Journal of Computational Mathematics
Vol.xx, No.x, 2026, 1–21.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2510-m2024-0290

ON THE SUPERCONVERGENCE OF COLLOCATION METHODS BASED ON TWO POST-PROCESSING TECHNIQUES FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH WEAKLY SINGULAR KERNELS*

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Abstract

The purpose of this paper is to investigate the superconvergence of collocation methods for fractional integro-differential equations (FIDEs) with weakly singular kernels and Caputo derivative of order $0 < \alpha < 1$. First, the initial value problem of FIDEs is reformulated as a weakly singular Volterra integral equation (VIE), and the existence, uniqueness, and regularity of the exact solution for the original FIDE are obtained with the help of the resolvent theory of VIEs, and it is shown that the singularity of the exact solution is governed by the Caputo derivative, not the weakly singular kernel. Next, the piecewise polynomial collocation method is employed to solve the reformulated VIE numerically, and the optimal convergence order of the collocation solution is obtained on graded meshes. In order to improve the numerical accuracy, two types of postprocessing techniques are used – one is the classical iterated technique for VIEs and another one is the interpolation postprocessing technique. The superconvergence is thoroughly investigated and the optimal superconvergence orders are obtained for both of these two postprocessing techniques. Compared to the classical iterated collocation method, the interpolation postprocessing method has a lower calculation cost. The theoretical results are illustrated by numerical experiments.

Mathematics subject classification: 65L05, 65L20, 65L50.
Key words: Fractional integro-differential equation, Volterra integral equations, Collocation methods, Interpolation postprocessing, Superconvergence.

1. Introduction

Over the past two decades, it has been shown that phenomena such as fluid flow in porous materials, anomalous diffusion transport, acoustic wave propagation in viscoelastic materials, dynamics in self-similar structures, signal processing, financial theory, and electric conductance

* Received December 23, 2024 / Revised version received April 4, 2025 / Accepted October 14, 2025 / Published online March 20, 2026 /
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in biological systems are more accurately modeled using fractional calculus (cf. [2, 7, 8, 11, 16–18, 20] and the references therein). However, for fractional differential equations (FDEs) and fractional integro-differential equations, it is usually impossible to get the exact solution. Consequently, significant effort has been devoted to develop numerical methods to solve these problems. For numerical methods of FDEs and FIDEs, the primary challenge is the weak singularity of the exact solution at the initial time. To address this issue, one must design the mesh carefully, and the graded mesh is a popular choice (see [3, 21]). In particular, some scholars reformulate FDEs or FIDEs as weakly singular Volterra integral equations, and then solve the reformulated VIEs numerically. For instance, in [6], the FDE is transform into an equivalent VIE, then an improved block-by-block approach is used to solve it numerically; In [12], a Caputo two-point boundary value problem is reformulated as VIEs, then the piecewise polynomial collocation method is used to solve these equations; In [15], a general class of two-point boundary value problems involving Caputo fractional-order derivatives is transformed into VIEs, and the transformed VIEs are solved numerically using piecewise polynomial collocation methods; In [9, 19, 23], the FIDE with weakly singular kernel is reformulated as VIEs, then a suitable smoothing transformation is used in order to overcome the weak singularity at the initial time, and the convergence and superconvergence of the collocation solution in piecewise polynomial space are investigated for the transformed equation.

For the numerical methods of VIEs, the piecewise polynomial collocation method is one of the most popular one. However, the collocation solution usually does not exhibit superconvergence properties for VIEs even if the collocation parameters satisfy certain orthogonality conditions, which is different from ordinary differential equations (ODEs). To improve the numerical accuracy, an iterated collocation solution is usually defined by substituting the obtained collocation solution back into the VIE (see [4]). However, obtaining the iterated collocation solution involves handling the "memory term" caused by integrals. Huang and Wang [10] introduced an interpolated postprocessing technique to enhance the accuracy based on collocation methods for weakly singular VIEs. Wang and Liang [22] first reformulated the initial value problem of FDEs as VIE, then applied the piecewise polynomial collocation method to solve the reformulated VIE numerically. In order to improve the numerical accuracy, both the iterated collocation technique and the interpolated postprocessing technique are used, and it is shown that both of these two postprocessing techniques can reach the same optimal superconvergence order, but the interpolation postprocessing technique is less expensive due to the absence of the "memory term".

In this paper, we consider the following initial value problem of Caputo FIDEs with weakly singular kernel:

$$\begin{cases} {}_0^C D_t^\alpha u(t) = p(t)u(t) + \displaystyle\int_0^t (t-s)^{-\beta} K(t,s)u(s)ds + g(t), \quad t \in I := [0,T], \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $0 < \alpha, \beta < 1, K \in C^m(D)$ with $D := \{(t,s) : 0 \le s \le t \le T\}$, p and g are given functions, and ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$ of a function f, defined by

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds. \tag{1.2}$$

Furthermore, the Riemann-Liouville fractional integral operator J^α is defined by

$$J^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds. \tag{1.3}$$

In estimates throughout the paper, C denotes a generic constant independent of the mesh diameter, and it can take different values in different places.

2. Reformulation to Weakly Singular VIEs with Two Kernels

Applying the integral operator J^α defined in (1.3) to the FIDE (1.1), yields the following weakly singular VIE:

$$\begin{aligned} u(t) &= w(t) + \int_0^t (t-s)^{\alpha-1} k_1(t,s) u(s) ds + \int_0^t (t-s)^{\alpha-\beta} k_2(t,s) u(s) ds \\ &= w(t) + \int_0^t (t-s)^{\alpha-1} H(t,s;\beta) u(s) ds, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} H(t,s;\beta) &:= k_1(t,s) + (t-s)^{1-\beta} k_2(t,s), \\ w(t) &:= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds \end{aligned}$$

with

$$k_1(t,s) := \frac{p(s)}{\Gamma(\alpha)}, \quad k_2(t,s) := \frac{1}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} x^{-\beta} K\big(x(t-s)+s, s\big) dx.$$

It is obvious that (2.1) is a special case of the weakly singular VIE with two kernels considered in [15, Section 4], so we can get the following results on the existence, uniqueness and regularity of VIE (2.1).

Theorem 2.1 ([15, Theorem 2]). *Assume that $p, g \in C^m(I)$ and $K \in C^m(D)$. Then the VIE (2.1) with weakly singular kernels has a unique solution $u \in C(I)$, and this solution has the following properties:*

(i) Its regularity is described by

$$u \in C^m(0,T] \cap C(I), \quad |u^{(l)}(t)| \le C t^{\alpha-l}, \quad l = 1,2,\ldots,m, \quad t \in I.$$

(ii) It has the following structure. Set $\alpha_{i_0} := -1$ and $A = \{\alpha-1, \alpha-\beta\}$. Then

$$u(t) = \sum_{(j,k)_\alpha} \gamma_{j,k} t^{k+1+\alpha_{i_0}+1+\alpha_{i_1}+\cdots+1+\alpha_{i_j}} + Y_m(t), \quad t \in I, \tag{2.2}$$

where $\gamma_{j,k}$ are some constants,

$$(j,k)_\alpha := \{(j,k) : j,\ k \in \mathbb{N}_0,\ \alpha_{i_n} \in A,\ 1 \le n \le j,\ k+1+\alpha_{i_0}+1+\alpha_{i_1}+\cdots+1+\alpha_{i_j} < m\}$$

and $Y_m \in C^m(I)$ with $|Y_m(t)| \le C t^m$ for all $t \in I$.

