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THE EXPONENTIAL SCALAR AUXILIARY VARIABLE APPROACH FOR THE LANDAU-LIFSHITZ EQUATION*

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Abstract

In this paper, we present two structure-preserving numerical schemes for the Landau-Lifshitz equation by combining the exponential scalar auxiliary variable method with the projection method. These schemes preserve both the length constraint and the modified energy dissipation law, ensuring numerical stability and accuracy. Moreover, they are particularly well-suited for studying the Landau-Lifshitz equation with higher-order energy terms, which have often been overlooked in earlier studies but have a significant impact on the stability, dynamics, and thermal behavior of magnetic skyrmions. We establish the unique solvability and energy stability of the schemes, and provide a rigorous error analysis. Numerical experiments are conducted to demonstrate the accuracy and effectiveness of the proposed schemes.

Mathematics subject classification: 65N15, 65N12, 35K61.
Key words: Landau-Lifshitz equation, Exponential scalar auxiliary variable, Error estimate, Micromagnetics, Structure-preserving.

1. Introduction

Magnetic skyrmions are a class of vortex-like spin structures found in certain magnetic materials. Owing to their topological stability, compact size, ease of manipulation, and particle-like behavior, skyrmions are considered as promising candidates for various applications, including next-generation information storage devices and neuromorphic computing systems [18, 27, 30]. The properties of skyrmions are generally attributed to the interplay between exchange interactions, the Dzyaloshinskii-Moriya interaction (DMI), and magnetocrystalline anisotropy energy [16, 20, 38]. Recent studies have revealed that the higher-order exchange interactions and anisotropies, which were previously neglected, play a crucial role in the stability, dynamics, and thermal behavior of skyrmions [13, 28, 29]. To accurately describe the skyrmion dynamics, the Landau-Lifshitz (LL) equation is often employed as a fundamental model [21]. In this work, we introduce a structure-preserving algorithm tailored for the numerical study of the LL equation including high-order terms, enabling more reliable and efficient simulations of skyrmion behavior in complex magnetic systems.

Let $\Omega = \mathbb{T}^2$ be the two-dimensional torus representing a periodic spatial domain and $[0, T]$ be the time interval. After suitable scaling, the magnetizations are described by a normalized vector field $\boldsymbol{m}(\boldsymbol{x}, t) = (m_1, m_2, m_3)$. We consider the dimensionless LL equation with the

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periodic boundary condition

$$\begin{cases} \boldsymbol{m}_t = -\boldsymbol{m} \times \boldsymbol{h}_{\text{eff}} - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{h}_{\text{eff}}) & \text{in } \Omega \times (0,T), \\ \boldsymbol{m}(\boldsymbol{x},0) = \boldsymbol{m}_0 \in \mathbb{S}^2 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\alpha > 0$ is a dimensionless damping coefficient, the effective field $\boldsymbol{h}_{\text{eff}} = -\delta E/\delta \boldsymbol{m}$ is derived from the gradient of the energy functional, and

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3,\ x^2+y^2+z^2 = 1\}.$$

In this study, we incorporate the fourth-order anisotropy as a representative example of higher-order terms. The energy functional including exchange energy, bulk DMI energy, anisotropy energy, and Zeeman energy is

$$\begin{aligned} E(\boldsymbol{m}) = &\frac{1}{2}\int_\Omega |\nabla \boldsymbol{m}|^2 \,\mathrm{d}\boldsymbol{x} + \kappa \int_\Omega (\nabla \times \boldsymbol{m}) \cdot \boldsymbol{m} \,\mathrm{d}\boldsymbol{x} \\ &+ \int_\Omega A_1\big(1 - m_3^2\big) + A_2\big(m_1^4 + m_2^4 + m_3^4\big) \,\mathrm{d}\boldsymbol{x} - \gamma \int_\Omega \boldsymbol{m} \cdot \hat{\boldsymbol{e}}_3 \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{1.2}$$

The exchange energy promotes the parallel alignment of neighboring magnetizations, whereas the DMI energy favors a perpendicular arrangement, thereby stabilizing non-collinear magnetic structures. Here, κ denotes the DMI constant, which is material-dependent. In 2D, the curl of the magnetization is given by

$$\nabla \times \boldsymbol{m} = (\partial_y m_3,\ -\partial_x m_3,\ \partial_x m_2 - \partial_y m_1).$$

The A_1 term corresponds to the uniaxial anisotropy and the A_2 term represents the fourth-order anisotropy. Both A_1 and A_2 are material-dependent constants. Finally, γ represents the strength of an external magnetic field along $\hat{\boldsymbol{e}}_3 = (0,0,1)$ direction.

It is straightforward to derive the following two key physical properties of the LL equation:

- Length preservation:
$$|\boldsymbol{m}(\boldsymbol{x},t)| = 1, \quad \forall \boldsymbol{x} \in \Omega, \quad t \in (0,T),$$
meaning that the magnetization vectors remain on the unit sphere throughout the evolution.

- Energy stability:
$$\frac{\mathrm{d}E\big(\boldsymbol{m}(t)\big)}{\mathrm{d}t} = -\alpha \|\boldsymbol{m} \times \boldsymbol{h}_{\text{eff}}\|^2,$$
where $\alpha > 0$ ensures energy dissipation, and $\alpha = 0$ implies energy conservation.

It is natural to aim for the development of structure-preserving algorithms that maintain these two properties. However, this has proven to be a significant challenge due to the strong nonlinearity of the LL equation. Over the past two decades, substantial efforts have been devoted to designing numerical methods for the LL equation, primarily focusing on models without DMI. In most numerical studies, the pointwise length constraint $|\boldsymbol{m}| = 1$ is enforced via a projection step [1,3,6,14,24,35]. Moreover, it has been shown that, after suitable linearization, such projection-based schemes can also satisfy a (modified) discrete energy dissipation law, see [5,8,10,11,25]. Specifically, the works in [5,10,11] employ the mid-point rule, but the need for a nonlinear solver at each time step poses significant challenges to computational efficiency.

By contrast, [8] adopts a Lagrange-multiplier strategy that leads to a linear scheme, but imposes strict limitations on the time step size. For a comprehensive review of numerical methods for the LL equation, readers are referred to [4, 9, 22, 36].

In contrast, only a handful of numerical methods treat the LL equation with DMI and certain higher-order nonlinear terms [12, 15, 23]. These contributions increase both the analytical and algorithmic complexity, making it difficult to retain the pointwise unit-length constraint and a (modified) discrete energy dissipation law within a fully linear scheme. Existing approaches often rely on nonlinear solvers or impose stringent time-step restrictions. To overcome this difficulty, we turn to the scalar auxiliary variable (SAV) framework, a widely used tool for gradient flows, which introduces a scalar variable via a square-root transformation of the nonlinear energy and enables linear, structure-preserving discretizations [17, 31, 32, 34]. However, the classical SAV method requires the nonlinear energy to be bounded from below, limiting its applicability. The exponential SAV (ESAV) method [19, 26, 37, 39] overcomes this by replacing the square root with an exponential, thereby removing the lower-bound requirement and offering greater flexibility and robustness.

In this paper, we propose two numerical schemes that combine the ESAV method with the projection method. These schemes have the following key features:

(i) Both schemes simultaneously preserve the length constraint and the modified energy dissipation law. Specifically, energy stability is achieved for an appropriate stabilizing constant and imposes no constraints on the physical parameters.

(ii) The schemes are semi-implicit, requiring only the solution of linear systems at each time step, ensuring computational efficiency without additional overhead.

(iii) The methods are highly flexible and can be extended to solve the LL equation with any additional higher-order energy terms.

The outline of this paper is as follows. In Section 2, we introduce the ESAV approach. In Section 3 we present the first- and second-order fully discrete schemes based on the ESAV approach and the main theoretical results of these schemes. We prove the unique solvability and stability in Section 4 and we provide a rigorous error analysis for the proposed projection schemes in Section 5. In Section 6, we conduct numerical experiments to validate the accuracy, convergence, and stability of the schemes, as well as to demonstrate their ability to capture key phenomenological features of magnetic skyrmions.

2. ESAV Scheme for Time Integration and Finite Difference Discretization

In this section, we present the ESAV approach and the corresponding first- and second-order ESAV schemes for the LL equation.

2.1. ESAV scheme for time integration

In the ESAV method for gradient flow equations, the energy functional is typically split into linear and nonlinear components. The key idea behind constructing the ESAV scheme for the LL equation is to introduce a stabilizing term in the linear part of the energy functional, while removing this term from the nonlinear part. This modification ensures that the resulting linear operator is positive definite, thereby guaranteeing energy stability.

Let $\lambda \geq 0$ denote the stabilizing constant. The energy functional (1.2) is splitted as follows:

$$E(\boldsymbol{m}) = E_1(\boldsymbol{m}) + E_2(\boldsymbol{m}), \tag{2.1}$$

where

$$\begin{aligned} E_1(\boldsymbol{m}) &= \int_\Omega \frac{1}{2}|\nabla \boldsymbol{m}|^2 + \kappa(\nabla \times \boldsymbol{m}) \cdot \boldsymbol{m} - A_1 m_3^2 + \frac{\lambda}{2}|\boldsymbol{m}|^2 \mathrm{d}\boldsymbol{x}, \\ E_2(\boldsymbol{m}) &= \int_\Omega A_2\left(m_1^4 + m_2^4 + m_3^4\right) - \gamma \boldsymbol{m} \cdot \hat{\boldsymbol{e}}_3 - \frac{\lambda}{2}|\boldsymbol{m}|^2 \mathrm{d}\boldsymbol{x}. \end{aligned}$$

Correspondingly, the energy gradient (2.1) is decomposed into a linear operator

$$\mathcal{L}\boldsymbol{m} = \frac{\delta E_1}{\delta \boldsymbol{m}} = -\Delta \boldsymbol{m} + 2\kappa \nabla \times \boldsymbol{m} - 2A_1(\hat{\boldsymbol{e}}_3 \otimes \hat{\boldsymbol{e}}_3)\boldsymbol{m} + \lambda \boldsymbol{m},$$

and a nonlinear part

$$N(\boldsymbol{m}) = \frac{\delta E_2}{\delta \boldsymbol{m}} = 4A_2\big(m_1^3, m_2^3, m_3^3\big) - \gamma \hat{\boldsymbol{e}}_3 - \lambda \boldsymbol{m}. \tag{2.2}$$

We define scalar auxiliary variable

$$r(t) = E_2\big(\boldsymbol{m}(t)\big),$$

and rewrite the LL equation (1.1) as

$$\boldsymbol{m}_t = \boldsymbol{m} \times \big(\mathcal{L}\boldsymbol{m} + g(\boldsymbol{m}, r)N(\boldsymbol{m})\big) + \alpha \boldsymbol{m} \times \Big(\boldsymbol{m} \times \big(\mathcal{L}\boldsymbol{m} + g(\boldsymbol{m}, r)N(\boldsymbol{m})\big)\Big), \tag{2.3a}$$

$$r_t = g(\boldsymbol{m}, r)\langle N(\boldsymbol{m}), \boldsymbol{m}_t \rangle, \tag{2.3b}$$

where

$$g(\boldsymbol{m}, r) = \frac{\exp(r)}{\exp\big(E_2(\boldsymbol{m})\big)}.$$

Note that this yields the following equivalent formulation of energy functional:

$$E(\boldsymbol{m}) = E(\boldsymbol{m}, r) = \frac{1}{2}\langle \boldsymbol{m}, \mathcal{L}\boldsymbol{m} \rangle + r. \tag{2.4}$$

2.2. Finite difference discretization

The finite difference method is used to approximate (2.3a)-(2.3b). For simplicity, we consider the two-dimensional square domain $\Omega = (0, L]^2$. Let M be a positive integer, and set $h = L/M$ as the uniform mesh size used to partition Ω. Denote the partition of the time interval $[0, T]$ by $\{t_n = n\tau\}_{n=0}^N$, where the time step size is $\tau = T/N$ for some positive integer N. Define Ω_h as the set of mesh points $(x_i, y_j) = (ih, jh)$ for $1 \leq i, j \leq M$.

For a grid function f on Ω_h, we use the shorthand $f_{ij} = f(x_i, y_j)$. Let $\mathcal{M}_h$ represent the space of all periodic grid functions on Ω_h, defined as

$$\mathcal{M}_h = \{f : \Omega_h \to \mathbb{R} \mid f_{i+kM, j+lM} = f_{ij},\ k, l \in \mathbb{Z},\ 1 \leq i, j \leq M\}.$$

This definition ensures that functions in $\mathcal{M}_h$ repeat periodically across the boundaries of Ω_h. For any $f \in \mathcal{M}_h$, we define the following discrete spatial differentiation and gradient operator:

$$\delta_x f_{ij} = \frac{f_{i+1,j} - f_{ij}}{h}, \quad \delta_y f_{ij} = \frac{f_{i,j+1} - f_{ij}}{h}, \quad \nabla_h f_{ij} = (\delta_x f_{ij}, \delta_y f_{ij})^\top.$$

For any $\boldsymbol{f} = (f^1, f^2, f^3)^\top \in \mathcal{M}_h^3$, the discrete gradient operator is defined by

$$\nabla_h \boldsymbol{f}_{i,j} = \begin{bmatrix} \delta_x f_{i,j}^1 & \delta_y f_{i,j}^1 \\ \delta_x f_{i,j}^2 & \delta_y f_{i,j}^2 \\ \delta_x f_{i,j}^3 & \delta_y f_{i,j}^3 \end{bmatrix}^\top.$$

The discrete curl operator using central differences is given by

$$\nabla_h \times \boldsymbol{f}_{i,j} = \frac{1}{2} \begin{bmatrix} \delta_y f_{i,j}^3 + \delta_y f_{i,j-1}^3 \\ -\left(\delta_x f_{i,j}^3 + \delta_x f_{i-1,j}^3\right) \\ \delta_x f_{i,j}^2 + \delta_x f_{i-1,j}^2 - \left(\delta_y f_{i,j}^1 + \delta_y f_{i,j-1}^1\right) \end{bmatrix}.$$

The discrete Laplacian operator is defined by the standard five-point stencil

$$\Delta_h \boldsymbol{f}_{i,j} = \frac{1}{h^2}(\boldsymbol{f}_{i+1,j} + \boldsymbol{f}_{i-1,j} + \boldsymbol{f}_{i,j+1} + \boldsymbol{f}_{i,j-1} - 4\boldsymbol{f}_{i,j}).$$

