq C\rho \left(\underbrace{(k-1) + \sum_{j=1}^{k-1} \ln(n - jN), \dots, (k-1) + \sum_{j=1}^{k-1} \ln((k-1-j)N + 1)}_{n-(k-1)N}, \right. \\ \left. \dots, \underbrace{1 + \ln(2N - 1)}_N, \dots, \underbrace{1, 0, \dots, 0}_N \right)^\top. \end{aligned} \quad (4.12)$$

Proof. First, it follows from the mathematical induction that (i) can be obtained easily. For $M^q Z_1$, we notice that (ii) is obviously true for $q = 0$. Suppose that (ii) holds for $q \leq l$. Then

$$\begin{aligned} M^{l+1} Z_1 &= M \cdot M^l Z_1 \leq Cp_1 M \left(1, 1, \dots, 1, \underbrace{0, \dots, 0}_{lN} \right)^\top \\ &\leq Cp_1 \rho \left(n - (l+1)N, n - (l+1)N - 1, \dots, 1, \underbrace{0, \dots, 0}_{(l+1)N} \right)^\top \\ &\leq Cp_1 \left(1, 1, \dots, 1, \underbrace{0, \dots, 0}_{(l+1)N} \right)^\top, \end{aligned} \quad (4.13)$$

$$\sum_{j=0}^{k-1} M^j Z_1 \leq Cp_1 \left(\underbrace{k, \dots, k}_{n-(k-1)N}, \underbrace{k-1, \dots, k-1}_N, \dots, \underbrace{1, \dots, 1}_N \right)^\top. \quad (4.14)$$

Therefore the mathematical induction implies that (ii) is true.

Now we discuss (iii). For $q = 1$, one has

$$\begin{aligned} MZ_2 &= \lambda p_2 \rho^2 \left(\sum_{j=1}^{n-N} (1 + \ln j) t_j^{\alpha-1}, \sum_{j=1}^{n-N-1} (1 + \ln j) t_j^{\alpha-1}, \dots, t_1^{\alpha-1}, \underbrace{0, \dots, 0}_N \right)^\top \\ &\leq C\rho^{1+\alpha} \left((1 + \ln(n-N)) \sum_{j=1}^{n-N} j^{\alpha-1}, (1 + \ln(n-N-1)) \sum_{j=1}^{n-N-1} j^{\alpha-1}, \dots, 1, \underbrace{0, \dots, 0}_N \right)^\top. \end{aligned} \quad (4.15)$$

Since

$$\rho^\alpha \sum_{j=1}^{n-N} j^{\alpha-1} \leq \rho^\alpha \left(1 + \int_0^{n-N} s^{\alpha-1} ds \right) \leq C,$$

MZ_2 can be bounded by

$$MZ_2 \leq C\rho \left(1 + \ln(n-N), 1 + \ln(n-N-1), \dots, 1, \underbrace{0, \dots, 0}_N \right)^\top. \quad (4.16)$$

Suppose that (4.11) holds for $q \leq l$. When $q = l+1$, then

$$\begin{aligned} M^{l+1}Z_2 &= M \cdot M^l Z_2 \\ &\leq C\rho^2 \left(\sum_{j=1}^{n-(l+1)N} (1 + \ln j), \sum_{j=1}^{n-(l+1)N-1} (1 + \ln j), \dots, 1, \underbrace{0, \dots, 0}_{(l+1)N} \right)^\top \\ &\leq C\rho \left(1 + \ln(n - (l+1)N), 1 + \ln(n - (l+1)N - 1), \dots, 1, \underbrace{0, \dots, 0}_{(l+1)N} \right)^\top. \end{aligned} \quad (4.17)$$

Thus it is clear that (iii) is true from the mathematical induction. \square

Using the above results, we state a new Grönwall inequality as follows.