3. Collocation Methods

Let $N \ge 2$ and $r \ge 1$ be given positive integers, and

$$I_h := \left\{ t_n := \left(\frac{n}{N}\right)^r T : n = 0,1,\ldots,N \right\}$$

be a graded mesh on $[0,T]$. Set $h_n := t_{n+1} - t_n$ for each n and $h := \max_n h_n = h_{N-1}$. Let $m \in \mathbb{N}$. We approximate the solution u of (2.1) by collocation in the piecewise polynomial space

$$S_{m-1}^{(-1)}(I_h) := \left\{v : v|_{(t_n,t_{n+1}]} \in \mathscr{P}_{m-1},\ 0 \le n \le N-1\right\}.$$

Here $\mathscr{P}_{m-1}$ denotes the space of real-valued polynomials whose degrees do not exceed $m-1$. We seek the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for (2.1), to satisfy the collocation equation

$$u_h(t) = w(t) + \int_0^t (t-s)^{\alpha-1} H(t,s;\beta) u_h(s) ds, \quad t \in X_h, \tag{3.1}$$

where the set of collocation points

$$X_h := \{t_n + c_i h_n : 0 < c_1 < \cdots < c_m \le 1,\ n = 0,1,\ldots,N-1\}$$

is determined by the mesh I_h and the collocation parameters $\{c_i\}$. On each mesh interval $(t_n, t_{n+1}]$,the collocation solution can be written as (see [4, Section 6.2.1])

$$u_h(t_n + s h_n) = \sum_{j=1}^{m} L_j(s) U_{n,j}, \quad s \in (0,1], \tag{3.2}$$

where $U_{n,i} := u_h(t_{n,i})$ with $t_{n,i} := t_n + c_i h_n$ for $i = 1,\ldots,m$, and the polynomials

$$L_j(s) := \prod_{k=1,k\ne j}^{m} \frac{s - c_k}{c_j - c_k}, \quad j = 1,\ldots,m$$

are the standard Lagrange basis functions for $\mathscr{P}_{m-1}$ on $[0,1]$ with respect to the (distinct) collocation parameters $\{c_i\}$.

Substituting (3.2) into (3.1), we obtain

$$\begin{aligned}
U_{n,i} &= w(t_{n,i}) + \int_0^{t_{n,i}} (t_{n,i} - s)^{\alpha-1} H(t_{n,i}, s; \beta) u_h(s) ds \\
&= w(t_{n,i}) + \sum_{l=0}^{n-1} h_l^{\alpha} \int_0^1 \left(\frac{t_{n,i} - t_l}{h_l} - s\right)^{\alpha-1} H(t_{n,i}, t_l + s h_l; \beta) u_h(t_l + s h_l) ds \\
&\quad + h_n^{\alpha} \int_0^{c_i} (c_i - s)^{\alpha-1} H(t_{n,i}, t_n + s h_n; \beta) u_h(t_n + s h_n) ds \\
&= w(t_{n,i}) + \sum_{j=1}^{m} h_n^{\alpha} \int_0^{c_i} (c_i - s)^{\alpha-1} H(t_{n,i}, t_n + s h_n; \beta) L_j(s) ds U_{n,j} \\
&\quad + \sum_{l=0}^{n-1} \sum_{j=1}^{m} h_l^{\alpha} \int_0^1 \left(\frac{t_{n,i} - t_l}{h_l} - s\right)^{\alpha-1} H(t_{n,i}, t_l + s h_l; \beta) L_j(s) ds U_{l,j}.
\end{aligned} \tag{3.3}$$

Define the $m \times m$ matrices $B^{(n)} := (B_{i,j}^{(n)})$ and $B^{(n,l)} := (B_{i,j}^{(n,l)})$ $(0 \le l < n \le N-1)$ with the elements

$$\begin{aligned}
B_{i,j}^{(n)} &:= \int_0^{c_i} H(t_{n,i}, t_n + s h_n; \beta)(c_i - s)^{\alpha-1} L_j(s) ds, \\
B_{i,j}^{(n,l)} &:= \int_0^1 H(t_{n,i}, t_l + s h_l; \beta) \left(\frac{t_{n,i} - t_l}{h_l} - s\right)^{\alpha-1} L_j(s) ds
\end{aligned}$$

for $i, j = 1, \ldots, m$, and set

$$U_n := \big(U(t_{n,1}), \ldots, U(t_{n,m})\big)^\top,$$
$$W_n := \big(w(t_{n,1}), \ldots, w(t_{n,m})\big)^\top.$$

Then the above system of equations for the $U_{n,i}$ can be written as

$$\Big[I_m - h_n^\alpha B^{(n)}\Big] U_n = \sum_{l=0}^{n-1} h_l^\alpha B^{(n,l)} U_l + W_n \tag{3.4}$$

for $n = 0, 1, \ldots, N-1$, where I_m is the $m \times m$ identity matrix.

It is obvious that the matrix $B^{(n)}$ is bounded, so the following result on the existence and uniqueness for the collocation solution is evident.

Theorem 3.1. *Assume that $p, g \in C(I)$ and $K \in C(D)$. Then there exists an $\bar{h} > 0$ so that, for every $\alpha \in (0,1)$ and any mesh I_h with mesh diameter h satisfying $h \in (0, \bar{h})$, each of the linear algebraic systems* (3.4) *has a unique solution $U_n \in \mathbb{R}^m, n = 0, 1, \ldots, N-1$. Hence, the collocation equation* (3.3) *defines a unique collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for the weakly singular VIE* (2.1) *with the local representation given by* (3.2).

4. Convergence Analysis

We first give two useful lemmas for the convergence analysis.

Lemma 4.1 ([14, Lemma 5.2]). *For $1 \le n \le N$ and $0 < \alpha < 1$,*

$$\sum_{l=1}^{n-1} l^{-\alpha}(n-l)^{\alpha-1} \le \frac{1}{\alpha} + \frac{1}{1-\alpha}.$$

Similarly to the proof of [22, Lemma 2], one can easily prove the following inequalities.

Lemma 4.2. *For $1 \le n \le N$, $0 < \alpha < 1$, $0 < \gamma < 1$ and $\delta \ge 0$,*

$$h_n^{m+\delta} t_n^{\alpha-m} \le CN^{-\min\{r(\alpha+\delta), m+\delta\}},$$
$$\sum_{l=1}^{n-1} h_l^{m+\gamma} t_l^{\alpha-m} (n-l)^{\gamma-1} \le CN^{-\min\{r(\alpha+\gamma), m\}},$$
$$\sum_{l=1}^{n-1} h_l^{m+\gamma} t_l^{\alpha-m} (n-l)^{\gamma-2} \le CN^{-\min\{r(\alpha+\gamma), m+\gamma\}}.$$

Now we state the convergence result.

Theorem 4.1. *Assume that $p, g \in C^m(I)$ and $K \in C^m(D)$. Let u be the unique solution of the VIE* (2.1)*, and $u_h \in S_{m-1}^{(-1)}(I_h)$ be the corresponding collocation solution defined by* (3.1) *with $h \in (0, \bar{h})$. Then*

$$\|u - u_h\|_\infty := \sup_{t \in I} |u(t) - u_h(t)| \le Ch^{\min\{r\alpha, m\}}$$

holds for any set X_h of collocation points with $0 < c_1 < \cdots < c_m \le 1$.

Proof. Let $e_h := u - u_h$. Then at $t = t_{n,i}$ for $0 \le n \le N-1$ and $i = 1, \ldots, m$, by (2.1) and (3.1), we have

$$e_h(t_{n,i}) = \int_0^{t_{n,i}} (t_{n,i} - s)^{\alpha-1} H(t_{n,i}, s; \beta) e_h(s) ds. \tag{4.1}$$

Let $\lambda(j,k,\alpha,\beta) := k + 1 + \alpha_{i_0} + 1 + \alpha_{i_1} + \cdots + 1 + \alpha_{i_j}$. Set

$$(j,k)' := \{(j,k) \in (j,k)_\alpha : \lambda(j,k,\alpha,\beta) \notin \mathbb{N}_0\}.$$