Using the definitions above, we define the discrete linear operator $\mathcal{L}_h : \mathcal{M}_h^3 \to \mathcal{M}_h^3$ as

$$\mathcal{L}_h \boldsymbol{f} = -\Delta_h \boldsymbol{f} + 2\kappa \nabla_h \times \boldsymbol{f} - 2A_1(\boldsymbol{e}_3 \otimes \boldsymbol{e}_3)\boldsymbol{f} + \lambda \boldsymbol{f}.$$

For any $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{M}_h^3$, the discrete inner product $\langle \cdot\,, \cdot \rangle$, discrete ℓ^2-norm $\|\cdot\|$, ℓ^∞-norm $\|\cdot\|_\infty$ and H^1-norm $\|\cdot\|_{H_h^1}$ can be defined as usual, namely,

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = h^2 \sum_{i,j=1}^{M} \boldsymbol{f}_{ij} \cdot \boldsymbol{g}_{ij}, \quad \|\boldsymbol{f}\| = \sqrt{\langle \boldsymbol{f}, \boldsymbol{f} \rangle},$$
$$\|\boldsymbol{f}\|_\infty = \max_{1 \le i,j \le M} |\boldsymbol{f}_{ij}|, \qquad \|\boldsymbol{f}\|_{H_h^1}^2 = \|\boldsymbol{f}\|^2 + \|\nabla_h \boldsymbol{f}\|^2.$$

Based on the definitions of the discrete operators above, we obtain the corresponding discrete form E_h of the energy functional (2.1) as follows:

$$E_h(\boldsymbol{m}_h) = E_{1h}(\boldsymbol{m}_h) + E_{2h}(\boldsymbol{m}_h) = \frac{1}{2}\langle \boldsymbol{m}_h, \mathcal{L}_h \boldsymbol{m}_h \rangle + \langle W(\boldsymbol{m}_h), 1 \rangle,$$

where $\boldsymbol{m}_h = (u_h, v_h, w_h) \in \mathcal{M}_h^3$, and

$$W(\boldsymbol{m}_h) = A_2\left(u_h^4 + v_h^4 + w_h^4\right) - \gamma \boldsymbol{m}_h \cdot \hat{\boldsymbol{e}}_3 - \frac{\lambda}{2}|\boldsymbol{m}_h|^2.$$

3. Semi-implicit ESAV Schemes and Main Results

3.1. Fully discrete schemes

For convenience, we define $\widetilde{\boldsymbol{m}}_h^{n+1/2} = (\widetilde{\boldsymbol{m}}_h^{n+1} + \widetilde{\boldsymbol{m}}_h^n)/2$, and based on (2.3), we propose the first-order scheme (Algorithm 3.1), and we present the second-order ESAV scheme (Algorithm 3.2), which utilizes a predictor-corrector structure.

Algorithm 3.1: First-order ESAV Scheme (ESAV-1).

Input: $\boldsymbol{m}_h^0 = \widetilde{\boldsymbol{m}}_h^0 = \boldsymbol{m}_0$, $r^0 = E(\boldsymbol{m}_0)$.

Loop: For $n = 0, \ldots, N-1$ iterate the following two steps:

Step 1: Compute $\widetilde{\boldsymbol{m}}_h^{n+1}$ and r^{n+1} by solving the following linear systems:

$$\frac{\widetilde{\boldsymbol{m}}_h^{n+1} - \widetilde{\boldsymbol{m}}_h^n}{\tau} = \boldsymbol{m}_h^n \times \Big(\mathcal{L}_h \widetilde{\boldsymbol{m}}_h^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_h^n, r^n\big) N\big(\boldsymbol{m}_h^n\big)\Big) + \alpha \boldsymbol{m}_h^n \times \Big(\boldsymbol{m}_h^n \times \Big(\mathcal{L}_h \widetilde{\boldsymbol{m}}_h^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_h^n, r^n\big) N\big(\boldsymbol{m}_h^n\big)\Big)\Big), \tag{3.1}$$

$$\frac{r^{n+1} - r^n}{\tau} = g\big(\boldsymbol{m}_h^n, r^n\big) \left\langle N\big(\boldsymbol{m}_h^n\big), \frac{\widetilde{\boldsymbol{m}}_h^{n+1} - \widetilde{\boldsymbol{m}}_h^n}{\tau} \right\rangle. \tag{3.2}$$

Step 2: Normalize the magnetic field $\widetilde{\boldsymbol{m}}_h^{n+1}$ to obtain $\boldsymbol{m}_h^{n+1}$

$$\boldsymbol{m}_h^{n+1} = \frac{\widetilde{\boldsymbol{m}}_h^{n+1}}{|\widetilde{\boldsymbol{m}}_h^{n+1}|}. \tag{3.3}$$

Output: $\boldsymbol{m}_h^n$ and r^n for all $n = 1, \ldots, N$.

Algorithm 3.2: Second-order ESAV Scheme (ESAV-2).

Input: $\boldsymbol{m}_h^0 = \widetilde{\boldsymbol{m}}_h^0 = \boldsymbol{m}_0$, $r^0 = E(\boldsymbol{m}_0)$.

Loop: For $n = 0, \cdots, N-1$ iterate the following three steps:

Step 1: Compute the intermediate layer $\widehat{\boldsymbol{m}}_h^{n+1/2}$ and $\widehat{r}^{n+1/2}$ by solving the following predictor step:

$$\frac{\widehat{\boldsymbol{m}}_h^{n+1/2} - \widetilde{\boldsymbol{m}}_h^n}{\tau/2} = \boldsymbol{m}_h^n \times \Big(\mathcal{L}_h \widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_h^n, r^n\big) N\big(\boldsymbol{m}_h^n\big)\Big) + \alpha \boldsymbol{m}_h^n \times \Big(\boldsymbol{m}_h^n \times \Big(\mathcal{L}_h \widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_h^n, r^n\big) N\big(\boldsymbol{m}_h^n\big)\Big)\Big), \tag{3.4a}$$

$$\frac{\widehat{r}^{n+1/2} - r^n}{\tau/2} = g\big(\boldsymbol{m}_h^n, r^n\big) \left\langle N\big(\boldsymbol{m}_h^n\big), \frac{\widehat{\boldsymbol{m}}_h^{n+1/2} - \widetilde{\boldsymbol{m}}_h^n}{\tau/2} \right\rangle. \tag{3.4b}$$

Step 2: Compute $\widetilde{\boldsymbol{m}}_h^{n+1}$ and r^{n+1} by solving the following linear system:

$$\frac{\widetilde{\boldsymbol{m}}_h^{n+1} - \widetilde{\boldsymbol{m}}_h^n}{\tau} = \widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}} \times \Big(\mathcal{L}_h \widetilde{\boldsymbol{m}}_h^{n+\frac{1}{2}} + g\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}, \widehat{r}^{n+\frac{1}{2}}\big) N\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\big)\Big) + \alpha \widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}} \times \Big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}} \times \Big(\mathcal{L}_h \widetilde{\boldsymbol{m}}_h^{n+\frac{1}{2}} + g\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}, \widehat{r}^{n+\frac{1}{2}}\big) N\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\big)\Big)\Big), \tag{3.5a}$$

$$\frac{r^{n+1} - r^n}{\tau} = g\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}, \widehat{r}^{n+\frac{1}{2}}\big) \left\langle N\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\big), \frac{\widetilde{\boldsymbol{m}}_h^{n+1} - \widetilde{\boldsymbol{m}}_h^n}{\tau} \right\rangle. \tag{3.5b}$$

Step 3: Normalize the magnetic field $\widetilde{\boldsymbol{m}}_h^{n+1}$ to obtain $\boldsymbol{m}_h^{n+1}$

$$\boldsymbol{m}_h^{n+1} = \frac{\widetilde{\boldsymbol{m}}_h^{n+1}}{|\widetilde{\boldsymbol{m}}_h^{n+1}|}. \tag{3.6}$$

Output: $\boldsymbol{m}_h^n$ and r^n for all $n = 1, \ldots, N$.

3.2. Some notations

Before proceeding with the error analysis, we assume that the exact solution $\boldsymbol{m}_e(x,t)$ of system (2.3) is sufficiently smooth on the domain $\Omega \times [0,T]$. Specifically, we assume that

$$\boldsymbol{m}_e \in C^3\big([0,T]; C^1(\Omega;\mathbb{R}^3)\big) \cap C^1\big([0,T]; C^3(\Omega;\mathbb{R}^3)\big) \cap C\big([0,T]; C^5(\Omega;\mathbb{R}^3)\big).$$

This regularity assumption is key to the subsequent consistency error analysis. In particular, there exists a constant $h^* > 0$ such that, for any $h < h^*$ and for all $t \in [0,T]$, the following uniform boundedness holds:

$$\begin{aligned}&\|\boldsymbol{m}_e(x,t)\|_\infty + \|\nabla_h \boldsymbol{m}_e(x,t)\|_\infty + \|\mathcal{L}_h \boldsymbol{m}_e(x,t)\|_\infty + \big\|\nabla_h\big(\mathcal{L}_h \boldsymbol{m}_e(x,t)\big)\big\|_\infty \\ &\quad + \big\|N\big(\boldsymbol{m}_e(x,t)\big)\big\| + \big\|\nabla_h N\big(\boldsymbol{m}_e(x,t)\big)\big\| \le L^*,\end{aligned}$$

where $L^* > 0$ is a constant independent of h and τ. For convenience, we denote

$$\boldsymbol{m}_e^n := \boldsymbol{m}_e(x,t_n), \quad \boldsymbol{m}_e^{n+\frac{1}{2}} := \frac{1}{2}\big(\boldsymbol{m}_e^n + \boldsymbol{m}_e^{n+1}\big).$$

Then the intermediate value $\boldsymbol{m}_e^{n+1/2}$ and its discrete derivatives satisfy similar uniform boundedness, namely,

$$\begin{aligned}&\big\|\boldsymbol{m}_e^{n+\frac{1}{2}}\big\|_\infty + \big\|\nabla_h \boldsymbol{m}_e^{n+\frac{1}{2}}\big\|_\infty + \big\|\mathcal{L}_h \boldsymbol{m}_e^{n+\frac{1}{2}}\big\|_\infty + \big\|\nabla_h\big(\mathcal{L}_h \boldsymbol{m}_e^{n+\frac{1}{2}}\big)\big\|_\infty \\ &\quad + \big\|N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big)\big\| + \big\|\nabla_h N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big)\big\| \le L^*.\end{aligned}$$

Next, we define the following error functions:

$$\begin{aligned}&\boldsymbol{e}_{\boldsymbol{m}}^n = \boldsymbol{m}_e^n - \boldsymbol{m}_h^n, && e_r^n = r_e^n - r^n, && \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n = \boldsymbol{m}_e^n - \widetilde{\boldsymbol{m}}_h^n, \\ &\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}} = \widetilde{\boldsymbol{m}}_e^{n+\frac{1}{2}} - \widetilde{\boldsymbol{m}}_h^{n+\frac{1}{2}}, && \widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}} = \boldsymbol{m}_e^{n+\frac{1}{2}} - \widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}, && \widehat{e}_r^{n+\frac{1}{2}} = r_e^{n+\frac{1}{2}} - \widehat{r}^{n+\frac{1}{2}}.\end{aligned}$$

3.3. Main theoretical results for ESAV schemes

Based on the energy functional (2.4) together with the proposed ESAV schemes, we define the discrete modified energy as

$$E_h^n = E_h\big(\widetilde{\boldsymbol{m}}_h^n, r^n\big) = \frac{1}{2}\big\langle \widetilde{\boldsymbol{m}}_h^n, \mathcal{L}\widetilde{\boldsymbol{m}}_h^n \big\rangle + r^n,$$

where r^n is computed implicitly via (3.2) for the first-order and (3.5b) for the second-order scheme.

We now establish the unique solvability and energy stability properties of the ESAV-1 and ESAV-2 schemes.

Theorem 3.1. *Let $\lambda > 2\kappa^2 + \max\{2A_1, 0\}$. For any $h > 0$ and $\tau > 0$, the ESAV-1 scheme satisfies the following properties for $n = 0, 1, \ldots, N-1$:*

(i) ***Unique Solvability:*** *The scheme admits a unique solution $\boldsymbol{m}_h^{n+1}$ satisfying $|\boldsymbol{m}_h^{n+1}| = 1$.*

(ii) ***Modified Energy Stability:***

- *If $\alpha > 0$, the scheme is energy dissipative, i.e., $E_h^{n+1} \le E_h^n$.*
- *If $\alpha = 0$, the scheme is energy conservative, i.e., $E_h^{n+1} = E_h^n$.*

Theorem 3.2. *Let $\lambda > 2\kappa^2 + \max\{2A_1, 0\}$. For any $h > 0$ and $\tau > 0$, the ESAV-2 scheme satisfies the following properties for $n = 0, 1, \dots, N-1$:*

(i) ***Unique Solvability:*** *The scheme admits a unique solution $\boldsymbol{m}_h^{n+1}$ satisfying $|\boldsymbol{m}_h^{n+1}| = 1$.*

(ii) ***Modified Energy Stability:***

- *If $\alpha > 0$, the scheme is energy dissipative, i.e., $E_h^{n+1} \le E_h^n$.*
- *If $\alpha = 0$, the scheme is energy conservative, i.e., $E_h^{n+1} = E_h^n$.*

We present the error estimate results in the following two theorems for the ESAV-1 and ESAV-2 schemes, respectively.

Theorem 3.3 (Error Estimate of ESAV-1). *Let $\lambda > 2\kappa^2 + \max\{2A_1, 0\}$. There exists a positive constant h^* such that when*

$$\tau \le h^{1+\epsilon_0}, \quad h \le h^*$$

for any $\epsilon_0 > 1$, the following error estimate for the ESAV-1 scheme holds:

$$\left\|\boldsymbol{e}_{\boldsymbol{m}}^n\right\|^2 + \left\|\nabla_h \boldsymbol{e}_{\boldsymbol{m}}^n\right\|^2 + \left|e_r^n\right|^2 \le C^*(\tau + h^2)^2, \quad 1 \le n \le N,$$

where $C^* > 0$ is a constant independent of h, τ, ϵ_0, and h^*.

Theorem 3.4 (Error Estimate of ESAV-2). *Let $\lambda > 2\kappa^2 + \max\{2A_1, 0\}$. There exists a positive constant τ^* such that when*

$$h = \mathcal{O}(\tau), \quad \tau \le \tau^*,$$

the following error estimate for the ESAV-2 scheme holds:

$$\left\|\boldsymbol{e}_{\boldsymbol{m}}^n\right\|^2 + \left\|\nabla_h \boldsymbol{e}_{\boldsymbol{m}}^n\right\|^2 + \left|e_r^n\right|^2 \le \widehat{C}^*(\tau^2 + h^2)^2, \quad 1 \le n \le N,$$

where $\widehat{C}^* > 0$ is a constant independent of h, τ, and τ^*.