Lemma 4.5. *Assume that $\{z^n\}$ is a non-negative real sequence and satisfies*

$$\begin{aligned} z^n &\leq p_3\rho \sum_{j=N+1}^n z^{j-N} + p_1 + p_2\rho(1 + \ln n)t_n^{\alpha-1}, \\ (k-1)N + 1 &\leq n \leq kN, \quad k = 1, 2, \dots, K. \end{aligned} \quad (4.18)$$

Then

$$z^n \leq C \left[kp_1 + \rho \left(\sum_{j=1}^{k-1} (1 + \ln(n - jN)) + p_2(1 + \ln n)t_n^{\alpha-1} \right) \right]. \quad (4.19)$$

Proof. For $(k-1)N + 1 \leq n \leq kN$, let $Z = (z^n, z^{n-1}, \dots, z^1)^\top$. Then it follows from (4.18) that

$$Z \leq MZ + Z_1 + Z_2. \quad (4.20)$$

Applying this inequality repeatedly and noting that the fact $M^k = \mathbb{O}$, it yields

$$\begin{aligned} Z &\leq M(MZ + Z_1 + Z_2) + Z_1 + Z_2 \\ &\leq M^2Z + \sum_{j=0}^1 M^j Z_1 + \sum_{j=0}^1 M^j Z_2 \\ &\vdots \\ &\leq \sum_{j=0}^{k-1} M^j Z_1 + \sum_{j=0}^{k-1} M^j Z_2. \end{aligned} \quad (4.21)$$

The combination of (4.10) and (4.12) gives

$$z^n \leq C \left[kp_1 + \rho \left(\sum_{j=1}^{k-1} (1 + \ln(n - jN)) + p_2(1 + \ln n)t_n^{\alpha-1} \right) \right]. \quad (4.22)$$

The proof is complete. \square

In order to discuss the spatial error, we introduce the orthogonal projection operator $P_h : H_0^1(0, L) \rightarrow X_h$ defined by

$$(\nabla P_h \phi, \nabla v_h) = (\nabla \phi, \nabla v_h), \quad \forall v_h \in X_h. \quad (4.23)$$

In fact, it has a well known property (see [36, Lemma 1.1]), i.e.

$$\|P_h \phi - \phi\|_0 \leq Ch^2 \|\phi\|_{H^2(0, L)}. \quad (4.24)$$

Now we state the convergence of the numerical scheme (3.15).

Theorem 4.1. *Assume that $u(x, t)$ is the solution of (1.1)-(1.3) which can be decomposed into (2.22)-(2.23), $u(x, t) \in H^2(0, L)$ for fixed t , the condition (4.5) is satisfied, and $u_h^n \in X_h$, $-N \leq n \leq 0$ in (3.15) is a suitable approximation of $\varphi(x, t_n)$ such that $\|u_h^n - \varphi(x, t_n)\| \leq Ch^2$. Then the numerical solution u_h^n to (3.15) satisfies*

$$\|u^n - u_h^n\|_0 \leq C \left[h^2 + \rho \left(\sum_{j=1}^{k-1} (1 + \ln(n - jN)) + (1 + \ln n) t_n^{\alpha-1} \right) \right], \quad (4.25)$$

where $(k-1)N + 1 \leq n \leq kN, k = 1, 2, \dots, K$.

Proof. First, (3.7) gives

$$({}^C D_t^\alpha u^n, v_h) = -p(\nabla u^n, \nabla v_h) + a(u^n, v_h) + b({}_0 \bar{I}_t^{1-\alpha} u^{n-N}, v_h) + (F^n, v_h). \quad (4.26)$$

Let $\varepsilon_h^n = u_h^n - P_h u^n$. The combination of (3.15) and (4.26) yields

$$\begin{aligned} (\bar{\partial}_\rho^\alpha \varepsilon_h^n, v_h) &= -p(\nabla \varepsilon_h^n, \nabla v_h) + a(\varepsilon_h^n, v_h) + b(J^{1-\alpha} \varepsilon_h^{n-N}, v_h) \\ &\quad + (r_1^n, v_h) + (\bar{\partial}_\rho^\alpha (u^n - P_h u^n), v_h) \\ &\quad - a(u^n - P_h u^n, v_h) - b(r_2^{n-N}, v_h) \\ &\quad + b(J^{1-\alpha} (P_h u^{n-N} - u^{n-N}), v_h). \end{aligned} \quad (4.27)$$