It is obvious that the minimum non-integer power of t in (2.2) is α. For $n = 0$, by Theorem 2.1, we know that on $\bar{\sigma}_0 = [t_0, t_1] = [0, h_0]$ the exact solution can be written as

$$u(t_0 + vh_0) = \sum_{j=0}^{m-1} c_{j,0} v^j + h_0^\alpha \sum_{(j,k)'} c_{j,k} v^{\lambda(j,k,\alpha,\beta)} + h_0^m \bar{Y}_{m,0}(v), \quad v \in [0,1],$$

where $c_{j,0}$ and $c_{j,k}$ are constants, which may depend on h_0 but are bounded. Since $u_h \in S_{m-1}^{(-1)}(I_h)$, we also have

$$u_h(t_0 + vh_0) = \sum_{j=0}^{m-1} d_{j,0} v^j, \quad v \in (0,1],$$

where the constants $d_{j,0}$ may depend on h_0, but they are bounded, so on $\bar{\sigma}_0$ the collocation error can be written as

$$e_h(t_0 + vh_0) = \sum_{j=0}^{m-1} \beta_{j,0} v^j + h_0^\alpha \sum_{(j,k)''} c_{j,k} v^{\lambda(j,k,\alpha,\beta)} + h_0^m R_{m,0}(v), \tag{4.2}$$

where $\beta_{j,0} := c_{j,0} - d_{j,0}$ and $R_{m,0}(v) := \bar{Y}_{m,0}(v)$. We now return to the error equation (4.1) corresponding to $n = 0$. By

$$e_h(t_0 + c_i h_0) = h_0^\alpha \int_0^{c_i} (c_i - s)^{\alpha-1} H(t_0 + c_i h_0, t_0 + s h_0; \beta) e_h(t_0 + s h_0) ds,$$

we know that the unknown coefficients $\beta_{j,0}(\alpha)$ in (4.2) solve the linear algebraic system

$$\begin{aligned}
&\sum_{j=0}^{m-1} \left[c_i^j - h_0^\alpha \int_0^{c_i} (c_i - s)^{\alpha-1} s^j H(t_0 + c_i h_0, t_0 + s h_0; \beta) ds \right] \beta_{j,0} \\
&= -h_0^\alpha \sum_{(j,k)'} \left[c_i^{\lambda(j,k,\alpha,\beta)} - h_0^\alpha \int_0^{c_i} (c_i - s)^{\alpha-1} H(t_0 + c_i h_0, t_0 + s h_0; \beta) s^{\lambda(j,k,\alpha,\beta)} ds \right] c_{j,k} \\
&\quad - h_0^m \left[R_{m,0}(c_i) - h_0^\alpha \int_0^{c_i} (c_i - s)^{\alpha-1} H(t_0 + c_i h_0, t_0 + s h_0; \beta) R_{m,0}(s) ds \right]
\end{aligned} \tag{4.3}$$

for $i = 1, \ldots, m$.

Let

$$\beta_0 := (\beta_{0,0}, \ldots, \beta_{m-1,0})^\top, \quad V_m := \begin{pmatrix} c_i^j \\ (i,j = 1,2,\ldots,m) \end{pmatrix},$$

$$\tilde{B}_0 := \begin{pmatrix} \int_0^{c_i} (c_i - s)^{\alpha-1} s^j H(t_0 + c_i h_0, t_0 + s h_0; \beta) ds \\ (i,j = 1,2,\ldots,m) \end{pmatrix}.$$

Then the above equation can be written compactly as

$$\big(V_m - h_0^\alpha \tilde{B}_0\big)\beta_0 = h_0^\alpha q_0 + h_0^m \rho_0 \tag{4.4}$$

with the obvious meanings of q_0 and ρ_0. Since V_m is the Vandermonde matrix based on the distinct collocation parameters c_i, so for sufficiently small h_0, the coefficient matrix $V_m - h_0^\alpha \tilde{B}_0$ is nonsingular for all $\alpha \in (0,1)$, with the uniformly bounded inverse matrix. Thus by (4.4),

$$\|\beta_0\|_1 \le C h_0^\alpha,$$

where $\|\cdot\|_1$ denotes the usual l_1 norm, which together with (4.2) yield that

$$|e_h(t_0 + v h_0)| \le C h_0^\alpha \le C h^{r\alpha} \tag{4.5}$$

since $h_0 = \mathcal{O}(h^r)$.

Now assume that $1 \le n \le N-1$. On $(t_n, t_{n+1}]$, by Peano's theorem (see [5, Corollary 1.8.4]), the collocation error has the local Lagrange representation

$$e_h(t_n + s h_n) = \sum_{j=1}^m L_j(s)\mathscr{E}_{n,j} + h_n^m R_{m,n}(s), \quad s \in (0,1], \tag{4.6}$$

where

$$\mathscr{E}_{n,j} := e_h\left(t_{n,j}\right), \quad R_{m,n}(v) := \int_0^1 k_m(v,z) u^{(m)}(t_n + z h_n) dz$$

with

$$k_m(v,z) := \frac{1}{(m-1)!}\left\{(v-s)_+^{m-1} - \sum_{k=1}^m L_k(v)(c_k - s)^{m-1}\right\}, \quad v \in [0,1].$$

In addition, by Theorem 2.1,

$$|R_{m,n}(s)| \le C t_n^{\alpha - m}. \tag{4.7}$$

By (4.1), we have for $i = 1, \ldots, m$,

$$\begin{aligned} &\mathscr{E}_{n,i} - h_n^\alpha \sum_{j=1}^m \int_0^{c_i} (c_i - s)^{\alpha-1} H(t_{n,i}, t_n + s h_n; \beta) L_j(s) ds \mathscr{E}_{n,j} \\ &= \sum_{l=1}^{n-1} h_l^\alpha \sum_{j=1}^m \int_0^1 \left(\frac{t_{n,i} - t_l}{h_l} - s\right)^{\alpha-1} H(t_{n,i}, t_l + s h_l; \beta) L_j(s) ds \mathscr{E}_{l,j} + E_{n,i}, \end{aligned}$$

where

$$E_{n,i} := h_0^\alpha q_{n,i}^{(0)} + h_n^{m+\alpha} \rho_{n,i} + \sum_{l=1}^{n-1} h_l^{m+\alpha} \rho_{n,i}^{(l)}$$

with

$$\begin{aligned} q_{n,i}^{(0)} &:= \int_0^1 \left(\frac{t_{n,i} - t_0}{h_0} - s\right)^{\alpha-1} H(t_{n,i}, t_0 + s h_0; \beta) e_h(t_0 + s h_0) ds, \\ \rho_{n,i} &:= \int_0^{c_i} (c_i - s)^{\alpha-1} H(t_{n,i}, t_n + s h_n; \beta) R_{m,n}(s) ds, \\ \rho_{n,i}^{(l)} &:= \int_0^1 \left(\frac{t_{n,i} - t_l}{h_l} - s\right)^{\alpha-1} H(t_{n,i}, t_l + s h_l; \beta) R_{m,l}(s) ds. \end{aligned}$$

In addition, by (4.5), we have

$$h_0^\alpha |q_{n,i}^{(0)}| \le Ch_0^{2\alpha} n^{\alpha-1} \le Ch_0^{2\alpha} \le Ch^{2r\alpha}.$$

By (4.7) and Lemma 4.2, we have

$$\begin{aligned} &h_n^{m+\alpha}|\rho_{n,i}| \le Ch_n^{m+\alpha} t_n^{\alpha-m} \le CN^{-\min\{2r\alpha, m+\alpha\}}, \\ &\sum_{l=1}^{n-1} h_l^{m+\alpha} |\rho_{n,i}^{(l)}| \le C\sum_{l=1}^{n-1} h_l^{m+\alpha} t_l^{\alpha-m}(n-l)^{\alpha-1} \le CN^{-\min\{2r\alpha, m\}}. \end{aligned}$$

In summary, for $1 \le n \le N-1$ and $i = 1, 2, \ldots, m$, we have the following estimate:

$$|E_{n,i}| \le CN^{-\min\{2r\alpha, m\}}. \tag{4.8}$$

Let $\mathscr{E}_n := (\mathscr{E}_{n,1}, \ldots, \mathscr{E}_{n,m})^\top$ and $E_n := (E_{n,1}, \ldots, E_{n,m})^\top$. Then we have the following compact form:

$$\left[I_m - h_n^\alpha B^{(n)}\right] \mathscr{E}_n = \sum_{l=1}^{n-1} h_l^\alpha B^{(n,l)} \mathscr{E}_l + E_n.$$

By Theorem 3.1, we have known that for $h \le \bar{h}$, the coefficient matrix is nonsingular, with $\|(I_m - h_n^\alpha B^{(n)})^{-1}\|_1 \le C$, and further by [4, Lemma 6.2.10] and (4.8),

$$\|\mathscr{E}_n\|_1 \le C\left[\sum_{l=1}^{n-1} h_l^\alpha (n-l)^{\alpha-1} \|\mathscr{E}_l\|_1 + h^{\min\{2r\alpha, m\}}\right]. \tag{4.9}$$

By the generalized discrete Gronwall inequality (see [4, Theorem 6.1.19]), we have

$$\|\mathscr{E}_n\|_1 \le Ch^{\min\{2r\alpha, m\}},$$

then by (4.6), (4.7) and Lemma 4.2,

$$|e_h(t_n + sh_n)| \le C\big[\|\mathscr{E}_n\|_1 + h_n^m t_n^{\alpha-m}\big] \le C\big[h^{\min\{2r\alpha, m\}} + h^{\min\{r\alpha, m\}}\big] \le Ch^{\min\{r\alpha, m\}},$$

which together with (4.5) yield the desired result. □

5. Iterated Collocation Methods

In this section, we will use the classical iterated collocation technique to improve the convergence of the collocation solution.