3.4. Preliminaries

First, we collect several lemmas used in the proof of the next lemma and in the subsequent proofs of the main results.

Lemma 3.1 (Summation by Parts and Young's Inequality). *For any grid functions $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{k} \in \mathcal{M}_h^3$, we have*

$$-\langle \boldsymbol{f}, \Delta_h \boldsymbol{g}\rangle = \langle \nabla_h \boldsymbol{f}, \nabla_h \boldsymbol{g}\rangle = -\langle \Delta_h \boldsymbol{f}, \boldsymbol{g}\rangle, \tag{3.7}$$

$$|\langle \boldsymbol{f}, \boldsymbol{g}\rangle| \le \|\boldsymbol{f}\|\|\boldsymbol{g}\| \le \frac{\epsilon^2}{2}\|\boldsymbol{f}\|^2 + \frac{1}{2\epsilon^2}\|\boldsymbol{g}\|^2, \quad \forall \epsilon \ne 0. \tag{3.8}$$

Lemma 3.2. *For any grid functions $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{M}_h^3$, the following inequalities hold:*

$$\|\nabla_h \times \boldsymbol{f_h}\| \le \sqrt{2}\|\nabla_h \boldsymbol{f_h}\|, \tag{3.9}$$

$$\|\boldsymbol{f} \times \boldsymbol{g}\|_\infty \le \|\boldsymbol{f}\|_\infty \|\boldsymbol{g}\|_\infty, \tag{3.10}$$

$$\langle \boldsymbol{f}, \mathcal{L}_h \boldsymbol{g}\rangle \le C\left(\|\boldsymbol{f}\|_{H_h^1}^2 + \|\boldsymbol{g}\|_{H_h^1}^2\right), \tag{3.11}$$

$$\langle \boldsymbol{f} \times \boldsymbol{g}, \boldsymbol{k}\rangle = \langle \boldsymbol{k} \times \boldsymbol{f}, \boldsymbol{g}\rangle, \tag{3.12}$$

$$\langle \boldsymbol{f} \times (\boldsymbol{f} \times \boldsymbol{g}), \boldsymbol{k}\rangle = \langle \boldsymbol{f} \times (\boldsymbol{f} \times \boldsymbol{k}), \boldsymbol{g}\rangle, \tag{3.13}$$

$$\langle \boldsymbol{f} \times (\boldsymbol{f} \times \boldsymbol{g}), \boldsymbol{g}\rangle = -\|\boldsymbol{f} \times \boldsymbol{g}\|^2. \tag{3.14}$$

The proofs of these lemmas are straightforward and are thus omitted.

Lemma 3.3 (Inverse Inequality, [7]). *The inverse inequality implies that, for any* $0\le n\le N$,

$$\|\boldsymbol{e}_{\boldsymbol{m}}^n\|_\infty \le h^{-\frac{d}{2}}\|\boldsymbol{e}_{\boldsymbol{m}}^n\|, \quad \|\nabla_h \boldsymbol{e}_{\boldsymbol{m}}^n\|_\infty \le h^{-\frac{d}{2}}\|\nabla_h \boldsymbol{e}_{\boldsymbol{m}}^n\|,$$

where d denotes the spatial dimension.

Lemma 3.4 (Discrete Gronwall's Inequality, [33]). *Suppose $\{Z^k\}_{k=1}^\infty$ is a non-negative sequence, and let ω be a non-negative constant, satisfying*

$$Z^{k+1} \le (1+C\tau)Z^k + \tau\omega, \quad k=1,2,3,\dots.$$

Then, we have

$$Z^k \le \exp(Ck\tau)\left(Z^1 + \frac{\omega}{C}\right), \quad k=2,3,4,\dots.$$

4. Unique Solvability and Energy Stability

In this section, we establish the unique solvability and energy stability of the two ESAV schemes proposed in the previous section.

4.1. Positivity of the linear operator

The unique solvability of both ESAV schemes relies on the discrete linear operator $\mathcal{L}_h$ being a positive operator, which necessitates imposing a lower bound on the stabilizing parameter λ.

Lemma 4.1. *Let $\lambda > 2\kappa^2 + \max\{2A_1, 0\}$. For any $\boldsymbol{f} \in \mathcal{M}_h^3$, we have*

$$\delta_1\|\boldsymbol{f}\|_{H_h^1}^2 \le \langle \boldsymbol{f}, \mathcal{L}_h\boldsymbol{f}\rangle \le \delta_2\|\boldsymbol{f}\|_{H_h^1}^2, \tag{4.1}$$

where δ_1 and δ_2 are positive constants independent of τ and h.

Proof. By the definition of the discrete curl operator $\nabla_h\times$ and identity (3.7), it is straightforward to verify that $\mathcal{L}_h$ is a linear operator. For any nonzero $\boldsymbol{f} \in \mathcal{M}_h^3$, applying (3.7) and (3.8), we obtain

$$\begin{aligned}\langle \boldsymbol{f}, \mathcal{L}_h\boldsymbol{f}\rangle &= \|\nabla_h\boldsymbol{f}\|^2 + 2\kappa\langle \boldsymbol{f}, \nabla_h\times\boldsymbol{f}\rangle + \lambda\|\boldsymbol{f}\|^2 - 2A_1\langle \boldsymbol{f}, (\hat{\boldsymbol{e}}_3\otimes\hat{\boldsymbol{e}}_3)\boldsymbol{f}\rangle \\ &\ge \|\nabla_h\boldsymbol{f}\|^2 + (\lambda - c)\|\boldsymbol{f}\|^2 + 2\kappa\langle \boldsymbol{f}, \nabla_h\times\boldsymbol{f}\rangle,\end{aligned}$$

where $c = \max\{2A_1, 0\}$. By applying (3.8) and (3.9), we get

$$2\kappa\langle \boldsymbol{f}, \nabla_h\times\boldsymbol{f}\rangle \ge -\left(2\kappa^2 + \frac{\delta_0}{2}\right)\|\boldsymbol{f}\|^2 - \frac{4\kappa^2}{4\kappa^2+\delta_0}\|\nabla_h\boldsymbol{f}\|^2,$$

where $\delta_0 := \lambda - 2\kappa^2 - c > 0$. Therefore,

$$\begin{aligned}\langle \boldsymbol{f}, \mathcal{L}_h\boldsymbol{f}\rangle &\ge \left(\lambda - c - 2\kappa^2 - \frac{\delta_0}{2}\right)\|\boldsymbol{f}\|^2 + \left(1 - \frac{4\kappa^2}{4\kappa^2+\delta_0}\right)\|\nabla_h\boldsymbol{f}\|^2 \\ &= \frac{\delta_0}{2}\|\boldsymbol{f}\|^2 + \frac{\delta_0}{4\kappa^2+\delta_0}\|\nabla_h\boldsymbol{f}\|^2 \ge \delta_1\|\boldsymbol{f}\|_{H_h^1}^2,\end{aligned}$$

where

$$\delta_1 := \min\left\{\frac{\delta_0}{2}, \frac{\delta_0}{4\kappa^2+\delta_0}\right\} > 0.$$

On the other hand, the upper bound directly follows from (3.11), which implies that there exists a constant $\delta_2 > 0$ such that

$$\langle \boldsymbol{f}, \mathcal{L}_h\boldsymbol{f}\rangle \le \delta_2\|\boldsymbol{f}\|_{H_h^1}^2,$$

which completes the proof. ∎

4.2. Proof of unique solvability and energy stability

Proof of Theorem 3.1. (i) First, we prove the unique solvability of (3.1). We rewrite (3.1) as

$$\begin{aligned}&\frac{\widetilde{\boldsymbol{m}}_h^{n+1}}{\tau}-\frac{1}{2}\boldsymbol{m}_h^n\times\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+1}-\frac{\alpha}{2}\boldsymbol{m}_h^n\times\left(\boldsymbol{m}_h^n\times\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+1}\right)\\&=\frac{\widetilde{\boldsymbol{m}}_h^{n}}{\tau}+\boldsymbol{m}_h^n\times\left(\frac{\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n}}{2}+g\left(\boldsymbol{m}_h^n,r^n\right)N\left(\boldsymbol{m}_h^n\right)\right)\\&\quad+\alpha\boldsymbol{m}_h^n\times\left(\boldsymbol{m}_h^n\times\left(\frac{\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n}}{2}+g\left(\boldsymbol{m}_h^n,r^n\right)N\left(\boldsymbol{m}_h^n\right)\right)\right).\end{aligned}$$

Thus, we only need to consider the corresponding homogeneous system

$$\frac{\widetilde{\boldsymbol{m}}_h^{n+1}}{\tau}-\frac{1}{2}\boldsymbol{m}_h^n\times\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+1}-\frac{\alpha}{2}\boldsymbol{m}_h^n\times\left(\boldsymbol{m}_h^n\times\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+1}\right)=\mathbf{0}. \tag{4.2}$$

Taking the discrete ℓ^2-inner product of both sides of (4.2) with $\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+1}$, and using the orthogonality of the vector triple product together with the identity (3.14), we obtain

$$\frac{1}{\tau}\langle\widetilde{\boldsymbol{m}}_h^{n+1},\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+1}\rangle+\frac{\alpha}{2}\left\|\boldsymbol{m}_h^n\times\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+1}\right\|^2=0. \tag{4.3}$$

Suppose $\widetilde{\boldsymbol{m}}_h^{n+1}\neq\mathbf{0}$. Then the first term on the left-hand side of (4.3) is strictly positive by Lemma 4.1, and the second term is nonnegative, a contradiction. Hence, we must have $\widetilde{\boldsymbol{m}}_h^{n+1}=\mathbf{0}$, which establishes the unique solvability of (3.1). Furthermore, once $\widetilde{\boldsymbol{m}}_h^{n+1}$ is obtained, it is evident that the explicit formulas (3.2) and (3.3) each admit a unique solution. Finally, the projection step (3.3) preserves the unit-length constraint, i.e., $|\boldsymbol{m}_h^{n+1}|=1$.

(ii) We take the discrete ℓ^2-inner product of both sides of (3.1) with $\tau(\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+1/2}+g(\boldsymbol{m}_h^n,r^n)\times N(\boldsymbol{m}_h^n))$. After simplification, the left-hand side simplifies to

$$\begin{aligned}&\left\langle\widetilde{\boldsymbol{m}}_h^{n+1}-\widetilde{\boldsymbol{m}}_h^{n},\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+\frac{1}{2}}+g\left(\boldsymbol{m}_h^n,r^n\right)N\left(\boldsymbol{m}_h^n\right)\right\rangle\\&=\frac{1}{2}\left\langle\widetilde{\boldsymbol{m}}_h^{n+1},\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+1}\right\rangle-\frac{1}{2}\langle\widetilde{\boldsymbol{m}}_h^{n},\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n}\rangle+r^{n+1}-r^n=E_h^{n+1}-E_h^n.\end{aligned}$$

On the right-hand side, we have

$$-\alpha\tau\left\|\boldsymbol{m}_h^{n+\frac{1}{2}}\times\left(\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n}+g\left(\boldsymbol{m}_h^n,r^n\right)N\left(\boldsymbol{m}_h^n\right)\right)\right\|^2\leq 0.$$

Therefore $E_h^{n+1}\leq E_h^n$ for $\alpha>0$, and $E_h^{n+1}=E_h^n$ for $\alpha=0$.

This completes the proof of unique solvability and modified energy stability. □

Proof of Theorem 3.2. Unique solvability follows exactly as in the proof of Theorem 3.1, we therefore omit the details. We now prove modified energy stability. We take the discrete ℓ^2-inner product of both sides of (3.5a) with $\tau(\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+1/2}+g(\widehat{\boldsymbol{m}}_h^{n+1/2},\widehat{r}^{n+1/2})N(\widehat{\boldsymbol{m}}_h^{n+1/2}))$. After simplification, the left-hand side simplifies to

$$\begin{aligned}&\left\langle\widetilde{\boldsymbol{m}}_h^{n+1}-\widetilde{\boldsymbol{m}}_h^{n},\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+\frac{1}{2}}+g\left(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}},\widehat{r}^{n+\frac{1}{2}}\right)N\left(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\right)\right\rangle\\&=\frac{1}{2}\langle\widetilde{\boldsymbol{m}}_h^{n+1},\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+1}\rangle-\frac{1}{2}\langle\widetilde{\boldsymbol{m}}_h^{n},\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n}\rangle+r^{n+1}-r^n=E_h^{n+1}-E_h^n.\end{aligned}$$

On the right-hand side, we have

$$-\alpha\tau\left\|\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\times\left(\mathcal{L}_h\widetilde{\boldsymbol{m}}_h^{n+\frac{1}{2}}+g\left(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}},\widehat{r}^{n+\frac{1}{2}}\right)N\left(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\right)\right)\right\|^2\leq 0.$$

Thus, the claim follows. □

5. Error Estimates

In this section, we present the proof of the error analysis stated in Theorems 3.3 and 3.4. For convenience, we use C to denote a generic positive constant independent of τ and h, which may vary from place to place.

5.1. A few preliminary estimates

The following lemmas will be frequently used throughout our analysis and can be obtained by straightforward calculations. Recall that L^* is a positive constant independent of h and τ defined in Section 3.2.