Taking $v_h = \varepsilon_h^n$ into (4.27) leads to

$$\begin{aligned} A_0 \|\varepsilon_h^n\|_0^2 &= A_{n-1} (\varepsilon_h^0, \varepsilon_h^n) + \sum_{k=1}^{n-1} (A_{n-k-1} - A_{n-k}) (\varepsilon_h^k, \varepsilon_h^n) - p \|\nabla \varepsilon_h^n\|_0^2 \\ &\quad + a \|\varepsilon_h^n\|_0^2 + b(J^{1-\alpha} \varepsilon_h^{n-N}, \varepsilon_h^n) + (r_1^n, \varepsilon_h^n) + (\bar{\partial}_\rho^\alpha (u^n - P_h u^n), \varepsilon_h^n) \\ &\quad - a(u^n - P_h u^n, \varepsilon_h^n) - b(r_2^{n-N}, \varepsilon_h^n) + b(J^{1-\alpha} (P_h u^{n-N} - u^{n-N}), \varepsilon_h^n) \\ &\leq \left[A_{n-1} \|\varepsilon_h^0\|_0 + \sum_{k=1}^{n-1} (A_{n-k-1} - A_{n-k}) \|\varepsilon_h^k\|_0 + |b| \|J^{1-\alpha} \varepsilon_h^{n-N}\|_0 + \|r_1^n\|_0 \right. \\ &\quad \left. + \|\bar{\partial}_\rho^\alpha (u^n - P_h u^n)\|_0 - a \|u^n - P_h u^n\|_0 + |b| \|r_2^{n-N}\|_0 \right. \\ &\quad \left. + |b| \|J^{1-\alpha} (P_h u^{n-N} - u^{n-N})\|_0 \right] \|\varepsilon_h^n\|_0. \end{aligned} \quad (4.28)$$

The above inequality implies that

$$\begin{aligned} \sum_{i=1}^j A_{j-i} \nabla_t \|\varepsilon_h^i\|_0 &\leq |b| \|J^{1-\alpha} \varepsilon_h^{j-N}\|_0 + \|r_1^j\|_0 + \|\bar{\partial}_\rho^\alpha (u^j - P_h u^j)\|_0 - a \|u^j - P_h u^j\|_0 \\ &\quad + |b| \|r_2^{j-N}\|_0 + |b| \|J^{1-\alpha} (P_h u^{j-N} - u^{j-N})\|_0. \end{aligned} \quad (4.29)$$

Multiplying P_{n-j} on both sides of (4.29) and summing on j from 1 to n give

$$\begin{aligned} \|\varepsilon_h^n\|_0 &\leq |b|\rho \sum_{j=0}^n \|\varepsilon_h^{j-N}\|_0 + \sum_{j=1}^n P_{n-j} \|r_1^j\|_0 + \|u^n - P_h u^n\|_0 - a \sum_{j=1}^n P_{n-j} \|u^j - P_h u^j\|_0 \\ &\quad + |b| \sum_{j=1}^n P_{n-j} \|r_2^{j-N}\|_0 + |b|\rho \sum_{j=0}^n \|P_h u^{j-N} - u^{j-N}\|_0 + \|\varepsilon_h^0\|_0. \end{aligned} \quad (4.30)$$