Let

$$u_h^{it}(t) = w(t) + \int_0^t (t-s)^{\alpha-1} H(t,s;\beta) u_h(s) ds, \quad t \in I \tag{5.1}$$

be the iterated collocation solution. Then we have the following superconvergence result.

Theorem 5.1. *Assume that $p, g \in C^{m+1}(I)$ and $K \in C^{m+1}(D)$. Let u and u_h be the exact and collocation solution of the VIE* (2.1)*, and u_h^{it} be the corresponding iterated collocation solution. If the collocation parameters satisfy the orthogonality condition*

$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_i) ds = 0,$$

then we have the following superconvergence results:

$$\left\|u-u_h^{it}\right\|_\infty := \max_{t\in I}\left|u(t)-u_h^{it}(t)\right| \le C\begin{cases} h^{2\alpha}, & \text{if} \quad r=1,\\ h^{m+\alpha}, & \text{if} \quad r\ge \dfrac{m}{\alpha}.\end{cases}$$

Proof. Let $e_h^{it} := u-u_h^{it}$ be the iterated collocation error. Then by (2.1) and (5.1), we have

$$e_h^{it}(t)=\int_0^t (t-s)^{\alpha-1}H(t,s;\beta)e_h(s)ds.$$

For the first interval, let $t\in\bar\sigma_0=[t_0,t_1]$. Then by (4.5), we have

$$\left|e_h^{it}(t)\right| \le Ch_0^\alpha\frac{t^\alpha}{\alpha}\le Ch_0^{2\alpha}. \tag{5.2}$$

Now we assume that $1\le n\le N-1$. On the subinterval σ_n, assume $t=t_n+vh_n, v\in(0,1]$, then we have

$$e_h^{it}(t)=\int_0^t (t-s)^{\alpha-1}H(t,s;\beta)e_h(s)ds=\sum_{k=0}^{2}A_n^{(k)}(v), \tag{5.3}$$

where

$$A_n^{(0)}(v) := h_0^\alpha\int_0^1\left(\frac{t-t_0}{h_0}-s\right)^{\alpha-1}H(t,t_0+sh_0;\beta)e_h(t_0+sh_0)ds,$$
$$A_n^{(1)}(v) := \sum_{l=1}^{n-1}h_l^\alpha\int_0^1\left(\frac{t-t_l}{h_l}-s\right)^{\alpha-1}H(t,t_l+sh_l;\beta)e_h(t_l+sh_l)ds,$$
$$A_n^{(2)}(v) := h_n^\alpha\int_0^v (v-s)^{\alpha-1}H(t,t_n+sh_n;\beta)e_h(t_n+sh_n)ds.$$

By (4.5), we have

$$\left|A_n^{(0)}(v)\right|\le Ch_0^{2\alpha}\int_0^1\left(\frac{t-t_0}{h_0}-s\right)^{\alpha-1}ds\le Ch_0^{2\alpha}\int_0^1\left(\frac{t_n-t_0}{h_0}-s\right)^{\alpha-1}ds.$$

For the special case $n=1$, the above inequality becomes

$$\left|A_1^{(0)}(v)\right|\le Ch_0^{2\alpha}\int_0^1\left(\frac{t_1-t_0}{h_0}-s\right)^{\alpha-1}ds=Ch_0^{2\alpha}\int_0^1(1-s)^{\alpha-1}ds=Ch_0^{2\alpha}\frac{1}{\alpha}\le Ch_0^{2\alpha}.$$

For $n\ge 2$, we have $n-1\ge 1$ and $(n-1)^{\alpha-1}\le 1$, so

$$\left|A_n^{(0)}(v)\right|\le Ch_0^{2\alpha}\left(\frac{t_n-t_0}{h_0}-1\right)^{\alpha-1}\le Ch_0^{2\alpha}(n-1)^{\alpha-1}\le Ch_0^{2\alpha}.$$

Combining the above two cases together and again observing that $h_0=\mathcal{O}(h^r)$, we have

$$\left|A_n^{(0)}(v)\right|\le Ch_0^{2\alpha}\le Ch^{2r\alpha}. \tag{5.4}$$

By Theorem 4.1, we have

$$\left|A_n^{(2)}(v)\right|\le C\|e_h\|_\infty h_n^\alpha\le Ch^{\min\{r\alpha,m\}+\alpha}. \tag{5.5}$$

To estimate $A_n^{(1)}(v)$, by Taylor's expansion, we write

$$A_n^{(1)}(v) = A_n^{(1,1)}(v) + A_n^{(1,2)}(v) + A_n^{(1,3)}(v)$$

with

$$\begin{aligned}
A_n^{(1,1)}(v) &:= h_{n-1}^{\alpha} \int_0^1 \left(\frac{t - t_{n-1}}{h_{n-1}} - s \right)^{\alpha-1} H(t, t_{n-1} + sh_{n-1}; \beta) e_h(t_{n-1} + sh_{n-1}) ds, \\
A_n^{(1,2)}(v) &:= \sum_{l=1}^{n-2} h_l^{\alpha} H(t, t_l; \beta) \int_0^1 \left(\frac{t - t_l}{h_l} - s \right)^{\alpha-1} e_h(t_l + sh_l) ds, \\
A_n^{(1,3)}(v) &:= \sum_{l=1}^{n-2} h_l^{\alpha+1} \int_0^1 \left(\frac{t - t_l}{h_l} - s \right)^{\alpha-1} s\partial_2 H(t, \xi_l; \beta) e_h(t_l + sh_l) ds,
\end{aligned}$$

where $\xi_l \in (t_l, t_{l+1})$.

By Theorem 4.1, it can be easily checked that

$$\begin{aligned}
\left| A_n^{(1,1)}(v) \right| &\le Ch \int_0^1 \left(\frac{t_n - t_{n-1}}{h_{n-1}} - s \right)^{\alpha-1} ds \|e_h\|_\infty = Ch \int_0^1 (1-s)^{\alpha-1} ds \|e_h\|_\infty \\
&= Ch \frac{1}{1-\alpha} \|e_h\|_\infty \le Ch^{\min\{r\alpha, m\}+\alpha},
\end{aligned} \tag{5.6}$$

further, by [4, p. 383], we know that

$$\sum_{l=1}^{n-1} h_l^{\alpha} (n-l)^{\alpha-1} \le \frac{T^\alpha}{1-\alpha}, \tag{5.7}$$

which together with Theorem 4.1 yields that

$$\begin{aligned}
\left| A_n^{(1,3)}(v) \right| &\le C \sum_{l=1}^{n-2} h_l^{\alpha-\beta+1} \int_0^1 \left(\frac{t_n - t_l}{h_l} - s \right)^{\alpha-1} \left(\frac{t_n - \xi_l}{h_l} \right)^{-\beta} ds \|e_h\|_\infty \\
&\le C \sum_{l=1}^{n-2} h_l^{\alpha-\beta+1} (n-l)^{\alpha-1} (n-l)^{-\beta} \|e_h\|_\infty \\
&\le Ch^{\alpha} \sum_{l=1}^{n-2} h_l^{1-\beta} (n-l)^{-\beta} \|e_h\|_\infty \le Ch^{\min\{r\alpha, m\}+\alpha}.
\end{aligned} \tag{5.8}$$

In order to estimate $A_n^{(1,2)}(v)$, we define an interpolation operator $P_h : C(I) \to S_{m-1}^{(-1)}(I_h)$ such that for $x \in C(I)$

$$(P_h x)(t_{n,i}) = x(t_{n,i}), \quad n = 0, 1, 2, \dots, N-1, \quad i = 1, 2, \dots, m. \tag{5.9}$$