Lemma 5.1. *Let $p_k, q_k \in \{2, \infty\}$ such that $1/p_k + 1/q_k = 1/2$ for $k = 1, 2$. For grid functions $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{M}_h^3$, we have*

$$\|\boldsymbol{f} \times \boldsymbol{g}\|^2 \le \|\boldsymbol{f}\|_{p_1}^2 \|\boldsymbol{g}\|_{q_1}^2, \tag{5.1}$$

$$\|\nabla_h(\boldsymbol{f} \times \boldsymbol{g})\|^2 \le C\left(\|\boldsymbol{f}\|_{p_1}^2 \|\nabla_h \boldsymbol{g}\|_{q_1}^2 + \|\nabla_h \boldsymbol{f}\|_{p_2}^2 \|\boldsymbol{g}\|_{q_2}^2\right). \tag{5.2}$$

Lemma 5.2. *For any grid functions $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{k} \in \mathcal{M}_h^3$, the following estimates hold:*

1. *Let $p, q, s \in \{2, \infty\}$ satisfy $1/p + 1/q + 1/s = 1/2$. Then, we have*

$$\langle \boldsymbol{f} \times \boldsymbol{g}, \boldsymbol{k} \rangle \le |\Omega|^{\frac{1}{2}} \|\boldsymbol{f}\|_p \|\boldsymbol{g}\|_q \|\boldsymbol{k}\|_s. \tag{5.3}$$

2. *Assume that $\boldsymbol{f}$ satisfies $\|\boldsymbol{f}\|_\infty, \|\nabla_h \boldsymbol{f}\|_\infty \le 2L^* + \widehat{G}^* + 1/4$. Then, the following inequality holds:*

$$\langle \boldsymbol{f} \times \boldsymbol{k}, \mathcal{L}_h \boldsymbol{g} \rangle \le C\left(\|\boldsymbol{g}\|_{H_h^1}^2 + \|\boldsymbol{k}\|_{H_h^1}^2\right). \tag{5.4}$$

3. *Assume that $\boldsymbol{f}$ and $\boldsymbol{k}$ satisfy $\|\boldsymbol{f}\|_\infty, \|\nabla_h \boldsymbol{f}\|_\infty, \|\boldsymbol{k}\|_\infty, \|\nabla_h \boldsymbol{k}\|_\infty \le 2L^* + \widehat{G}^* + 1/4$. Then, the following inequalities hold:*

$$\langle \boldsymbol{f} \times \boldsymbol{k}, \mathcal{L}_h \boldsymbol{g} \rangle \le C(\|\boldsymbol{g}\| + \|\nabla_h \boldsymbol{g}\|), \tag{5.5}$$

$$\langle \boldsymbol{f} \times (\boldsymbol{f} \times \boldsymbol{k}), \mathcal{L}_h \boldsymbol{g} \rangle \le C(\|\boldsymbol{g}\| + \|\nabla_h \boldsymbol{g}\|), \tag{5.6}$$

$$\|\boldsymbol{f} \times (\boldsymbol{g} \times \boldsymbol{k})\|_{H_h^1}^2 \le C\|\boldsymbol{g}\|_{H_h^1}^2. \tag{5.7}$$

Remark 5.1. (i) The upper bound $2L^* + \widehat{G}^* + 1/4$ may be any positive constant, but we adopt this particular form for convenience in the subsequent proof. $\widehat{G}^*$ is a positive constant, independent of h and τ, as defined below in (5.15).

(ii) In (5.7), the order of $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{k}$ in the cross product can be interchanged without altering the inequality.

There are two types of error function $\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n$ and $\boldsymbol{e}_{\boldsymbol{m}}^n$, corresponding to the errors of unprojected solution $\widetilde{\boldsymbol{m}}_h^n$ and the projected solution $\boldsymbol{m}_h^n$, respectively. Based on [3], we derive the following essential lemma for error estimates, which quantifies the difference between these two error functions.

Lemma 5.3. *For $1 \le n \le N$, under the condition*

$$\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\|_\infty \le \epsilon$$

for some small $\epsilon > 0$, we have

$$\begin{aligned}
&\|\boldsymbol{e}_{\boldsymbol{m}}^n\|^2 \le 2\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\|^2,\\
&\|\nabla_h \boldsymbol{e}_{\boldsymbol{m}}^n\|^2 \le \|\nabla_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\|^2 + C\left(\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\|^2 + h^{-1}\|\nabla_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\|^4\right).
\end{aligned}$$

Lemma 5.4. *For the ESAV-1 and ESAV-2 schemes, we have*

$$r^n \le E(\boldsymbol{m}_0), \quad \forall\, n = 0, \dots, N.$$

Proof. By Theorem 3.1 or 3.2 for $n = 0, 1, \dots, N$, we have

$$E_h^n = \frac{1}{2}\langle \widetilde{\boldsymbol{m}}_h^n, \mathcal{L}_h \widetilde{\boldsymbol{m}}_h^n\rangle + r^n \le E_h^{n-1} \le \cdots \le E_h^0.$$

Since $\mathcal{L}_h$ is a positive operator, it follows that $r^n \le E_h^0 = E(\boldsymbol{m}_0)$. □

The deviation of the nonlinear operator $N(\cdot)$ is controlled by the error functions as follows.

Lemma 5.5. *Let $\boldsymbol{m}_h^n$ and r^n be the solutions obtained from the ESAV-1 or ESAV-2 scheme. Assume $\boldsymbol{m}_h^n$ satisfies*

$$\max\left\{\|\boldsymbol{m}_h^n\|_\infty, \|\nabla_h \boldsymbol{m}_h^n\|_\infty\right\} \le L^* + \frac{1}{4}. \tag{5.8}$$

Then, the following inequalities hold:

$$\|\nabla_h N(\boldsymbol{m}_h^n)\|_\infty \le \widehat{G}^*, \tag{5.9}$$

$$0 < g(\boldsymbol{m}_h^n, r^n) \le C, \tag{5.10}$$

$$\|g(\boldsymbol{m}_e^n, r_e^n)N(\boldsymbol{m}_e^n) - g(\boldsymbol{m}_h^n, r^n)N(\boldsymbol{m}_h^n)\| \le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^n\| + |\boldsymbol{e}_r^n|\right), \tag{5.11}$$

$$\begin{aligned}
&\|\nabla_h\left[g(\boldsymbol{m}_e^n, r_e^n)N(\boldsymbol{m}_e^n)\right] - \nabla_h\left[g(\boldsymbol{m}_h^n, r^n)N(\boldsymbol{m}_h^n)\right]\|\\
\le\, & C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^n\| + \|\nabla_h \boldsymbol{e}_{\boldsymbol{m}}^n\| + |\boldsymbol{e}_r^n| + b_n\right),
\end{aligned} \tag{5.12}$$

where $\widehat{G}^*$ is a positive constant independent of h and τ, and

$$b_n = h^{-1}\|\boldsymbol{e}_{\boldsymbol{m}}^n\|\left(\|\boldsymbol{e}_{\boldsymbol{m}}^n\| + \|\nabla_h \boldsymbol{e}_{\boldsymbol{m}}^n\|\right).$$

Proof. Due to (5.8), we derive that

$$|E_{2h}(\boldsymbol{m}_h^n)| \le C. \tag{5.13}$$

For any $\theta \in [0, 1]$, we define

$$\begin{aligned}
&\xi_{\boldsymbol{m}}^n = \theta \boldsymbol{m}_e^n + (1-\theta)\boldsymbol{m}_h^n,\\
&\xi_r^n = \theta r_e^n + (1-\theta) r^n,\\
&\xi_E^n = \theta E_{2h}(\boldsymbol{m}_e^n) + (1-\theta)E_{2h}(\boldsymbol{m}_h^n).
\end{aligned}$$

Combing (5.8) with Lemma 5.4 and the smoothness assumption on the exact solution, we obtain

$$\max\left\{\|\xi_{\boldsymbol{m}}^n\|, |\xi_r^n|, |\xi_E^n|\right\} \le C.$$

The nonlinear term $N(\boldsymbol{m})$, explicitly defined in (2.2), has derivatives up to the third order that can be computed analytically and are uniformly bounded when $\boldsymbol{m}$ lies in a bounded region. Therefore, there exists a positive constant G^*, independent of h and τ and depending only on the uniform bound L^*, such that the following bound holds:

$$\max\left\{\left\|N(\xi_{\boldsymbol{m}}^n)\right\|_\infty, \left\|\nabla_{\boldsymbol{m}} N(\xi_{\boldsymbol{m}}^n)\right\|_\infty, \left\|\nabla_{\boldsymbol{m}}^2 N(\xi_{\boldsymbol{m}}^n)\right\|_\infty, \left\|\nabla_{\boldsymbol{m}}^3 N(\xi_{\boldsymbol{m}}^n)\right\|_\infty\right\} \leq G^*, \tag{5.14}$$

where $\nabla_{\boldsymbol{m}}, \nabla_{\boldsymbol{m}}^2$, and $\nabla_{\boldsymbol{m}}^3$ denote the first, second, and third derivatives with respect to $\boldsymbol{m}$, respectively. As a consequence, we can derive

$$\left\|\nabla_h N(\boldsymbol{m}_h^n)\right\|_\infty \leq \left\|\nabla_{\boldsymbol{m}} N(\boldsymbol{m}_h^n)\right\|_\infty \left\|\nabla_h \boldsymbol{m}_h^n\right\|_\infty \leq G^*\left(L^* + \frac{1}{4}\right) =: \widehat{G}^*. \tag{5.15}$$

It follows from Lemma 5.4 and (5.13) that

$$0 < g(\boldsymbol{m}_h^n, r^n) = \frac{\exp(r^n)}{\exp\left(E_{2h}(\boldsymbol{m}_h^n)\right)} \leq C,$$

and thus (5.10). Now we claim that the deviation of g is bounded in the terms of the error functions

$$\left|g(\boldsymbol{m}_e^n, r_e^n) - g(\boldsymbol{m}_h^n, r^n)\right| \leq C\left(\left\|\boldsymbol{e}_{\boldsymbol{m}}^n\right\| + \left|\boldsymbol{e}_r^n\right|\right). \tag{5.16}$$

This is indeed correct. Applying the triangle inequality, we have

$$\begin{aligned}
&\left|g(\boldsymbol{m}_e^n, r_e^n) - g(\boldsymbol{m}_h^n, r^n)\right| \\
\leq\ &\left|g(\boldsymbol{m}_e^n, r_e^n) - g(\boldsymbol{m}_h^n, r_e^n)\right| + \left|g(\boldsymbol{m}_h^n, r_e^n) - g(\boldsymbol{m}_h^n, r^n)\right| \\
=:\ &H_1 + H_2.
\end{aligned}$$

It follows by the mean value theorem, (5.13) and (5.14) that

$$\begin{aligned}
H_1 &= \exp\left(r_e^n\right)\left|\frac{1}{\exp\left(E_{2h}(\boldsymbol{m}_e^n)\right)} - \frac{1}{\exp\left(E_{2h}(\boldsymbol{m}_h^n)\right)}\right| \\
&\leq \frac{\exp\left(r_e^n\right)}{\exp\left(\xi_E^n\right)}\left|E_{2h}(\boldsymbol{m}_e^n) - E_{2h}(\boldsymbol{m}_h^n)\right| \\
&\leq C\left|\left\langle W(\boldsymbol{m}_e^n) - W(\boldsymbol{m}_h^n), 1\right\rangle\right| \\
&\leq C|\Omega|^{\frac{1}{2}}\left\|N(\xi_{\boldsymbol{m}}^n)\right\|_\infty \left\|\boldsymbol{m}_e^n - \boldsymbol{m}_h^n\right\| \leq C\left\|\boldsymbol{e}_{\boldsymbol{m}}^n\right\| \\
H_2 &= \frac{1}{\exp\left(E_{2h}(\boldsymbol{m}_h^n)\right)}\left|\exp\left(r_e^n\right) - \exp(r^n)\right| \\
&\leq \frac{\exp\left(\xi_r^n\right)}{\exp\left(E_{2h}(\boldsymbol{m}_h^n)\right)}\left|r_e^n - r^n\right| \leq C\left|\boldsymbol{e}_r^n\right|.
\end{aligned}$$

Hence (5.16) follows.

We now prove (5.11). By applying the triangle inequality, we obtain

$$\begin{aligned}
&\left\|g(\boldsymbol{m}_e^n, r_e^n)N(\boldsymbol{m}_e^n) - g(\boldsymbol{m}_h^n, r^n)N(\boldsymbol{m}_h^n)\right\| \\
\leq\ &\left|g(\boldsymbol{m}_e^n, r_e^n) - g(\boldsymbol{m}_h^n, r^n)\right|\left\|N(\boldsymbol{m}_e^n)\right\| + g(\boldsymbol{m}_h^n, r^n)\left\|N(\boldsymbol{m}_e^n) - N(\boldsymbol{m}_h^n)\right\| \\
=:\ &H_3 + H_4.
\end{aligned}$$

It follows by (5.14) and (5.16) that

$$H_3 \le G^*|\Omega|^{\frac{1}{2}}\big|g\big(\boldsymbol{m}_e^n, r_e^n\big) - g\big(\boldsymbol{m}_h^n, r^n\big)\big| \le C\big(\|\boldsymbol{e}_{\boldsymbol{m}}^n\| + |\boldsymbol{e}_r^n|\big),$$

and by the mean value theorem, (5.10) and (5.14) that

$$H_4 \le CG^*\|\boldsymbol{m}_e^n - \boldsymbol{m}_h^n\| \le C\|\boldsymbol{e}_{\boldsymbol{m}}^n\|,$$

thus (5.11) holds. (5.12) follows by the same strategy. Applying the triangle inequality once more, we obtain

$$\begin{aligned}
&\big\|\nabla_h\big[g\big(\boldsymbol{m}_e^n, r_e^n\big)N\big(\boldsymbol{m}_e^n\big)\big] - \nabla_h\big[g\big(\boldsymbol{m}_h^n, r^n\big)N\big(\boldsymbol{m}_h^n\big)\big]\big\| \\
\le\ &\big\|\nabla_h\big[N\big(\boldsymbol{m}_e^n\big) - N\big(\boldsymbol{m}_h^n\big)\big]\big\| + \big|g\big(\boldsymbol{m}_e^n, r_e^n\big) - g\big(\boldsymbol{m}_h^n, r^n\big)\big|\big\|\nabla_h N\big(\boldsymbol{m}_h^n\big)\big\| \\
=:\ &H_5 + H_6.
\end{aligned}$$

For H_5, based on the Taylor expansion and the properties of the function N, we obtain

$$N\big(\boldsymbol{m}_e^n\big) - N\big(\boldsymbol{m}_h^n\big) = \nabla_{\boldsymbol{m}} N\big(\boldsymbol{m}_h^n\big)\boldsymbol{e}_{\boldsymbol{m}}^n + \frac{1}{2}\boldsymbol{e}_{\boldsymbol{m}}^{n\top}\nabla_{\boldsymbol{m}}^2 N\big(\xi_{\boldsymbol{m}}^n\big)\boldsymbol{e}_{\boldsymbol{m}}^n,$$

and hence

$$H_5 \le C\left(\big\|\nabla_h\big[\nabla_{\boldsymbol{m}} N\big(\boldsymbol{m}_h^n\big)\boldsymbol{e}_{\boldsymbol{m}}^n\big]\big\| + \big\|\nabla_h\big[\boldsymbol{e}_{\boldsymbol{m}}^{n\top}\nabla_{\boldsymbol{m}}^2 N\big(\xi_{\boldsymbol{m}}^n\big)\boldsymbol{e}_{\boldsymbol{m}}^n\big]\big\|\right).$$