For $\sum_{j=1}^n P_{n-j} \|r_1^j\|_0$, Lemma 3.3 and (4.3) imply that

$$\begin{aligned} \sum_{j=1}^n P_{n-j} \|r_1^j\|_0 &\leq C \sum_{j=1}^n P_{n-j} j^{-1} \leq C \rho^\alpha \left[n^{-1} + \frac{1}{\Gamma(\alpha)} \left((1 + \ln n) \left(\frac{n}{2}\right)^{\alpha-1} + \frac{1}{\alpha} \left(\frac{n}{2}\right)^{\alpha-1} \right) \right] \\ &\leq C \rho (1 + \ln n) t_n^{\alpha-1}. \end{aligned} \quad (4.31)$$

Similarly we can also easily obtain that

$$\sum_{j=1}^n P_{n-j} \|r_2^{j-N}\|_0 \leq C \rho. \quad (4.32)$$

Taking the estimates (4.31) and (4.32) into (4.30), and noting that the fact $\|u^j - u_h^j\|_0 \leq Ch^2$ and the assumption $\|\varepsilon_h^{j-N}\|_0 \leq Ch^2$ for $j = 0, 1, \dots, N$, one has

$$\|\varepsilon_h^n\|_0 \leq |b|\rho \sum_{j=N+1}^n \|\varepsilon_h^{j-N}\|_0 + Ch^2 + C\rho(1 + \ln n)t_n^{\alpha-1}. \quad (4.33)$$

Therefore it follows from Lemma 4.5 that

$$\|\varepsilon_h^n\|_0 \leq C \left[h^2 + \rho \left(\sum_{j=1}^{k-1} (1 + \ln(n - jN)) + (1 + \ln n)t_n^{\alpha-1} \right) \right]. \quad (4.34)$$

The proof is complete. \square

In view of Theorem 2.1, we also have an alternative version of the error estimate.

Theorem 4.2. *Assume that $u(x, t)$ is the solution of (1.1)-(1.3), the conditions of Theorem 2.1 and (4.5) hold, and $u_h^n \in X_h$, $-N \leq n \leq 0$ in (3.15) is a suitable approximation of $\varphi(x, t_n)$ such that $\|u_h^n - \varphi(x, t_n)\| \leq Ch^2$. Then the numerical solution u_h^n to (3.15) satisfies*

$$\|u^n - u_h^n\|_0 \leq C \left[h^2 + \rho \left(\sum_{j=1}^{k-1} (1 + \ln(n - jN)) + (1 + \ln n)t_n^{\alpha-1} \right) \right], \quad (4.35)$$

where $(k-1)N + 1 \leq n \leq kN$, $k = 1, 2, \dots, K$.

5. Numerical Experiment

In this section, some numerical tests are presented to support the achieved theoretical results. In the first case, an exact solution is provided for the considered problem, which can obviously be decomposed into (2.22)-(2.23). Therefore we will use it to check the Theorem 4.1. Then,

in the next case, we do not provide the exact solution, but give suitable right hand function and initial value function which satisfy the conditions of Theorem 4.2. The obtained numerical results will be used to test the Theorem 4.2. Define $E(h, N, k) = \max_{(k-1)N+1 \leq n \leq kN} \|\bar{u}^n - u_h^n\|_0$ and

$$rate_t := \log_2 \left(\frac{E(h, N, k)}{E(h, 2N, k)} \right), \quad rate_s := \log_2 \left(\frac{E(h, N, k)}{E(h/2, N, k)} \right),$$

where $\bar{u}^n = u^n$ when the exact solution is known, otherwise \bar{u}^n represents an approximation of u^n on a sufficiently fine mesh.

Example 5.1. For the problem (1.1)-(1.3) on the time and space domain $(0, 3]$ and $[0, 1]$, two cases are considered as follows:

- In the first case, let $\tau = 1, p = 1/5, a = 0, b = 1$, and $\varphi(x, t) = -\Gamma(\alpha + 1)\sin(\pi x)/\pi^2$. We take

$$u(x, t) = \begin{cases} \left(\frac{-\Gamma(\alpha + 1)}{\pi^2} + t^\alpha \right) \sin(\pi x), & 0 < t \leq 1, \\ \left(\frac{-\Gamma(\alpha + 1)}{\pi^2} + t^\alpha + (t - 1)^{\alpha+1} \right) \sin(\pi x), & 1 < t \leq 2, \\ \left(\frac{-\Gamma(\alpha + 1)}{\pi^2} + t^\alpha + (t - 1)^{\alpha+1} + (t - 2)^{\alpha+2} \right) \sin(\pi x), & 2 < t \leq 3 \end{cases}$$

as the exact solution. The right hand function $f(x, t)$ can be computed by the exact solution and initial value function.