By the definition of the iterated collocation solution, we know that $u_h^{it}(t_{n,i}) = u_h(t_{n,i})$, which implies $P_h u_h^{it} = u_h$. Then

$$e_h = u - u_h = u - P_h u + P_h u - u_h = u - P_h u + P_h u - P_h u_h^{it} = u - P_h u + P_h e_h^{it},$$

and we write

$$A_n^{(1,2)}(v) = A_n^{(1,2,1)}(v) + A_n^{(1,2,2)}(v),$$

where

$$A_n^{(1,2,1)}(v) := \sum_{l=1}^{n-2} h_l^\alpha H(t,t_l;\beta)\int_0^1 \left(\frac{t-t_l}{h_l}-s\right)^{\alpha-1}(u-P_hu)(t_l+sh_l)ds,$$
$$A_n^{(1,2,2)}(v) := \sum_{l=1}^{n-2} h_l^\alpha H(t,t_l;\beta)\int_0^1 \left(\frac{t-t_l}{h_l}-s\right)^{\alpha-1}\left(P_h e_h^{it}\right)(t_l+sh_l)ds.$$

By Taylor's expansion and noting that $P_hu(t_{n,i})=u(t_{n,i})$, we have

$$\begin{aligned}&(u-P_hu)(t_l+sh_l)\\ &=\sum_{j=1}^m L_j(s)(u-P_hu)(t_{l,j})+\frac{(u-P_hu)^{(m)}(\eta_l)}{m!}h_l^m\prod_{j=1}^m(s-c_j)\\ &=\frac{(u-P_hu)^{(m)}(\eta_l)}{m!}h_l^m\prod_{j=1}^m(s-c_j)=\frac{h_l^m}{m!}u^{(m)}(\eta_l)\prod_{j=1}^m(s-c_j)\\ &=\frac{h_l^m}{m!}\left[u^{(m)}(t_l)+u^{(m+1)}(\zeta_l)(\eta_l-t_l)\right]\prod_{j=1}^m(s-c_j),\end{aligned}$$

where $\eta_l\in(t_l,t_{l+1})$ and $\zeta_l\in(t_l,\eta_l)$. Again by Taylor's expansion,

$$\left(\frac{t-t_l}{h_l}-s\right)^{\alpha-1}=\left(\frac{t-t_l}{h_l}\right)^{\alpha-1}-(\alpha-1)s\left(\frac{t-t_l}{h_l}-\hat{c}_{n,l}\right)^{\alpha-2},$$

where $\hat{c}_{n,l}\in(0,1)$. Then by the orthogonality condition,

$$\begin{aligned}A_n^{(1,2,1)}(v)=&(1-\alpha)\sum_{l=0}^{n-2}\frac{H(t,t_l;\beta)\,u^{(m)}(t_l)}{m!}h_l^{m+\alpha}\int_0^1\left(\frac{t-t_l}{h_l}-\hat{c}_{n,l}\right)^{\alpha-2}s\prod_{j=1}^m(s-c_j)\,ds\\ &+\sum_{l=1}^{n-2}\frac{H(t,t_l;\beta)}{m!}h_l^{m+\alpha}\int_0^1\left(\frac{t-t_l}{h_l}-s\right)^{\alpha-1}u^{(m+1)}(\zeta_l)(\eta_l-t_l)\prod_{j=1}^m(s-c_j)\,ds.\end{aligned}$$

Recalling Theorem 2.1, Lemma 4.2 and [4, Lemma 6.2.10], we have

$$\begin{aligned}\left|A_n^{(1,2,1)}(v)\right|&\le C\sum_{l=1}^{n-2}h_l^{m+\alpha}t_l^{\alpha-m}(n-l)^{\alpha-2}+C\sum_{l=1}^{n-2}h_l^{m+1+\alpha}t_l^{\alpha-(m+1)}(n-l)^{\alpha-1}\\ &\le Ch^{\min\{2r\alpha,m+\alpha\}}+Ch^{\min\{2r\alpha,m+1\}}\le Ch^{\min\{2r\alpha,m+\alpha\}}.\end{aligned}\tag{5.10}$$

For $A_n^{(1,2,2)}(v)$, due to the boundedness of the operator P_h, we have

$$\left|A_n^{(1,2,2)}(v)\right|\le C\sum_{l=1}^{n-2}h_l^\alpha(n-l)^{\alpha-1}\left\|P_he_h^{it}\right\|_{l,\infty}\le C\sum_{l=1}^{n-2}h_l^\alpha(n-l)^{\alpha-1}\left\|e_h^{it}\right\|_{l,\infty}.\tag{5.11}$$

Then, substituting the estimates (5.4),(5.5), (5.8), (5.10) and (5.11) into (5.3), we have

$$\begin{aligned}\left\|e_h^{it}\right\|_{n,\infty}&\le Ch^{2r\alpha}+Ch^{\min\{r\alpha,m\}+\alpha}+C\sum_{l=1}^{n-1}h_l^\alpha(n-l)^{\alpha-1}\left\|e_h^{it}\right\|_{l,\infty}\\ &\le Ch^{\min\{r\alpha,m\}+\alpha}+C\sum_{l=1}^{n-1}h_l^\alpha(n-l)^{\alpha-1}\left\|e_h^{it}\right\|_{l,\infty}.\end{aligned}$$

By the generalized Gronwall inequality (see [4, Theorem 6.1.19]), we have

$$\left\|e_h^{it}\right\|_{n,\infty} \leq Ch^{\min\{r\alpha,m\}+\alpha},$$

which together with (5.2) yields the desired result. □

6. Interpolated Postprocessing

In this section, we will use the interpolated postprocessing technique to improve the convergence of the collocation solution.

6.1. Supercloseness analysis

Define the weakly singular Volterra integral operator $\mathscr{V}_{\alpha,\beta}: C(I) \to C(I)$ as

$$(\mathscr{V}_{\alpha,\beta}u)(t) := \int_0^t (t-s)^{\alpha-1} H(t,s;\beta)u(s)ds,$$

then the VIE (2.1) can also be written as

$$u(t) = w(t) + (\mathscr{V}_{\alpha,\beta}u)(t). \tag{6.1}$$

Recalling the operator defined in (5.9), we can write the collocation Eq. (3.1) as the following operator form:

$$u_h = P_h w + P_h(\mathscr{V}_{\alpha,\beta}u_h). \tag{6.2}$$

Definition 6.1 ([10]). *If the error between the numerical solution and some interpolant of the exact solution is much smaller than that between the numerical solution and the exact solution, that is, if*

$$\|P_h u - u_h\|_\infty := \sup_{t\in I}|P_h u(t) - u_h(t)| \ll \|u - u_h\|_\infty := \sup_{t\in I}|u(t) - u_h(t)|,$$

then this phenomenon is called supercloseness.

For $0 < \alpha < 1$, we introduce the following Hölder space (see [13]):

$$C^{(0,\alpha)}[a,b] = \left\{ g \in C[a,b] \middle| \sup_{a\leq t,\tau\leq b} \frac{|g(t)-g(\tau)|}{|t-\tau|^\alpha} < \infty \right\}.$$

If $u \in C^{(0,\alpha)}[a,b]$, then u is called α-Hölder continuous.

Lemma 6.1 ([1, p. 128]). *For $N, m > 0$ and grading exponent $r \geq 1$, let P_h be the interpolation projection operator defined by (5.9). Assume that $u \in C^{(0,\alpha)}[0,T] \cap C^m(0,T]$ with $0 < \alpha < 1$ and*

$$|u^{(m)}(t)| \leq c_{\alpha,m} t^{\alpha-m}, \quad 0 < t \leq T,$$

where $c_{\alpha,m}$ are the constants relevant with α and m. Then for $r \geq m/\alpha$,

$$\|u - P_h u\|_\infty := \sup_{t\in I}|u(t) - P_h u(t)| \leq CN^{-m}.$$

Remark 6.1. Under the condition of Theorem 3.1, on uniform meshes, we only have the following lower order:

$$\|u - P_h u\|_\infty = \sup_{t\in I} |u(t) - P_h u(t)| \le CN^{-\alpha}.$$

Similarly to the proof of [22, Lemma 4], we can obtain the following lemma.