For the first term, using the definition of the discrete gradient and Hölder's inequality, we obtain

$$\begin{aligned}
&\big\|\nabla_h\big[\nabla_{\boldsymbol{m}} N\big(\boldsymbol{m}_h^n\big)\boldsymbol{e}_{\boldsymbol{m}}^n\big]\big\| \\
&\le C\left(\big\|\nabla_h\big[\nabla_{\boldsymbol{m}} N\big(\boldsymbol{m}_h^n\big)\big]\big\|_\infty\|\boldsymbol{e}_{\boldsymbol{m}}^n\| + \big\|\nabla_{\boldsymbol{m}} N\big(\boldsymbol{m}_h^n\big)\big\|_\infty\|\nabla_h\boldsymbol{e}_{\boldsymbol{m}}^n\|\right) \\
&\le C\left(\big\|\nabla_{\boldsymbol{m}}^2 N\big(\boldsymbol{m}_h^n\big)\big\|_\infty\|\nabla_h\boldsymbol{m}_h^n\|_\infty\|\boldsymbol{e}_{\boldsymbol{m}}^n\| + G^*\|\nabla_h\boldsymbol{e}_{\boldsymbol{m}}^n\|\right) \\
&\le C\left(G^*\left(L^* + \frac{1}{4}\right)\|\boldsymbol{e}_{\boldsymbol{m}}^n\| + G^*\|\nabla_h\boldsymbol{e}_{\boldsymbol{m}}^n\|\right) \\
&\le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^n\| + \|\nabla_h\boldsymbol{e}_{\boldsymbol{m}}^n\|\right).
\end{aligned}$$

For the second term, applying Hölder's inequality and Lemma 3.3, we have

$$\begin{aligned}
&\big\|\nabla_h\big[\boldsymbol{e}_{\boldsymbol{m}}^{n\top}\nabla_{\boldsymbol{m}}^2 N\big(\xi_{\boldsymbol{m}}^n\big)\boldsymbol{e}_{\boldsymbol{m}}^n\big]\big\| \\
&\le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^n\|\|\boldsymbol{e}_{\boldsymbol{m}}^n\|_\infty\big\|\nabla_h\nabla_{\boldsymbol{m}}^2 N\big(\xi_{\boldsymbol{m}}^n\big)\big\|_\infty + 2\|\nabla_h\boldsymbol{e}_{\boldsymbol{m}}^n\|\|\boldsymbol{e}_{\boldsymbol{m}}^n\|_\infty\big\|\nabla_{\boldsymbol{m}}^2 N\big(\xi_{\boldsymbol{m}}^n\big)\big\|_\infty\right) \\
&\le C\left(h^{-1}\|\boldsymbol{e}_{\boldsymbol{m}}^n\|^2\big\|\nabla_{\boldsymbol{m}}^3 N\big(\xi_{\boldsymbol{m}}^n\big)\big\|_\infty\|\nabla_h\xi_{\boldsymbol{m}}^n\|_\infty + 2G^*h^{-1}\|\nabla_h\boldsymbol{e}_{\boldsymbol{m}}^n\|\|\boldsymbol{e}_{\boldsymbol{m}}^n\|\right) \\
&\le Ch^{-1}\|\boldsymbol{e}_{\boldsymbol{m}}^n\|\left(\|\boldsymbol{e}_{\boldsymbol{m}}^n\| + \|\nabla_h\boldsymbol{e}_{\boldsymbol{m}}^n\|\right).
\end{aligned}$$

For H_6, using(5.8), (5.14) and (5.16), we obtain

$$H_6 \le C|\Omega|^{\frac{1}{2}}\big\|\nabla_{\boldsymbol{m}} N\big(\boldsymbol{m}_h^n\big)\big\|_\infty\|\nabla_h\boldsymbol{m}_h^n\|_\infty\left(\|\boldsymbol{e}_{\boldsymbol{m}}^n\| + |\boldsymbol{e}_r^n|\right) \le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^n\| + |\boldsymbol{e}_r^n|\right).$$

Combining the estimates for H_5 and H_6, we conclude that (5.12) holds. □

We state Lemma 5.6, whose proof is similar to Lemma 5.5 and is therefore omitted for brevity.

Lemma 5.6. *Let $\widehat{\boldsymbol{m}}_h^{n+1/2}$ and $\widehat{r}^{n+1/2}$ be the solutions obtained from* (3.4a) *and* (3.4b) *in the ESAV-2 scheme. Assume $\widehat{\boldsymbol{m}}_h^{n+1/2}$ satisfies*

$$\max\left\{\left\|\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\right\|_\infty, \left\|\nabla_h \widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\right\|_\infty\right\} \le L^* + \frac{1}{4}.$$

Then, the following inequalities hold:

$$\left\|\nabla_h N\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\big)\right\|_\infty \le \widehat{G}^*, \tag{5.17}$$

$$0 < g\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}, \widehat{r}^{n+\frac{1}{2}}\big) \le C, \tag{5.18}$$

$$\begin{aligned}&\left\|g\big(\boldsymbol{m}_e^{n+\frac{1}{2}}, r_e^{n+\frac{1}{2}}\big)N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big) - g\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}, \widehat{r}^{n+\frac{1}{2}}\big)N\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\big)\right\| \\ \le\ & C\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\right\| + \left|\widehat{e}_r^{n+\frac{1}{2}}\right|\right),\end{aligned} \tag{5.19}$$

$$\begin{aligned}&\left\|\nabla_h\left[g\big(\boldsymbol{m}_e^{n+\frac{1}{2}}, r_e^{n+\frac{1}{2}}\big)N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big)\right] - \nabla_h\left[g\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}, \widehat{r}^{n+\frac{1}{2}}\big)N\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\big)\right]\right\| \\ \le\ & C\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\right\| + \left\|\nabla_h \widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\right\| + \left|\widehat{e}_r^{n+\frac{1}{2}}\right| + \widehat{b}_{n+\frac{1}{2}}\right),\end{aligned} \tag{5.20}$$

where $\widehat{G}^*$ is a positive constant independent of h and τ, and

$$\widehat{b}_{n+\frac{1}{2}} = h^{-1}\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\right\|\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\right\| + \left\|\nabla_h \widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\right\|\right).$$

5.2. Proof of Theorem 3.3

We now provide a rigorous proof of Theorem 3.3. It is easy to verify that, the exact solution $\boldsymbol{m}_e$ of (2.3a) and the exact solution r_e of (2.3b) satisfies

$$\begin{aligned}\frac{\boldsymbol{m}_e^{n+1} - \boldsymbol{m}_e^n}{\tau} &= \boldsymbol{m}_e^n \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_e^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_e^n, r_e^n\big)N\big(\boldsymbol{m}_e^n\big)\right) \\ &\quad + \alpha \boldsymbol{m}_e^n \times \left(\boldsymbol{m}_e^n \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_e^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_e^n, r_e^n\big)N\big(\boldsymbol{m}_e^n\big)\right)\right) + \boldsymbol{T}_{\boldsymbol{m}}^n,\end{aligned} \tag{5.21}$$

$$\frac{r_e^{n+1} - r_e^n}{\tau} = g\big(\boldsymbol{m}_e^n, r_e^n\big)\left\langle N\big(\boldsymbol{m}_e^n\big), \frac{\boldsymbol{m}_e^{n+1} - \boldsymbol{m}_e^n}{\tau}\right\rangle + \boldsymbol{T}_r^n. \tag{5.22}$$

where the truncation errors $\boldsymbol{T}_{\boldsymbol{m}}^n$ and $\boldsymbol{T}_r^n$ satisfy

$$\left\|\boldsymbol{T}_{\boldsymbol{m}}^n\right\| \le C(\tau + h^2), \quad \left|\boldsymbol{T}_r^n\right| \le C(\tau + h^2).$$

By applying Taylor expansion in both time and space, along with the mean value theorem and the definition of the discrete gradient, we obtain

$$\left\|\nabla_h \boldsymbol{T}_{\boldsymbol{m}}^n\right\| \le C(\tau + h^2).$$

Subtracting (3.1) from (5.21) and (3.2) from (5.22) yields the following error equations:

$$\begin{aligned}\frac{\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1} - \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n}{\tau} &= \boldsymbol{e}_{\boldsymbol{m}}^n \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_e^{n+\frac{1}{2}} + N\big(\boldsymbol{m}_e^n\big)\right) \\ &\quad + \boldsymbol{m}_h^n \times \left(\mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_e^n, r_e^n\big)N\big(\boldsymbol{m}_e^n\big) - g\big(\boldsymbol{m}_h^n, r^n\big)N\big(\boldsymbol{m}_h^n\big)\right) \\ &\quad + \alpha \boldsymbol{m}_h^n \times \left(\boldsymbol{e}_{\boldsymbol{m}}^n \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_e^{n+\frac{1}{2}} + N\big(\boldsymbol{m}_e^n\big)\right)\right)\end{aligned}$$

$$
\begin{aligned}
&+\alpha \boldsymbol{e}_{\boldsymbol{m}}^n \times \Big(\boldsymbol{m}_e^n \times \Big(\mathcal{L}_h \widetilde{\boldsymbol{m}}_e^{n+\frac{1}{2}} + N\big(\boldsymbol{m}_e^n\big)\Big)\Big) \\
&+\alpha \boldsymbol{m}_h^n \times \Big(\boldsymbol{m}_h^n \times \Big(\mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_e^n, r_e^n\big) N\big(\boldsymbol{m}_e^n\big) - g\big(\boldsymbol{m}_h^n, r^n\big) N\big(\boldsymbol{m}_h^n\big)\Big)\Big) + \boldsymbol{T}_{\boldsymbol{m}}^n \\
=:& \sum_{i=1}^{6} A_i^n,
\end{aligned} \tag{5.23}
$$

$$
\begin{aligned}
\boldsymbol{e}_r^{n+1} - \boldsymbol{e}_r^n =& \left\langle g\big(\boldsymbol{m}_e^n, r_e^n\big) N\big(\boldsymbol{m}_e^n\big) - g\big(\boldsymbol{m}_h^n, r^n\big) N\big(\boldsymbol{m}_h^n\big), \boldsymbol{m}_e^{n+1} - \boldsymbol{m}_e^n \right\rangle \\
&+ \left\langle g\big(\boldsymbol{m}_h^n, r^n\big) N\big(\boldsymbol{m}_h^n\big), \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1} - \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n \right\rangle + \tau \boldsymbol{T}_r^n.
\end{aligned} \tag{5.24}
$$

We first establish the primary error estimate

$$
\big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2 + \big|\boldsymbol{e}_r^n\big|^2 \le \widehat{C}(\tau + h^2)^2, \quad 1 \le n \le N, \tag{5.25}
$$

where $\widehat{C}$ is a positive constant independent of τ and h. The proof is carried out by mathematical induction and consists of the following two steps:

Step 1. We prove that (5.25) holds for $n = 1$. Taking $n = 0$ in (5.23) and noting that $\boldsymbol{e}_{\boldsymbol{m}}^0 = \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^0 = \boldsymbol{0}$ and $\boldsymbol{e}_r^0 = 0$, we have

$$
\frac{\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1}{\tau} = \boldsymbol{m}_h^0 \times \frac{1}{2} \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 + \alpha \boldsymbol{m}_h^0 \times \left(\boldsymbol{m}_h^0 \times \frac{1}{2} \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\right) + \boldsymbol{T}_{\boldsymbol{m}}^0.
$$

Taking the ℓ^2-inner product of both sides of above equation with $\tau \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1$ and using the orthogonality of the vector triple product, we obtain

$$
\left\langle \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \right\rangle = \frac{\tau\alpha}{2} \left\langle \boldsymbol{m}_h^0 \times \big(\boldsymbol{m}_h^0 \times \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big)\right\rangle + \tau \left\langle \boldsymbol{T}_{\boldsymbol{m}}^0, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \right\rangle. \tag{5.26}
$$

Using (3.14) for the first term, we get

$$
\frac{\tau\alpha}{2} \left\langle \boldsymbol{m}_h^0 \times \big(\boldsymbol{m}_h^0 \times \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big), \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \right\rangle = -\frac{\tau\alpha}{2} \big\|\boldsymbol{m}_h^0 \times \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\|^2 \le 0.
$$

For the second term, we obtain by (3.11) that

$$
\tau \left\langle \boldsymbol{T}_{\boldsymbol{m}}^0, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \right\rangle \le C\tau \left(\big\|\boldsymbol{T}_{\boldsymbol{m}}^0\big\|_{H_h^1}^2 + \big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\|_{H_h^1}^2\right) \le + C\tau(\tau + h^2)^2 + C\tau \big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\|_{H_h^1}^2.
$$

Combining these estimates with (5.26), we obtain

$$
\left\langle \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \right\rangle \le C\tau \big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\|_{H_h^1}^2 + C\tau(\tau + h^2)^2.
$$

When $\tau \le h^{1+\epsilon_0} \le h_0^{1+\epsilon_0} \le \delta_1/(2C)$ for a positive constant $h_0 > 0$, using (4.1), we derive

$$
\big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\|_{H_h^1}^2 \le (\tau + h^2)^2.
$$

By setting $n = 0$ in (5.24) and noticing that $\boldsymbol{e}_r^0 = 0$, we multiply both sides by $\boldsymbol{e}_r^1$ and get

$$
\big|\boldsymbol{e}_r^1\big|^2 = \boldsymbol{e}_r^1 \left\langle N\big(\boldsymbol{m}_h^0\big), \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \right\rangle + \tau \boldsymbol{e}_r^1 \boldsymbol{T}_r^0.
$$

It follows by Hölder's inequality and (3.8) that

$$
\begin{aligned}
\big|\boldsymbol{e}_r^1\big|^2 &\le |\Omega|^{\frac{1}{2}} \big|\boldsymbol{e}_r^1\big| \big\|N\big(\boldsymbol{m}_h^0\big)\big\|_\infty \big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\| + \frac{1}{4}\big|\boldsymbol{e}_r^1\big|^2 + \tau^2 \big|\boldsymbol{T}_r^0\big|^2 \\
&\le \frac{1}{2}\big|\boldsymbol{e}_r^1\big|^2 + C\big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\|^2 + C\tau^2(\tau + h^2)^2,
\end{aligned}
$$

and thus,

$$|\boldsymbol{e}_r^1|^2 \leq C\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|^2 + C\tau^2(\tau+h^2)^2.$$

Therefore, there exists a positive constant C_1 such that the following inequality holds:

$$\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 + |\boldsymbol{e}_r^1|^2 \leq C_1(\tau+h^2)^2.$$

Thus, (5.25) holds for $n = 1$.