- In the second case, instead of giving the exact solution, we give

$$f(x, t) = \begin{cases} t\sin(\pi x), & 0 < t \leq 1, \\ (t + (t - 1)^2) \sin(\pi x), & 1 < t \leq 2, \\ (t + (t - 1)^2 + (t - 2)^2) \sin(\pi x), & 2 < t \leq 3, \end{cases}$$

$\varphi(x, t) = (1 + 0.1t) \sin(\pi x)$, and choose the parameters $\tau = 1, p = 1/100, a = -1, b = 1$.

The numerical results of the first case are presented in Tables 5.1 and 5.2, and the numerical tests of the second case are shown in Tables 5.3 and 5.4. In Tables 5.1 and 5.3, we fix $h = 1/1000$

Table 5.1: Numerical temporal accuracy for different α .

α	N	$k = 1$		$k = 2$		$k = 3$	
		$E(h, N, k)$	$rate_t$	$E(h, N, k)$	$rate_t$	$E(h, N, k)$	$rate_t$
0.7	400	9.4632e-04		4.3779e-04		1.5550e-03	
	800	5.8905e-04	0.683	2.1786e-04	1.006	7.7459e-04	1.005
	1600	3.6511e-04	0.690	1.0774e-04	1.015	3.8416e-04	1.011
	3200	2.2570e-04	0.693	5.2634e-05	1.033	1.8888e-04	1.024
0.9	400	1.2244e-04		6.3351e-04		2.2168e-03	
	800	6.6105e-05	0.889	3.1553e-04	1.005	1.1050e-03	1.004
	1600	3.5548e-05	0.894	1.5644e-04	1.012	5.4887e-04	1.009
	3200	1.9066e-05	0.898	7.6877e-05	1.024	2.7075e-04	1.019

Table 5.2: The temporal accuracy and spatial accuracy for $\alpha = 0.7$ at t_{kN} .

N	$k = 1$		$k = 2$		$k = 3$	
	$\ u^N - u_h^N\ _0$	$rate_t$	$\ u^{2N} - u_h^{2N}\ _0$	$rate_t$	$\ u^{3N} - u_h^{3N}\ _0$	$rate_t$
400	1.9718e-05		4.4021e-04		1.5612e-03	
800	9.7579e-06	1.014	2.2028e-04	0.998	7.8079e-04	0.999
1600	4.8601e-06	1.005	1.1016e-04	0.999	3.9035e-04	1.000
3200	2.4361e-06	0.996	5.5051e-05	1.000	1.9507e-04	1.000
h	$\ u^N - u_h^N\ _0$	$rate_s$	$\ u^{2N} - u_h^{2N}\ _0$	$rate_s$	$\ u^{3N} - u_h^{3N}\ _0$	$rate_s$
1/8	1.3786e-02		4.0332e-02		1.0297e-01	
1/16	3.4693e-03	1.990	1.0142e-02	1.991	2.5891e-02	1.991
1/32	8.6928e-04	1.996	2.5273e-03	2.004	6.4395e-03	2.007
1/64	2.1796e-04	1.996	6.1931e-04	2.028	1.5652e-03	2.040

Table 5.3: Numerical temporal accuracy for different α .