Lemma 6.2. *Let $p \in C(I)$ and $K \in C(D)$. Then the linear operator $[I - P_h \mathscr{V}_{\alpha,\beta}]^{-1} P_h$: $C(I) \to S_{m-1}^{(-1)}(I_h)$ is bounded.*

Now we state the supeclose result.

Theorem 6.1. *Under the assumption of Theorem* 5.1*, we have the following global supercloseness result:*

$$\|u_h - P_h u\|_\infty := \sup_{t\in I} |u_h(t) - P_h u(t)| \le C \begin{cases} h^{2\alpha}, & \textit{if} \quad r = 1, \\ h^{m+\alpha}, & \textit{if} \quad r \ge \dfrac{m}{\alpha}. \end{cases}$$

Proof. By (6.1) and (6.2), we have

$$\begin{aligned} u_h - P_h u &= P_h w + P_h(\mathscr{V}_{\alpha,\beta} u_h) - \big(P_h w + P_h(\mathscr{V}_{\alpha,\beta} u)\big) \\ &= P_h \mathscr{V}_{\alpha,\beta}(u_h - u) = P_h \mathscr{V}_{\alpha,\beta}(u_h - P_h u) + P_h \mathscr{V}_{\alpha,\beta}(P_h u - u). \end{aligned}$$

Then

$$u_h - P_h u = (I - P_h \mathscr{V}_{\alpha,\beta})^{-1} P_h \mathscr{V}_{\alpha,\beta}(P_h u - u),$$

and by Lemma 6.2,

$$\|u_h - P_h u\|_\infty \le C \|\mathscr{V}_{\alpha,\beta}(P_h u - u)\|_\infty, \tag{6.3}$$

so in the following, we only need to estimate $\|\mathscr{V}_{\alpha,\beta}(P_h u - u)\|_\infty$.

For $n = 0, t \in [t_0, t_1]$. By Lemma 6.1 and Remark 6.1,

$$\begin{aligned} |\mathscr{V}_{\alpha,\beta}(P_h u - u)(t)| &= \left| \int_0^t (t-s)^{\alpha-1} H(t,s;\beta)(P_h u - u)(s) ds \right| \\ &\le C \|P_h u - u\|_\infty \frac{t^\alpha}{\alpha} \\ &\le C h_0^\alpha \begin{cases} N^{-\alpha}, & \text{if} \quad r = 1, \\ N^{-m}, & \text{if} \quad r \ge \dfrac{m}{\alpha} \end{cases} \\ &\le C \begin{cases} h^{2\alpha}, & \text{if} \quad r = 1, \\ h^{m+\alpha}, & \text{if} \quad r \ge \dfrac{m}{\alpha}. \end{cases} \end{aligned}$$

Now we assume that $1 \le n \le N-1$ and $t = t_n + vh_n, v \in (0,1]$, and write

$$\mathscr{V}_{\alpha,\beta}(P_h u - u)(t_n + vh_n) = \sum_{k=0}^{2} B_n^{(k)}(v),$$

where

$$B_n^{(0)}(v) := h_0^\alpha \int_0^1 \left(\frac{t - t_0}{h_0} - s \right)^{\alpha-1} H(t, t_0 + sh_0; \beta)(P_h u - u)(t_0 + sh_0) ds,$$

$$B_n^{(1)}(v) := \sum_{l=1}^{n-2} h_l^\alpha \int_0^1 \left(\frac{t-t_l}{h_l} - s\right)^{\alpha-1} H(t, t_l + sh_l; \beta)(P_h u - u)(t_l + sh_l) ds,$$

$$B_n^{(2)}(v) := h_n^\alpha \int_0^v (v-s)^{\alpha-1} H(t, t_n + sh_n; \beta)(P_h u - u)(t_n + sh_n) ds$$

$$+ h_{n-1}^\alpha \int_0^1 \left(\frac{t-t_{n-1}}{h_{n-1}} - s\right)^{\alpha-1} H(t, t_{n-1} + sh_{n-1}; \beta)(P_h u - u)(t_{n-1} + sh_{n-1}) ds.$$

By Lemma 6.1 and Remark 6.1, we have the following estimates:

$$\left|B_n^{(0)}(v)\right| \le C h_0^\alpha n^{\alpha-1} \|P_h u - u\|_\infty \le C \begin{cases} h^{2\alpha}, & \text{if} \quad r = 1, \\ h^{m+\alpha}, & \text{if} \quad r \ge \dfrac{m}{\alpha}, \end{cases}$$

$$\left|B_n^{(2)}(v)\right| \le C \frac{h_n^\alpha + h_{n-1}^\alpha}{\alpha} \|P_h u - u\|_\infty \le C \begin{cases} h^{2\alpha}, & \text{if} \quad r = 1, \\ h^{m+\alpha}, & \text{if} \quad r \ge \dfrac{m}{\alpha}. \end{cases}$$

For $B_n^{(1)}(v)$, by Taylor's expansion, we can write

$$B_n^{(1)}(v) = -A_n^{(1,2,1)}(v) + B_n^{(1,2)}(v),$$

where $A_n^{(1,2,1)}(v)$ is defined in the proof of Theorem 5.1, and

$$B_n^{(1,2)}(v) := \sum_{l=1}^{n-2} h_l^{\alpha+1} \int_0^1 \left(\frac{t-t_l}{h_l} - s\right)^{\alpha-1} s\partial_2 H(t, \xi_l'; \beta)(P_h u - u)(t_l + sh_l) ds,$$

where $\xi_l' \in (t_l, t_{l+1})$.

By (5.7), Remark 6.1 and Lemma 6.1, we have

$$\begin{aligned} \left|B_n^{(1,2)}(v)\right| &\le C \sum_{l=1}^{n-2} h_l^{\alpha-\beta+1} \int_0^1 \left(\frac{t-t_l}{h_l} - s\right)^{\alpha-1} \left(\frac{t-\xi_l}{h_l}\right)^{-\beta} ds \|P_h u - u\|_\infty \\ &\le C \sum_{l=1}^{n-2} h_l^{\alpha-\beta+1} (n-l)^{\alpha-1} (n-l)^{-\beta} \|P_h u - u\|_\infty \\ &\le C h^\alpha \sum_{l=1}^{n-2} h_l^{1-\beta} (n-l)^{-\beta} \|P_h u - u\|_\infty \\ &\le C h^\alpha \|P_h u - u\|_\infty \le C \begin{cases} h^{2\alpha}, & \text{if} \quad r = 1, \\ h^{m+\alpha}, & \text{if} \quad r \ge \dfrac{m}{\alpha}, \end{cases} \end{aligned}$$

which together with (5.10) yields that

$$\left|B_n^{(1)}(v)\right| \le C \begin{cases} h^{2\alpha}, & \text{if} \quad r = 1, \\ h^{m+\alpha}, & \text{if} \quad r \ge \dfrac{m}{\alpha}. \end{cases}$$

Combining the above estimates for $B_n^{(0)}(v), B_n^{(1)}(v)$ and $B_n^{(2)}(v)$, we obtain

$$|\mathscr{V}_{\alpha,\beta}(P_h u - u)(t)| \le C \begin{cases} h^{2\alpha}, & \text{if} \quad r = 1, \\ h^{m+\alpha}, & \text{if} \quad r \ge \dfrac{m}{\alpha}, \end{cases}$$

which together with (6.3) yield the desired result. □

6.2. Superconvergence analysis

In order to ensure that the number of elements N for I_h is even, we define $N = 2\widetilde{N}$ and $I_n := \sigma_{n-1}$, for $n = 1, 2, \dots, N$. Assuming that I_{2h} is obtained from I_h by merging two adjacent intervals $I_n \cup I_{n+1}$, where $n = 1, 3, \dots, 2\widetilde{N} - 1$. We define a higher-order interpolation operator I_{2h}^{2m-1} of degree $2m - 1$ associated with I_{2h}, which satisfies the following conditions:

$$I_{2h}^{2m-1}u\big|_{I_n \cup I_{n+1}} \in \mathscr{P}_{2m-1}, \quad n = 1, 3, \dots, 2\tilde{N} - 1$$

with

$$I_{2h}^{2m-1}u(t_{n,i}) = u(t_{n,i}), \quad n = 0, 1, \dots, 2\tilde{N} - 1, \quad i = 1, 2, \dots, m.$$

We first state some results on the interpolation operator in [22, Lemma 5].