Step 2. To complete the induction, assume that (5.25) holds for $n = 1, 2, \ldots, k$. We now prove that it also holds for $n = k + 1$. Using Lemma 5.3 and the mesh ratio condition, we have the following bounds:

$$\begin{aligned}
&\|\boldsymbol{e}_{\boldsymbol{m}}^k\|^2 \leq 2\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\|^2,\\
&\|\nabla_h \boldsymbol{e}_{\boldsymbol{m}}^k\|^2 \leq \|\nabla_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\|^2 + C(\tau+h^2)^2 \leq C\|\nabla_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\|^2.
\end{aligned}$$

Therefore

$$\|\boldsymbol{e}_{\boldsymbol{m}}^k\|_{H_h^1}^2 \leq C_2\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\|_{H_h^1}^2 \leq C_2\widehat{C}(\tau+h^2)^2, \tag{5.27}$$

where C_2 is a positive constant and $C_2 > 2$. Using Lemma 3.3 and the mesh ratio condition, we have

$$\begin{aligned}
&\|\boldsymbol{e}_{\boldsymbol{m}}^k\|_\infty \leq h^{-1}\|\boldsymbol{e}_{\boldsymbol{m}}^k\| \leq Ch^{\epsilon_0} \leq \frac{1}{4},\\
&\|\nabla_h \boldsymbol{e}_{\boldsymbol{m}}^k\|_\infty \leq h^{-1}\|\nabla_h \boldsymbol{e}_{\boldsymbol{m}}^k\| \leq Ch^{\epsilon_0} \leq \frac{1}{4},
\end{aligned}$$

when $h^{\epsilon_0} \leq h_1^{\epsilon_0} \leq 1/(4C)$ for a positive constant $h_1 > 0$. By applying the triangle inequality, we obtain

$$\begin{aligned}
&\|\boldsymbol{m}_h^k\|_\infty \leq \|\boldsymbol{m}_e^k\|_\infty + \|\boldsymbol{e}_{\boldsymbol{m}}^k\|_\infty \leq L^* + \frac{1}{4},\\
&\|\nabla_h \boldsymbol{m}_h^k\|_\infty \leq \|\nabla_h \boldsymbol{m}_e^k\|_\infty + \|\nabla_h \boldsymbol{e}_{\boldsymbol{m}}^k\|_\infty \leq L^* + \frac{1}{4}.
\end{aligned}$$

Thus, in the remainder of the proof, we can apply Lemmas 5.2 and 5.5.

By setting $n = k$ in (5.23) and taking the ℓ^2-inner product on both sides of (5.23) with $2\tau\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1/2}$, we have

$$\langle \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}, \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}\rangle - \langle \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k}, \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k}\rangle = 2\tau\left\langle \sum_{i=1}^{6} A_i^k, \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\rangle.$$

We denote $I_i := \langle A_i^k, \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1/2}\rangle$ for simplicity and estimate I_i term by term below. For I_1, it follows by (5.4) that

$$I_1 = \left\langle \boldsymbol{e}_{\boldsymbol{m}}^k \times \left(\mathcal{L}_h\widetilde{\boldsymbol{m}}_e^{k+\frac{1}{2}} + N(\boldsymbol{m}_e^k)\right), \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\rangle \leq C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2\right).$$

For I_2, it follows by the orthogonality of the vector triple product, (5.4), Lemma 5.5 and (5.27) that

$$\begin{aligned}
I_2 &\leq C\left(\|g(\boldsymbol{m}_e^k, r_e^k)N(\boldsymbol{m}_e^k) - g(\boldsymbol{m}_h^k, r^k)N(\boldsymbol{m}_h^k)\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2\right)\\
&\leq C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + |\boldsymbol{e}_r^k|^2 + b_k^2\right).
\end{aligned}$$

From the definition of b_k in (5.12), and by applying the estimate (5.27) under the mesh ratio condition $\tau \le h^{1+\epsilon_0}$, we have

$$b_k^2 \le 2h^{-2}\|\boldsymbol{e}_{\boldsymbol{m}}^k\|^2\|\boldsymbol{e}_{\boldsymbol{m}}^k\|_{H_h^1}^2 \le Ch^{-2}(\tau+h^2)^4 \le C(\tau+h^2)^2, \tag{5.28}$$

provided that $\tau \le h^{1+\epsilon_0} \le h_1^{1+\epsilon_0} \le 1$ for some constant $h_1 > 0$. Then we derive

$$I_2 \le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + |\boldsymbol{e}_r^k|^2\right) + C(\tau+h^2)^2.$$

For I_3 and I_4, it follows by (3.11) and (5.7) that

$$I_3 \le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2\right),$$
$$I_4 \le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2\right).$$

By (3.14), we have

$$\alpha\left\langle \boldsymbol{m}_h^k \times \left(\boldsymbol{m}_h^k \times \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right), \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\rangle = -\alpha\|\boldsymbol{m}_h^k \times \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|^2 \le 0.$$

Hence for I_5, similar to the estimate of I_2, we apply (3.11), (5.7) and Lemma 5.5 and obtain

$$\begin{aligned}I_5 &\le C\left(\|g(\boldsymbol{m}_e^k, r_e^k)N(\boldsymbol{m}_e^k) - g(\boldsymbol{m}_h^k, r^k)N(\boldsymbol{m}_h^k)\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2\right)\\ &\le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + |\boldsymbol{e}_r^k|^2\right) + C(\tau+h^2)^2.\end{aligned}$$

For I_6, using the Hölder's inequality and (3.11), we have

$$I_6 \le C\left(\|\boldsymbol{T}_{\boldsymbol{m}}^k\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2\right) \le C\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + C(\tau+h^2)^2.$$

Putting all the estimates for I_i together, we conclude that

$$\begin{aligned}&\langle\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}, \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}\rangle - \langle\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k}, \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k}\rangle\\ &\le C\tau\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + |\boldsymbol{e}_r^k|^2\right) + C\tau(\tau+h^2)^2.\end{aligned} \tag{5.29}$$

Next, we derive an estimate of the same type for $\boldsymbol{e}_r^{k+1}$. By setting $n = k$ in (5.24), multiplying both sides by $2\boldsymbol{e}_r^{k+1}$ and using the identity

$$2(a-b)a = a^2 - b^2 + (a-b)^2, \quad \forall\, a, b \in \mathbb{R}, \tag{5.30}$$

we obtain

$$\begin{aligned}&|\boldsymbol{e}_r^{k+1}|^2 - |\boldsymbol{e}_r^k|^2 + |\boldsymbol{e}_r^{k+1} - \boldsymbol{e}_r^k|^2\\ &= 2\boldsymbol{e}_r^{k+1}\langle g(\boldsymbol{m}_e^k, r_e^k)N(\boldsymbol{m}_e^k) - g(\boldsymbol{m}_h^k, r^k)N(\boldsymbol{m}_h^k),\, \boldsymbol{m}_e^{k+1} - \boldsymbol{m}_e^k\rangle\\ &\quad + 2g(\boldsymbol{m}_h^k, r^k)\boldsymbol{e}_r^{k+1}\langle N(\boldsymbol{m}_h^k),\, \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1} - \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\rangle + 2\tau\boldsymbol{e}_r^{k+1}\boldsymbol{T}_r^k.\end{aligned} \tag{5.31}$$

For the first term on the right-hand side of (5.31), using (3.8), (5.11), and the mean value theorem, we derive

$$\begin{aligned}&|2\boldsymbol{e}_r^{k+1}\langle g(\boldsymbol{m}_e^k, r_e^k)N(\boldsymbol{m}_e^k) - g(\boldsymbol{m}_h^k, r^k)N(\boldsymbol{m}_h^k), \boldsymbol{m}_e^{k+1} - \boldsymbol{m}_e^k\rangle|\\ &\le 2|\boldsymbol{e}_r^{k+1}|\,\|g(\boldsymbol{m}_e^k, r_e^k)N(\boldsymbol{m}_e^k) - g(\boldsymbol{m}_h^k, r^k)N(\boldsymbol{m}_h^k)\|\,\|\boldsymbol{m}_e^{k+1} - \boldsymbol{m}_e^k\|\\ &\le C\tau|\boldsymbol{e}_r^{k+1}|\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\| + |\boldsymbol{e}_r^k|\right) \le C\tau\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|^2 + |\boldsymbol{e}_r^k|^2 + |\boldsymbol{e}_r^{k+1}|^2\right).\end{aligned} \tag{5.32}$$

For the second term on the right-hand side of (5.31), it follows by (5.10) and (5.23) that

$$\begin{aligned}
&\left|2\boldsymbol{e}_r^{k+1} g(\boldsymbol{m}_h^k, r^k)\left\langle N(\boldsymbol{m}_h^k), \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1} - \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\right\rangle\right| \\
&\le 2\tau\left|\boldsymbol{e}_r^{k+1} g(\boldsymbol{m}_h^k, r^k)\left\langle N(\boldsymbol{m}_h^k), \frac{\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1} - \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k}{\tau}\right\rangle\right| \\
&\le C\tau\left|\boldsymbol{e}_r^{k+1}\right|\left|\left\langle N(\boldsymbol{m}_h^k), \sum_{i=1}^{6} A_i^k\right\rangle\right|.
\end{aligned}$$

We denote $J_i := |\boldsymbol{e}_r^{k+1}|\,|\langle N(\boldsymbol{m}_h^k), A_i^k\rangle|$ for simplicity and estimate I_i term by term below. For J_1, it follows by (5.3) and (5.14) that

$$\begin{aligned}
J_1 &\le |\Omega|^{\frac{1}{2}}|\boldsymbol{e}_r^{k+1}|\|\boldsymbol{e}_{\boldsymbol{m}}^k\|\|\mathcal{L}_h\widetilde{\boldsymbol{m}}_e^{k+\frac{1}{2}} + g(\boldsymbol{m}_e^k, r_e^k)N(\boldsymbol{m}_e^k)\|_\infty\|N(\boldsymbol{m}_h^k)\|_\infty \\
&\le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|^2 + |\boldsymbol{e}_r^{k+1}|^2\right).
\end{aligned}$$

It follows by (3.12) and (5.5) that

$$\begin{aligned}
&|\boldsymbol{e}_r^{k+1}|\left|\left\langle \boldsymbol{m}_h^k \times \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}, N(\boldsymbol{m}_h^k)\right\rangle\right| \\
&= |\boldsymbol{e}_r^{k+1}|\left|\left\langle N(\boldsymbol{m}_h^k)\times\boldsymbol{m}_h^k, \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\rangle\right| \\
&\le C|\boldsymbol{e}_r^{k+1}|\left(\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\| + \|\nabla_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|\right) \\
&\le C\left(\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + |\boldsymbol{e}_r^{k+1}|^2\right).
\end{aligned}$$

Then, for J_2, applying (5.3) and the bound in (5.11), we obtain

$$\begin{aligned}
J_2 &\le |\Omega|^{\frac{1}{2}}|\boldsymbol{e}_r^{k+1}|\|\boldsymbol{m}_h^k\|_\infty\,\|g(\boldsymbol{m}_e^k, r_e^k)N(\boldsymbol{m}_e^k) - g(\boldsymbol{m}_h^k, r^k)N(\boldsymbol{m}_h^k)\|\,\|N(\boldsymbol{m}_h^k)\|_\infty \\
&\quad + C\left(\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + |\boldsymbol{e}_r^{k+1}|^2\right) \\
&\le C|\boldsymbol{e}_r^{k+1}|\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\| + |\boldsymbol{e}_r^k|\right) + C\left(\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + |\boldsymbol{e}_r^{k+1}|^2\right) \\
&\le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + |\boldsymbol{e}_r^k|^2 + |\boldsymbol{e}_r^{k+1}|^2\right).
\end{aligned}$$

For J_3 and J_4, it follows by (3.10) and (5.3) that

$$\begin{aligned}
J_3, J_4 &\le \alpha|\Omega|^{\frac{1}{2}}|\boldsymbol{e}_r^{k+1}|\|N(\boldsymbol{m}_h^k)\|_\infty\|\boldsymbol{e}_{\boldsymbol{m}}^k\|\|\boldsymbol{m}_e^k\|_\infty\|\mathcal{L}_h\widetilde{\boldsymbol{m}}_e^{k+\frac{1}{2}} + g(\boldsymbol{m}_e^k, r_e^k)N(\boldsymbol{m}_e^k)\|_\infty \\
&\le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|^2 + |\boldsymbol{e}_r^{k+1}|^2\right).
\end{aligned}$$

It follows by (3.13) and (5.6) that

$$\begin{aligned}
&|\boldsymbol{e}_r^{k+1}|\left|\left\langle \boldsymbol{m}_h^k \times \left(\boldsymbol{m}_h^k \times \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right), N\left(\boldsymbol{m}_h^k\right)\right\rangle\right| \\
&= |\boldsymbol{e}_r^{k+1}|\left|\left\langle \boldsymbol{m}_h^k \times \left(\boldsymbol{m}_h^k \times N(\boldsymbol{m}_h^k)\right), \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\rangle\right| \\
&\le C|\boldsymbol{e}_r^{k+1}|\left(\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\| + \|\nabla_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|\right) \le C\left(\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + |\boldsymbol{e}_r^{k+1}|^2\right).
\end{aligned}$$

Applying the same type of estimate as for J_2, we derive the following result for J_5:

$$J_5 \le C\left(\|\boldsymbol{e}_{\boldsymbol{m}}^k\|^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + |\boldsymbol{e}_r^k|^2 + |\boldsymbol{e}_r^{k+1}|^2\right).$$

For J_6, the Hölder's inequality and (3.8) yield

$$J_6 \le C\left|\boldsymbol{e}_r^{k+1}\right|^2 + C(\tau+h^2)^2.$$

Putting all the estimates for J_i together, we conclude that

$$\begin{aligned}&2\boldsymbol{e}_r^{k+1}\left\langle g(\boldsymbol{m}_h^k, r^k)N(\boldsymbol{m}_h^k), \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1} - \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\right\rangle \\ &\le C\tau\left(\left\|\boldsymbol{e}_{\boldsymbol{m}}^k\right\|^2 + \left\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2 + \left|\boldsymbol{e}_r^k\right|^2 + \left|\boldsymbol{e}_r^{k+1}\right|^2\right) + C\tau(\tau+h^2)^2. \end{aligned} \tag{5.33}$$

For the third term on the right-hand side of (5.31), we have

$$\left|2\tau\boldsymbol{e}_r^{k+1}\boldsymbol{T}_r^k\right| \le \tau\left|\boldsymbol{e}_r^{k+1}\right|^2 + C\tau(\tau+h^2)^2. \tag{5.34}$$

Putting (5.33), (5.32) and (5.34) into (5.31), we obtain

$$\left|\boldsymbol{e}_r^{k+1}\right|^2 - \left|\boldsymbol{e}_r^k\right|^2 \le C\tau\left(\left\|\boldsymbol{e}_{\boldsymbol{m}}^k\right\|^2 + \left\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2 + \left|\boldsymbol{e}_r^k\right|^2 + \left|\boldsymbol{e}_r^{k+1}\right|^2\right) + C\tau(\tau+h^2)^2. \tag{5.35}$$