α	N	$k = 1$		$k = 2$		$k = 3$	
		$E(h, N, k)$	$rate_t$	$E(h, N, k)$	$rate_t$	$E(h, N, k)$	$rate_t$
0.4	400	4.1103e-03		3.8838e-04		8.9101e-04	
	800	3.1522e-03	0.382	1.9445e-04	0.998	4.4578e-04	0.999
	1600	2.4100e-03	0.387	9.7313e-05	0.998	2.2299e-04	0.999
	3200	1.8385e-03	0.390	4.8685e-05	0.999	1.1153e-04	0.999
0.6	400	1.2744e-03		4.7951e-04		1.0567e-03	
	800	8.2696e-04	0.623	2.4074e-04	0.994	5.2949e-04	0.996
	1600	5.3841e-04	0.619	1.2074e-04	0.995	2.6518e-04	0.997
	3200	3.5159e-04	0.614	6.0513e-05	0.996	1.3275e-04	0.998

Table 5.4: The temporal accuracy and spatial accuracy for $\alpha = 0.4$ at t_{kN} .

N	$k = 1$		$k = 2$		$k = 3$	
	$\ u^N - u_h^N\ _0$	$rate_t$	$\ u^{2N} - u_h^{2N}\ _0$	$rate_t$	$\ u^{3N} - u_h^{3N}\ _0$	$rate_t$
400	1.3858e-04		3.8838e-04		8.9101e-04	
800	6.9385e-05	0.998	1.9445e-04	0.990	4.4578e-04	0.999
1600	3.4722e-05	0.998	9.7313e-05	0.998	2.2299e-04	0.999
3200	1.7371e-05	0.999	4.8685e-05	0.999	1.1153e-04	0.999
h	$\ u^N - u_h^N\ _0$	$rate_s$	$\ u^{2N} - u_h^{2N}\ _0$	$rate_s$	$\ u^{3N} - u_h^{3N}\ _0$	$rate_s$
1/8	9.7302e-03		1.9471e-02		4.5101e-02	
1/16	2.4393e-03	1.996	4.8819e-03	1.995	1.1308e-02	1.995
1/32	6.1026e-04	1.998	1.2213e-03	1.999	2.8290e-03	1.998
1/64	1.5259e-04	1.999	3.0539e-04	1.999	7.0739e-04	1.999

and $h = 1/5000$ respectively to ignore the spatial error. Then, $\alpha = 0.7$ and $\alpha = 0.9$ are chosen for the former, and $\alpha = 0.4$ and $\alpha = 0.6$ are taken for the latter to examine the temporal convergence rate. The numerical results of these two tables show that the time convergence

order is near α in the first time interval, and the convergence order is 1 in the following two time intervals. In Tables 5.2 and 5.4, we test the convergence rate at $t = 1, 2, 3$. In these two tables, $h = 1/5000$ is chosen to test the convergence in time. It is clear that the results of these two tables are different from those of Tables 5.1 and 5.3 because these results show that the convergence order at $t = 1, 2, 3$ is 1. Then we fix $\rho = 1/10000$ and choose different h in Tables 5.2 and 5.4, the results show that the spatial convergence rate is near 2. The above results are obviously agree with the expected cases mentioned in Theorems 4.1 and 4.2.

6. Conclusion

In this paper, we derive the exact solution of a time-fractional diffusion equation with constant time delay, investigate its regularity, and decompose the solution into its singular and regular parts. Unlike the previous works which show that the solutions of those fractional differential equations with constant time delay have multi-singularity at $t = k\tau^+$, the solution we consider is only exhibits singularity at $t = 0^+$ for its first time derivative, and at both $t = 0^+$ and τ^+ for its second time derivative. Therefore, we choose the Grünwald-Letnikov type approximation to handle the time-fractional operators, and establish the fully discrete finite element scheme. For the developed numerical scheme, it is proved to be stable. Then, in order to discuss the convergence, the temporal local truncation error based on the decomposition of the solution mentioned in Theorem 2.1 is discussed. Under a established new discrete Grönwall inequality, we obtain the error estimate of the proposed numerical scheme. Finally, some numerical tests are provided to verify our theoretical results.

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