Lemma 6.3. *For any $u \in C[0, T] \cap C^{m+1}(0, T]$, we have the following results:*

(1) $\left\|u - I_{2h}^{2m-1}u\right\|_\infty := \sup_{t \in I} \left|u(t) - I_{2h}^{2m-1}u(t)\right| \le Ch^{\min\{r\alpha, m+1\}}$.

(2) $I_{2h}^{2m-1}P_h = I_{2h}^{2m-1}$.

(3) $\left\|I_{2h}^{2m-1}u\right\|_\infty \le C\|u\|_\infty$.

With the help of the supercloseness relationship between u_h and $P_h u$, and the above results on the higher-order interpolation operator, we can achieve the following global superconvergence result based on the interpolation postprocessing technique.

Theorem 6.2. *Under the assumption of Theorem* 5.1*, we have the following superconvergence result:*

$$\left\|u - I_{2h}^{2m-1}u_h\right\|_\infty := \sup_{t \in I} \left|u(t) - I_{2h}^{2m-1}u_h(t)\right| \le C \begin{cases} h^\alpha, & \text{if} \quad r = 1, \\ h^{r\alpha}, & \text{if} \quad \dfrac{m}{\alpha} \le r < \dfrac{m}{\alpha} + 1, \\ h^{m+\alpha}, & \text{if} \quad r \ge \dfrac{m}{\alpha} + 1. \end{cases}$$

Proof. By Lemma 6.3 and Theorem 6.1, we have

$$\begin{aligned} \left\|u - I_{2h}^{2m-1}u_h\right\|_\infty &\le \left\|u - I_{2h}^{2m-1}P_h u\right\|_\infty + \left\|I_{2h}^{2m-1}P_h u - I_{2h}^{2m-1}u_h\right\|_\infty \\ &= \left\|u - I_{2h}^{2m-1}u\right\|_\infty + \left\|I_{2h}^{2m-1}(P_h u - u_h)\right\|_\infty \\ &\le Ch^{\min\{r\alpha, m+1\}} + C \begin{cases} h^{2\alpha}, & \text{if} \quad r = 1, \\ h^{m+\alpha}, & \text{if} \quad r \ge \dfrac{m}{\alpha} \end{cases} \\ &\le C \begin{cases} h^\alpha, & \text{if} \quad r = 1, \\ h^{r\alpha}, & \text{if} \quad \dfrac{m}{\alpha} \le r < \dfrac{m}{\alpha} + 1, \\ h^{m+\alpha}, & \text{if} \quad r \ge \dfrac{m}{\alpha} + 1. \end{cases} \end{aligned}$$

The proof is complete. □

7. Numerical Experiments

In this section, we present several examples for $m = 2$ to numerically verify our theoretical results. We use Gauss collocation parameters $c_1 = (3-\sqrt{3})/6, c_2 = (3+\sqrt{3})/6$; Radau II collocation parameters $c_1 = 1/3, c_2 = 1$; as well as two other sets of collocation parameters: $c_1 = 1/4, c_2 = 5/6$ ($J_0 = 0$); $c_1 = 1/3, c_2 = 2/3$. To assess the global convergence results discussed in the preceding section, we follow the methodology of [15] by computing the maximum error at the points $\tau_{k,i} := t_k + ih_k/10$, where $k = 0, 1, \ldots, N-1$ and $i = 0, 1, \ldots, 10$. In each table, we estimate the convergence order of the error using the expression $\ln(\text{error}(N_1)/\text{error}(N_2))/\ln(N_2/N_1)$, allowing us to irregularly utilize the number N of subintervals (see [24]). We then select the two largest values N_1 and N_2 of N from the table, and use the corresponding errors observed for those values to calculate the convergence order.

Example 7.1. In order to illustrate the result for the case $\alpha > \beta$, in (1.1), we set $\alpha = 1/2$, $\beta = 1/3, p(t) = 1/2, K(t,s) = 1, u_0 = 1$, and choose $g(t)$ such that the exact solution $u(t) = t^{1/2} + t^{7/6} + 1$.

In order to illustrate the convergence result of Theorem 4.1, in Tables 7.1 and 7.2, we present the errors and convergence orders of the collocation solution both for $r = 1$ (uniform mesh) and $r = m/\alpha$ (graded mesh) respectively, and it demonstrates that the convergence order is α on uniform meshes and m on graded meshes, which agrees with our theoretical convergence result.

In order to illustrate the superconvergence result of Theorem 5.1 for the iterated postprocessing method, in Tables 7.3 and 7.4, we present the errors and orders of the iterated postprocessing solution, and it shows that the superconvergence order is 2α on the uniform mesh ($r = 1$), and $m + \alpha$ on the graded mesh ($r = m/\alpha$) for the collocation parameter satisfying the orthogonal condition $J_0 = 0$, which agrees with the theoretical superconvergence results of the iterated postprocessing method.

Table 7.1: The collocation errors for Example 7.1 with $r = 1$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
64	3.7729e-02	4.5533e-02	4.0224e-02	4.2109e-02
128	2.6725e-02	3.2271e-02	2.8497e-02	2.9834e-02
256	1.8917e-02	2.2850e-02	2.0173e-02	2.1120e-02
512	1.3383e-02	1.6169e-02	1.4273e-02	1.4943e-02
1024	9.4661e-03	1.1438e-02	1.0095e-02	1.0569e-02
Order	0.50	0.50	0.50	0.50

Table 7.2: The collocation errors for Example 7.1 with $r = m/\alpha$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
64	1.1680e-04	1.7238e-04	1.3156e-04	1.4219e-04
128	2.9199e-05	4.3800e-05	3.2888e-05	3.5545e-05
256	7.2997e-06	1.0773e-05	8.2219e-06	8.8861e-06
512	1.8249e-06	2.6931e-06	2.0555e-06	2.2215e-06
Order	2.00	2.00	2.01	2.00

Table 7.3: The iterated collocation errors for Example 7.1 with $r = 1$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
128	1.4895e-04	2.2810e-04	1.7426e-04	3.7407e-04
256	7.3802e-05	1.1319e-04	8.6395e-05	1.3672e-04
512	3.6681e-05	5.6309e-05	4.2956e-05	4.9501e-05
1024	1.8268e-05	2.8060e-05	2.1397e-05	2.4164e-05
Order	1.01	1.01	1.01	1.04

Table 7.4: The iterated collocation errors for Example 7.1 with $r = m/\alpha$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
64	1.3001e-06	8.9383e-06	2.9868e-06	1.0697e-04
128	2.2258e-07	1.3436e-06	4.5559e-07	2.6844e-05
256	3.8742e-08	2.0349e-07	7.0987e-08	6.7299e-06
512	6.7988e-09	3.1422e-08	1.1330e-08	1.6859e-06
Order	2.52	2.70	2.65	2.00

Table 7.5: The interpolation postprocessing results for Example 7.1 with $r = 1$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
128	2.1754e-02	2.5382e-02	2.2916e-02	2.3758e-02
256	1.5353e-02	1.7895e-02	1.6170e-02	1.6765e-02
512	1.0840e-02	1.2627e-02	1.1416e-02	1.1836e-02
1024	7.6571e-03	8.9143e-03	8.0631e-03	8.3600e-03
Order	0.50	0.50	0.50	0.50

Table 7.6: The interpolation postprocessing results for Example 7.1 with $r = m/\alpha$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
64	8.7841e-04	1.7684e-04	5.1872e-04	9.3601e-04
128	2.1962e-04	4.4217e-05	1.2969e-04	2.3402e-04
256	5.4905e-05	1.1055e-05	3.2423e-05	5.8505e-05
512	1.3726e-05	2.7637e-06	8.1058e-06	1.4626e-05
Order	2.00	2.00	2.01	2.00

In order to illustrate the superconvergence result of Theorem 6.2 for the interpolation postprocessing method, in Tables 7.5-7.7, we present the errors and orders of the interpolation postprocessing solution, and it shows that there are no superconvergence phenomena for the uniform mesh with $r = 1$ and graded mesh with $r = m/\alpha$, but for the graded mesh with $r = m/\alpha + 1$, the superconvergence phenomena happen, and the superconvergence order is $m + \alpha$ for the collocation parameter satisfying the orthogonal condition $J_0 = 0$, which agrees with the theoretical superconvergence results of the iterated postprocessing method.