In order to obtain the estimates for error functions $\boldsymbol{e}_{\boldsymbol{m}}^{k+1}$ and $\boldsymbol{e}_r^{k+1}$, we define

$$Z^k = \langle\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k, \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\rangle + \left|\boldsymbol{e}_r^k\right|^2.$$

Since

$$\begin{aligned}\left\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2 &= \left\|\frac{1}{2}\left(\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1} + \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\right)\right\|^2 + \left\|\frac{1}{2}\left(\nabla_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1} + \nabla_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\right)\right\|^2 \\ &\le \left\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}\right\|_{H_h^1}^2 + \left\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\right\|_{H_h^1}^2,\end{aligned} \tag{5.36}$$

it follows by (5.29), (5.35) and (5.27) that

$$Z^{k+1} - Z^k \le C\tau\left(\left\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\right\|_{H_h^1}^2 + \left\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}\right\|_{H_h^1}^2 + \left|\boldsymbol{e}_r^k\right|^2 + \left|\boldsymbol{e}_r^{k+1}\right|^2\right) + C\tau(\tau+h^2)^2.$$

It follows by (4.1) that

$$Z^k \ge \delta_3\left(\left\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k\right\|_{H_h^1}^2 + \left|\boldsymbol{e}_r^k\right|^2\right), \tag{5.37}$$

where $\delta_3 = \min\{\delta_1, 1\}$. Thus,

$$Z^{k+1} - Z^k \le C\tau(Z^{k+1} + Z^k) + C\tau(\tau+h^2)^2. \tag{5.38}$$

When $\tau \le h^{1+\epsilon_0} \le h_2^{1+\epsilon_0} \le 1/(2C)$ for a positive constant h_2, noting that

$$\frac{1+C\tau}{1-C\tau} \le 1 + 4C\tau, \quad \frac{1}{1-C\tau} \le 2,$$

we obtain

$$Z^{k+1} \le (1+4C\tau)Z^k + 2C\tau(\tau+h^2)^2.$$

It follows by Lemma 3.4 that

$$Z^{k+1} \le \exp(4CT)\left[Z^1 + \frac{1}{2}(\tau+h^2)^2\right] \le \exp(4CT)\left(C_1 + \frac{1}{2}\right)(\tau+h^2)^2.$$

Then, by (5.37), we obtain

$$\left\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}\right\|_{H_h^1}^2 + \left|\boldsymbol{e}_r^{k+1}\right|^2 \le \frac{\exp(4CT)(C_1+1/2)}{\delta_3}(\tau+h^2)^2.$$

Thus, by mathematical induction, (5.25) holds for all $1 \le n \le N$, provided

$$\widehat{C} = \max\left\{C_1, \frac{\exp(4CT)(C_1 + 1/2)}{\delta_3}\right\}.$$

Finally, it follows by (5.25) and (5.27) that

$$\left\|\boldsymbol{e}_{\boldsymbol{m}}^n\right\|_{H_h^1}^2 + \left|\boldsymbol{e}_r^n\right|^2 \le C_2\left(\left\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\right\|_{H_h^1}^2 + \left|\boldsymbol{e}_r^n\right|^2\right) \le \widehat{C}C_2(\tau + h^2)^2.$$

This completes the proof of Theorem 3.3 for $C^* = \widehat{C}C_2$ and $h^* = \min_{0\le i\le 2} h_i$.

5.3. Proof of Theorem 3.4

As the first step for proving Theorem 3.4, we establish the following estimate for the error functions of the intermediate layer $\widehat{\boldsymbol{m}}^{n+1/2}$ and $\widehat{r}^{n+1/2}$.

Lemma 5.7. *Let $\lambda > 2\kappa^2 + \max\{2A_1, 0\}$. Assume $\boldsymbol{m}_h^n$ satisfies*

$$\left\|\boldsymbol{m}_h^n\right\|_\infty, \left\|\nabla_h \boldsymbol{m}_h^n\right\|_\infty \le L^* + \frac{1}{4}. \tag{5.39}$$

Then, there exists a positive constant τ_0 such that, when $\tau \le \tau_0$, the following error estimate holds:

$$\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\right\|_{H_h^1}^2 + \left|\widehat{\boldsymbol{e}}_r^{n+\frac{1}{2}}\right|^2 \le C_0^*\left(\left\|\boldsymbol{e}_{\boldsymbol{m}}^n\right\|_{H_h^1}^2 + \left\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\right\|_{H_h^1}^2 + \left|\boldsymbol{e}_r^n\right|^2 + b_n^2\right) + C_0^*\tau^2(\tau + h^2)^2,$$

where $C_0^* > 0$ is independent of h, τ, and τ_0.

Proof. It is easy to verify that the exact solution $\boldsymbol{m}_e$ of (2.3a), along with r_e, the exact solution of (2.3b), satisfies

$$\begin{aligned} \frac{\boldsymbol{m}_e^{n+1/2} - \boldsymbol{m}_e^n}{\tau/2} &= \boldsymbol{m}_e^n \times \left(\mathcal{L}_h \boldsymbol{m}_e^{n+\frac{1}{2}} + g\left(\boldsymbol{m}_e^n, r_e^n\right) N\left(\boldsymbol{m}_e^n\right)\right) \\ &\quad + \alpha\, \boldsymbol{m}_e^n \times \left(\boldsymbol{m}_e^n \times \left(\mathcal{L}_h \boldsymbol{m}_e^{n+\frac{1}{2}} + g\left(\boldsymbol{m}_e^n, r_e^n\right) N\left(\boldsymbol{m}_e^n\right)\right)\right) + \boldsymbol{R}_{\boldsymbol{m}}^n, \end{aligned} \tag{5.40}$$

$$\frac{r_e^{n+1/2} - r_e^n}{\tau/2} = g\left(\boldsymbol{m}_e^n, r_e^n\right)\left\langle N\left(\boldsymbol{m}_e^n\right), \frac{\boldsymbol{m}_e^{n+1/2} - \boldsymbol{m}_e^n}{\tau/2}\right\rangle + \boldsymbol{R}_r^n. \tag{5.41}$$

where the truncation errors $\boldsymbol{R}_{\boldsymbol{m}}^n$ and $\boldsymbol{R}_r^n$ satisfy

$$\left\|\boldsymbol{R}_{\boldsymbol{m}}^n\right\| \le C(\tau + h^2), \quad \left|\boldsymbol{R}_r^n\right| \le C(\tau + h^2).$$

Similar to the analysis for $\boldsymbol{T}_{\boldsymbol{m}}^n$ in (5.21), we have

$$\left\|\nabla_h \boldsymbol{R}_{\boldsymbol{m}}^n\right\| \le C(\tau + h^2).$$

Subtracting (3.4a) from (5.40) and (3.4b) from (5.41) yields the following error equations:

$$\begin{aligned}\frac{\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1/2}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{n}}{\tau/2} &= \boldsymbol{e}_{\boldsymbol{m}}^{n}\times\Big(\mathcal{L}_h\boldsymbol{m}_e^{n+\frac{1}{2}}+N\big(\boldsymbol{m}_e^n\big)\Big)\\ &\quad+\boldsymbol{m}_h^n\times\Big(\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}+g\big(\boldsymbol{m}_e^n,r_e^n\big)N\big(\boldsymbol{m}_e^n\big)-g\big(\boldsymbol{m}_h^n,r^n\big)N\big(\boldsymbol{m}_h^n\big)\Big)\\ &\quad+\alpha\,\boldsymbol{m}_h^n\times\Big(\boldsymbol{e}_{\boldsymbol{m}}^n\times\Big(\mathcal{L}_h\boldsymbol{m}_e^{n+\frac{1}{2}}+N\big(\boldsymbol{m}_e^n\big)\Big)\Big)\\ &\quad+\alpha\,\boldsymbol{e}_{\boldsymbol{m}}^n\times\Big(\boldsymbol{m}_e^n\times\Big(\mathcal{L}_h\boldsymbol{m}_e^{n+\frac{1}{2}}+N\big(\boldsymbol{m}_e^n\big)\Big)\Big)\\ &\quad+\alpha\,\boldsymbol{m}_h^n\times\Big(\boldsymbol{m}_h^n\times\Big(\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}+g\big(\boldsymbol{m}_e^n,r_e^n\big)N\big(\boldsymbol{m}_e^n\big)-g\big(\boldsymbol{m}_h^n,r^n\big)N\big(\boldsymbol{m}_h^n\big)\Big)\Big)+\boldsymbol{R}_{\boldsymbol{m}}^n\\ &=:\sum_{i=1}^{6}B_i,\end{aligned} \tag{5.42}$$

$$\begin{aligned}\widehat{e}_r^{n+\frac{1}{2}}-e_r^n &= \Big\langle g\big(\boldsymbol{m}_e^n,r_e^n\big)N\big(\boldsymbol{m}_e^n\big)-g\big(\boldsymbol{m}_h^n,r^n\big)N\big(\boldsymbol{m}_h^n\big),\,\boldsymbol{m}_e^{n+\frac{1}{2}}-\boldsymbol{m}_e^n\Big\rangle\\ &\quad+g\big(\boldsymbol{m}_h^n,r^n\big)\Big\langle N\big(\boldsymbol{m}_h^n\big),\,\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\Big\rangle+\frac{\tau}{2}\boldsymbol{R}_r^n.\end{aligned} \tag{5.43}$$

We first consider the estimate of $\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1/2}$. Taking the ℓ^2-inner product on both sides of (5.42) with $\tau\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1/2}$ yield

$$2\big\langle\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\rangle=\tau\left\langle\sum_{i=1}^{6}B_i,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\right\rangle. \tag{5.44}$$

For the left-hand side of (5.44), it follows by Lemma 4.1 that

$$\begin{aligned}&2\big\langle\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\rangle\\ &=\big\langle\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}},\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\rangle-\big\langle\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n,\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\big\rangle\\ &\qquad+\Big\langle\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n,\mathcal{L}_h\big(\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\big)\Big\rangle\\ &\geq\big\langle\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}},\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\rangle-\big\langle\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n,\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\big\rangle.\end{aligned} \tag{5.45}$$

By virtue of (5.39), Lemmas 5.2 and 5.5 can be directly applied. Consequently, we proceed to estimate the terms on the right-hand side of (5.44). We begin with the term $\langle B_1,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1/2}\rangle$, for which it follows from (5.4) that

$$\big\langle B_1,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\rangle=\Big\langle\boldsymbol{e}_{\boldsymbol{m}}^n\times\big(\mathcal{L}_h\boldsymbol{m}_e^{n+\frac{1}{2}}+N\big(\boldsymbol{m}_e^n\big)\big),\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\Big\rangle\leq C\left(\big\|\boldsymbol{e}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2+\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\|_{H_h^1}^2\right).$$

For $\langle B_2,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1/2}\rangle$, it follows by the orthogonality of the vector triple product, (5.4), Lemma 5.5 and (5.27) that

$$\begin{aligned}\big\langle B_2,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\rangle&\leq C\left(\big\|g\big(\boldsymbol{m}_e^n,r_e^n\big)N\big(\boldsymbol{m}_e^n\big)-g\big(\boldsymbol{m}_h^n,r^n\big)N\big(\boldsymbol{m}_h^n\big)\big\|_{H_h^1}^2+\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\|_{H_h^1}^2\right)\\ &\leq C\left(\big\|\boldsymbol{e}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2+\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\|_{H_h^1}^2+\big|e_r^n\big|^2\right)+Cb_n^2.\end{aligned}$$

For $\langle B_3,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1/2}\rangle$ and $\langle B_4,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1/2}\rangle$, it follows by (3.11) and (5.7) that

$$\begin{aligned}\big\langle B_3,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\rangle&\leq C\left(\big\|\boldsymbol{e}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2+\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\|_{H_h^1}^2\right),\\ \big\langle B_4,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\rangle&\leq C\left(\big\|\boldsymbol{e}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2+\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\|_{H_h^1}^2\right).\end{aligned}$$

By (3.14), we have

$$\alpha\left\langle \boldsymbol{m}_h^n\times\big(\boldsymbol{m}_h^n\times\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big),\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\right\rangle=-\alpha\big\|\boldsymbol{m}_h^n\times\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\|^2\le 0.$$

We apply (3.11), (5.7) and Lemma 5.5 and obtain

$$\begin{aligned}\big\langle B_5,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\rangle&\le C\left(\big\|g\big(\boldsymbol{m}_e^n,r_e^n\big)N\big(\boldsymbol{m}_e^n\big)-g\big(\boldsymbol{m}_h^n,r^n\big)N\big(\boldsymbol{m}_h^n\big)\big\|_{H_h^1}^2+\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\|_{H_h^1}^2\right)\\&\le C\left(\big\|\boldsymbol{e}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2+\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\|_{H_h^1}^2+\big|\boldsymbol{e}_r^n\big|^2\right)+Cb_n^2.\end{aligned}$$

For $\tau\langle B_6,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1/2}\rangle$, using (3.7) and (3.8), we have

$$\begin{aligned}\tau\big\langle B_6,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\rangle&=\big\langle\tau\nabla_h\boldsymbol{R}_{\boldsymbol{m}}^n,\nabla_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\rangle+2\kappa\big\langle\tau\boldsymbol{R}_{\boldsymbol{m}}^n,\nabla_h\times\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\rangle\\&\quad+\big\langle\tau\boldsymbol{R}_{\boldsymbol{m}}^n,\lambda\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}-2A_1(\hat{\boldsymbol{e}}_3\otimes\hat{\boldsymbol{e}}_3)\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\rangle\\&\le\frac{\delta_1}{4}\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\|_{H_h^1}^2+C\tau^2\big\|\boldsymbol{R}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2\\&\le\frac{\delta_1}{4}\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\|_{H_h^1}^2+C\tau^2(\tau+h^2)^2.\end{aligned}$$

Combining all these estimates for the terms the right-hand side of (5.44), we conclude that

$$\begin{aligned}\tau\left\langle\sum_{i=1}^6 B_i,\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\right\rangle&\le C\tau\left(\big\|\boldsymbol{e}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2+\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\|_{H_h^1}^2+\big|\boldsymbol{e}_r^n\big|^2+b_n^2\right)\\&\quad+\frac{\delta_1}{4}\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\|_{H_h^1}^2+C\tau^2(\tau+h^2)^2.\end{aligned}\tag{5.46}$$