In Table 7.8, we use the Gauss collocation parameter to compare the superconvergence order and CPU calculation time for interpolation postprocessing with $r = m/\alpha + 1$ and the iterated collocation method with $r = m/\alpha$. The results show that both postprocessing techniques achieve the same optimal global superconvergence order. However, the computational cost

Table 7.7: The interpolation postprocessing results for Example 7.1 with $r = m/\alpha + 1$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
64	3.2649e-04	6.4874e-05	1.9059e-04	3.3737e-04
128	5.7717e-05	1.1468e-05	3.3692e-05	5.9639e-05
256	1.0203e-05	2.0274e-06	5.9560e-06	1.0543e-05
512	1.8036e-06	3.5839e-07	1.0529e-06	2.1884e-06
Order	2.51	2.51	2.51	2.27

Table 7.8: Iterated collocation method versus interpolation postprocessing for Example 7.1.

N	$u - u_h^{it}$ $(r = m/\alpha)$	CPU	$u - I_{2h}^{2m-1} u_h$ $(r = m/\alpha + 1)$	CPU
64	1.3001e-06	0.70s	3.2649e-04	0.27s
128	2.2258e-07	2.02s	5.7717e-05	0.40s
256	3.8742e-08	7.56s	1.0203e-05	0.92s
512	6.7988e-09	29.49s	1.8036e-06	3.08s
Order	2.52	–	2.51	–

for the interpolation postprocessing method is significantly lower than that for the iterated collocation method.

Example 7.2. In order to illustrate the result for the case $\alpha < \beta$, in (1.1), we set $\alpha = 3/10$, $\beta = 4/10, p(t) = 1/2, K(t,s) = 1, u_0 = 1$, and choose $g(t)$ such that the exact solution $u(t) = t^{3/10} + t^{9/10} + 1$.

Tables 7.9-7.11 list the errors and orders of the collocation solution, iterated collocation solution and interpolation postprocessing solution, respectively on graded meshes for Example 7.2, and it observes the optimal convergence and superconvergence orders, which means that our theoretical results still hold for the case $\alpha < \beta$.

Table 7.9: The collocation errors for Example 7.2 with $r = m/\alpha$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
24	2.6363e-03	6.2225e-03	3.5617e-03	1.0789e-02
44	7.1735e-04	1.7505e-03	9.7587e-04	3.3714e-03
90	1.5951e-04	3.7856e-04	2.1400e-04	8.3525e-04
124	8.2732e-05	1.9079e-04	1.0759e-04	4.4532e-04
Order	2.07	2.16	2.16	1.98

Table 7.10: The iterated collocation errors for Example 7.2 with $r = m/\alpha$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
24	5.7537e-04	3.3518e-03	1.2860e-03	9.4263e-03
44	1.2340e-04	6.2414e-04	2.4412e-04	2.8527e-03
90	2.1075e-05	1.0538e-04	4.1819e-05	6.9275e-04
124	9.5701e-06	4.7559e-05	1.8648e-05	3.6643e-04
Order	2.49	2.51	2.54	2.00

Table 7.11: The interpolation postprocessing results for Example 7.2 with $r = m/\alpha + 1$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
16	1.5407e-01	2.8464e-02	8.6795e-02	1.4846e-01
28	4.2539e-02	7.8589e-03	2.3964e-02	4.0991e-02
44	1.5042e-02	2.7790e-03	8.4739e-03	1.4495e-02
90	2.9007e-03	5.3590e-04	1.6340e-03	2.7951e-03
Order	2.33	2.33	2.33	2.34

Example 7.3. In order to illustrate the result for the case $\alpha = \beta$, in (1.1), we set $\alpha = \beta = 1/2$, $p(t) = 1/2$, $K(t, s) = 1$, $u_0 = 1$, and choose $g(t)$ such that the exact solution $u(t) = t^{1/2} + t + 1$.

In Tables 7.12-7.14, we also list the errors and orders of the collocation solution, iterated collocation solution and interpolation postprocessing solution, respectively on graded meshes for Example 7.3, and it observes the optimal convergence and superconvergence orders, which means that our theoretical results still hold for the case $\alpha = \beta$.

Table 7.12: The collocation errors for Example 7.3 with $r = m/\alpha$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
64	1.1680e-04	2.0316e-04	1.3156e-04	1.5883e-04
128	2.9199e-05	4.6851e-05	3.2888e-05	3.9644e-05
256	7.2997e-06	1.1106e-05	8.2219e-06	9.8926e-06
512	1.8249e-06	2.6931e-06	2.0555e-06	2.4684e-06
Order	2.01	2.05	2.01	2.01

Table 7.13: The iterated collocation errors for Example 7.3 with $r = m/\alpha$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
64	4.9545e-06	3.7760e-05	1.2737e-05	6.5905e-05
128	7.9600e-07	5.7135e-06	1.9550e-06	1.5744e-05
256	1.3302e-07	8.6349e-07	3.0502e-07	3.8392e-06
512	2.2753e-08	1.3241e-07	4.8505e-08	9.4616e-07
Order	2.55	2.71	2.66	2.03

Table 7.14: The interpolation postprocessing results for Example 7.3 with $r = m/\alpha + 1$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
64	3.2649e-04	1.0574e-04	1.9059e-04	2.4227e-04
128	5.7717e-05	1.7366e-05	3.3692e-05	4.2828e-05
256	1.0203e-05	2.7912e-06	5.9560e-06	7.5709e-06
512	1.8036e-06	4.5224e-07	1.0529e-06	1.3384e-06
Order	2.51	2.63	2.51	2.51

Example 7.4. In order to show that our methods still hold for nonlinear fractional integro-differential equations with weakly singular kernel, we consider the following initial-value problem:

$$\begin{cases} {}_0^C D_t^{\frac{1}{2}} u(t) = \dfrac{1}{2}u(t) + \displaystyle\int_0^t (t-s)^{-\frac{1}{3}} s^2 \sin\big(u(s)\big) ds + e^t, \quad t \in (0,1], \\ u(0) = 1. \end{cases} \tag{7.1}$$

Now the exact solution is unknown, so we compute a reference solution using a uniform or graded mesh with $N = 2990$. In Tables 7.15 and 7.16, we present the errors of the collocation method and the interpolation postprocessing method on graded meshes with graded exponent $r = m/\alpha + 1$. The results show that the convergence order for the collocation method is m, while the order for the interpolation postprocessing method improves to $m + \alpha$, which indicates that our interpolation postprocessing technique still works for nonlinear FIDEs.

Table 7.15: The collocation errors for Example 7.4 with $r = m/\alpha + 1$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
62	1.0924e-03	2.1467e-03	1.3558e-03	1.6409e-03
118	2.9763e-04	5.8857e-04	3.7025e-04	4.5125e-04
152	1.7879e-04	3.5406e-04	2.2262e-04	2.7156e-04
212	9.1751e-05	1.8199e-04	1.1430e-04	1.3930e-04
Order	2.01	2.00	2.00	2.01

Table 7.16: The interpolation postprocessing results for Example 7.4 with $r = m/\alpha + 1$.

N	Gauss	Radau IIA	$(1/4, 5/6)$	$(1/3, 2/3)$
62	2.2360e-04	5.1840e-04	2.4456e-04	3.7983e-04
118	4.3059e-05	1.0960e-04	5.1744e-05	9.8708e-05
152	2.3080e-05	5.9530e-05	2.8060e-05	5.7966e-05
212	9.8999e-06	2.6023e-05	1.2137e-05	2.8276e-05
Order	2.54	2.49	2.52	2.16

Acknowledgements. The authors thank the anonymous referees for their constructive suggestions.

This work is supported by the Visiting scholar program of National Natural Science Foundation of China (Grant No. 12426616), by the Natural Science Research Start-up Foundation of Recruiting Talents of Nanjing University of Posts and Telecommunications (Grant No. NY223127) and by the National Natural Science Foundation of China (Grant No. 12271233).

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