Substituting (5.45) and (5.46) into (5.44), we obtain

$$\begin{aligned}&\big\langle\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12},\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\rangle-\frac{\delta_1}{2}\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\|_{H_h^1}^2\\&\le\big\langle\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n,\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\big\rangle+C\left(\big\|\boldsymbol{e}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2+\big|\boldsymbol{e}_r^n\big|^2+b_n^2\right)+C\tau^2(\tau+h^2)^2,\end{aligned}$$

provided $\tau\le\tau_1\le\delta_1/(4C)$ for a positive constant $\tau_1>0$. According to Lemma 4.1, we have

$$\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\|_{H_h^1}^2\le C\left(\big\|\boldsymbol{e}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2+\big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2+\big|\boldsymbol{e}_r^n\big|^2+b_n^2\right)+C\tau^2(\tau+h^2)^2.\tag{5.47}$$

Next, we consider the error function $\widehat{\boldsymbol{e}}_r^{n+1/2}$. Multiplying (5.43) by $2\widehat{\boldsymbol{e}}_r^{n+\frac12}$, and using (5.30), we have

$$\begin{aligned}&\big|\widehat{\boldsymbol{e}}_r^{n+\frac12}\big|^2-\big|\boldsymbol{e}_r^n\big|^2+\big|\widehat{\boldsymbol{e}}_r^{n+\frac12}-\boldsymbol{e}_r^n\big|^2\\&=2\widehat{\boldsymbol{e}}_r^{n+\frac12}\left\langle g\big(\boldsymbol{m}_e^n,r_e^n\big)N\big(\boldsymbol{m}_e^n\big)-g\big(\boldsymbol{m}_h^n,r^n\big)N\big(\boldsymbol{m}_h^n\big),\ \boldsymbol{m}_e^{n+\frac12}-\boldsymbol{m}_e^n\right\rangle\\&\quad+2\widehat{\boldsymbol{e}}_r^{n+\frac12}g\big(\boldsymbol{m}_h^n,r^n\big)\left\langle N\big(\boldsymbol{m}_h^n\big),\ \widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\right\rangle+\tau\widehat{\boldsymbol{e}}_r^{n+\frac12}\boldsymbol{R}_r^n.\end{aligned}$$

Using the Young's inequality (3.8), (5.11) and the mean value theorem, the first term on the right-hand side of above equation can be estimated as

$$\begin{aligned}&2\widehat{\boldsymbol{e}}_r^{n+\frac12}\left\langle g\big(\boldsymbol{m}_e^n,r_e^n\big)N\big(\boldsymbol{m}_e^n\big)-g\big(\boldsymbol{m}_h^n,r^n\big)N\big(\boldsymbol{m}_h^n\big),\ \boldsymbol{m}_e^{n+\frac12}-\boldsymbol{m}_e^n\right\rangle\\&\le 2\big|\widehat{\boldsymbol{e}}_r^{n+\frac12}\big|\,\big\|g\big(\boldsymbol{m}_e^n,r_e^n\big)N\big(\boldsymbol{m}_e^n\big)-g\big(\boldsymbol{m}_h^n,r^n\big)N\big(\boldsymbol{m}_h^n\big)\big\|\,\big\|\boldsymbol{m}_e^{n+\frac12}-\boldsymbol{m}_e^n\big\|\\&\le C\tau\big|\widehat{\boldsymbol{e}}_r^{n+\frac12}\big|\,\big(\big\|\boldsymbol{e}_{\boldsymbol{m}}^n\big\|+\big|\boldsymbol{e}_r^n\big|\big)\\&\le C\tau\left(\big\|\boldsymbol{e}_{\boldsymbol{m}}^n\big\|^2+\big|\boldsymbol{e}_r^n\big|^2+\big|\widehat{\boldsymbol{e}}_r^{n+\frac12}\big|^2\right).\end{aligned}$$

By applying (5.10) and (5.14), the second term can be estimated as follows:

$$\begin{aligned}
&2\widehat{e}_r^{n+\frac{1}{2}} g(\boldsymbol{m}_h^n, r^n)\langle N(\boldsymbol{m}_h^n),\, \widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}} - \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\rangle \\
\le\,& 2|\Omega|^{\frac{1}{2}} g(\boldsymbol{m}_h^n, r^n)\big|\widehat{e}_r^{n+\frac{1}{2}}\big|\big\|N(\boldsymbol{m}_h^n)\big\|_\infty \|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}} - \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\| \\
\le\,& \frac{1}{4}\big|\widehat{e}_r^{n+\frac{1}{2}}\big|^2 + C\left(\big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\big\|^2 + \big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\|^2\right).
\end{aligned}$$

With (3.8), the third term can be estimated as

$$\tau \widehat{e}_r^{n+\frac{1}{2}} \boldsymbol{R}_r^n \le \frac{1}{4}\big|\widehat{e}_r^{n+\frac{1}{2}}\big|^2 + C\tau^2(\tau + h^2)^2.$$

Altogether, when $\tau \le \tau_2 \le 1/(4C)$ for a positive constant τ_2, we have

$$\frac{1}{4}\big|\widehat{e}_r^{n+\frac{1}{2}}\big|^2 \le C\left(\big\|\boldsymbol{e}_{\boldsymbol{m}}^n\big\|^2 + \big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\big\|^2 + \big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\|^2 + \big|e_r^n\big|^2\right) + C\tau^2(\tau + h^2)^2. \tag{5.48}$$

Thus, the claim follows by combining (5.47) with (5.48), where $\tau_0 = \min_{1\le i\le 2}\tau_i$. □

Based on the above estimate for the error functions of the intermediate layer, we now consider the error functions $\boldsymbol{e}_{\boldsymbol{m}}^n$ and $\boldsymbol{e}_r^n$. It is easy to verify that the exact solution $\boldsymbol{m}_e$ of (2.3a) and the exact solution r_e of (2.3b) satisfy

$$\begin{aligned}
\frac{\boldsymbol{m}_e^{n+1} - \boldsymbol{m}_e^n}{\tau} =\,& \boldsymbol{m}_e^{n+\frac{1}{2}} \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_e^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_e^{n+\frac{1}{2}}, r_e^{n+\frac{1}{2}}\big)N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big)\right) \qquad (5.49)\\
&+ \alpha\, \boldsymbol{m}_e^{n+\frac{1}{2}} \times \left(\boldsymbol{m}_e^{n+\frac{1}{2}} \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_e^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_e^{n+\frac{1}{2}}, r_e^{n+\frac{1}{2}}\big)N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big)\right)\right) + \boldsymbol{R}_{\boldsymbol{m}}^{n+\frac{1}{2}},
\end{aligned}$$

$$\frac{r_e^{n+1} - r_e^n}{\tau} = g\big(\boldsymbol{m}_e^{n+\frac{1}{2}}, r_e^{n+\frac{1}{2}}\big)\left\langle N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big),\, \frac{\boldsymbol{m}_e^{n+1} - \boldsymbol{m}_e^n}{\tau}\right\rangle + \boldsymbol{R}_r^{n+\frac{1}{2}}, \tag{5.50}$$

where the truncation errors $\boldsymbol{R}_{\boldsymbol{m}}^{n+1/2}$ and $\boldsymbol{R}_r^{n+\frac{1}{2}}$ satisfy

$$\big\|\boldsymbol{R}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\| \le C(\tau^2 + h^2), \quad \big|\boldsymbol{R}_r^{n+\frac{1}{2}}\big| \le C(\tau^2 + h^2).$$

Similar to the analysis for $\boldsymbol{T}_{\boldsymbol{m}}^n$ in (5.21), we have

$$\big\|\nabla_h \boldsymbol{R}_{\boldsymbol{m}}^{n+\frac{1}{2}}\big\| \le C(\tau^2 + h^2).$$

Subtracting (3.5a) from (5.49) and (3.5b) from (5.50) yields the error equations

$$\begin{aligned}
\frac{\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1} - \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n}{\tau} =\,& \widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}} \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_e^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_e^{n+\frac{1}{2}}, r_e^{n+\frac{1}{2}}\big)N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big)\right) \\
&+ \widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}} \times \left(\mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_e^{n+\frac{1}{2}}, r_e^{n+\frac{1}{2}}\big)N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big) - g\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}, \widehat{r}^{n+\frac{1}{2}}\big)N\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\big)\right) \\
&+ \alpha\, \widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}} \times \left(\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}} \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_e^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_e^{n+\frac{1}{2}}, r_e^{n+\frac{1}{2}}\big)N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big)\right)\right) \\
&+ \alpha\, \widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}} \times \left(\boldsymbol{m}_e^{n+\frac{1}{2}} \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_e^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_e^{n+\frac{1}{2}}, r_e^{n+\frac{1}{2}}\big)N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big)\right)\right) \\
&+ \alpha\, \widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}} \times \left(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}} \times \left(\mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac{1}{2}} + g\big(\boldsymbol{m}_e^{n+\frac{1}{2}}, r_e^{n+\frac{1}{2}}\big)N\big(\boldsymbol{m}_e^{n+\frac{1}{2}}\big)\right.\right. \\
&\left.\left. - g\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}, \widehat{r}^{n+\frac{1}{2}}\big)N\big(\widehat{\boldsymbol{m}}_h^{n+\frac{1}{2}}\big)\right)\right) + \boldsymbol{R}_{\boldsymbol{m}}^{n+\frac{1}{2}} =: \sum_{i=1}^{6} \widetilde{B}_i^n, \qquad (5.51)
\end{aligned}$$

$$\frac{\boldsymbol{e}_r^{n+1}-\boldsymbol{e}_r^n}{\tau}=\left\langle g\big(\boldsymbol{m}_e^{n+\frac12},r_e^{n+\frac12}\big)N\big(\boldsymbol{m}_e^{n+\frac12}\big)-g\big(\widehat{\boldsymbol{m}}_h^{n+\frac12},\widehat{r}^{n+\frac12}\big)N\big(\widehat{\boldsymbol{m}}_h^{n+\frac12}\big),\ \frac{\boldsymbol{m}_e^{n+1}-\boldsymbol{m}_e^n}{\tau}\right\rangle$$
$$+\,g\big(\widehat{\boldsymbol{m}}_h^{n+\frac12},\widehat{r}^{n+\frac12}\big)\left\langle N\big(\widehat{\boldsymbol{m}}_h^{n+\frac12}\big),\ \frac{\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{n+1}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n}{\tau}\right\rangle+\boldsymbol{R}_r^{n+\frac12}. \tag{5.52}$$

We first establish the primary error estimate

$$\big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n\big\|_{H_h^1}^2+\big|\boldsymbol{e}_r^n\big|^2\le\widehat{C}_0(\tau^2+h^2)^2,\quad 1\le n\le N, \tag{5.53}$$

where $\widehat{C}_0$ is a positive constant independent of τ and h. The proof is carried out by mathematical induction and consists of the following two steps:

Step 1. We prove that (5.53) holds for $n=1$. Since $\boldsymbol{m}_h^0=\boldsymbol{m}_0$, we have

$$\big\|\boldsymbol{m}_h^0\big\|_\infty,\big\|\nabla_h\boldsymbol{m}_h^0\big\|_\infty\le L^*.$$

Applying Lemma 5.7 with $\boldsymbol{e}_{\boldsymbol{m}}^0=\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^0=\boldsymbol{0}$ and $\boldsymbol{e}_r^0=0$, we obtain

$$\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\big\|_{H_h^1}^2+\big|\widehat{\boldsymbol{e}}_r^{\frac12}\big|^2\le C_0^*\tau^2(\tau+h^2)^2. \tag{5.54}$$

It follows by Lemma 3.3 and the mesh ratio condition that

$$\begin{aligned}\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\big\|_\infty&\le h^{-1}\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\big\|\le C\tau\le\frac14,\\ \big\|\nabla_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\big\|_\infty&\le h^{-1}\big\|\nabla_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\big\|\le C\tau\le\frac14,\end{aligned}$$

when $\tau\le\tau_3\le1/(4C)$ for a positive constant $\tau_3>0$. Thus, by the triangle inequality, we get

$$\begin{aligned}\big\|\widehat{\boldsymbol{m}}_h^{\frac12}\big\|_\infty&\le\big\|\widehat{\boldsymbol{m}}_e^{\frac12}\big\|_\infty+\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\big\|_\infty\le L^*+\frac14,\\ \big\|\nabla_h\widehat{\boldsymbol{m}}_h^{\frac12}\big\|_\infty&\le\big\|\nabla_h\widehat{\boldsymbol{m}}_e^{\frac12}\big\|_\infty+\big\|\nabla_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\big\|_\infty\le L^*+\frac14.\end{aligned}$$

Therefore, we can apply Lemmas 5.2 and 5.6 directly. Setting $n=0$ in (5.51) and taking the ℓ^2-inner product of both sides of (5.51) with $\tau\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1$ yield

$$\big\langle\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1,\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\rangle=\tau\left\langle\sum_{i=1}^6\widetilde{B}_i^0,\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\right\rangle. \tag{5.55}$$

We now estimate the right-hand side of above equation term by term. For $\langle\widetilde{B}_1^0,\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\rangle$, using (5.4), we have

$$\big\langle\widetilde{B}_1^0,\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\rangle\le C\left(\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\big\|_{H_h^1}^2+\big\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\|_{H_h^1}^2\right).$$

Due to $\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^0=\boldsymbol{0}$, using the orthogonality of the vector triple product, we have

$$\big\langle\widehat{\boldsymbol{m}}_h^{\frac12}\times\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12},\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\rangle=\frac12\big\langle\widehat{\boldsymbol{m}}_h^{\frac12}\times\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1,\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\big\rangle=0.$$

Recall that $\widehat{b}_{1/2}$ is defined in (5.20). By applying the estimate (5.54) under the mesh ratio condition $h=O(\tau)$, we obtain

$$\widehat{b}_{1/2}^2\le2h^{-2}\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\|^2\big\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{n+\frac12}\big\|_{H_h^1}^2\le Ch^{-2}(\tau^2+h^2)^4\le C(\tau^2+h^2)^2, \tag{5.56}$$

provided that $\tau \le 1$. Then it follows by (5.4) and Lemma 5.6 that

$$\begin{aligned}\langle \widetilde{B}_2^0,\, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \rangle &\le C \left(\left\| g(\boldsymbol{m}_e^{\frac12}, r_e^{\frac12}) N(\boldsymbol{m}_e^{\frac12}) - g(\widehat{\boldsymbol{m}}_h^{\frac12}, \widehat{r}^{\frac12}) N(\widehat{\boldsymbol{m}}_h^{\frac12}) \right\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 \right) \\ &\le C \left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 + |\widehat{\boldsymbol{e}}_r^{\frac12}|^2 \right) + C(\tau^2 + h^2)^2.\end{aligned}$$

Applying (3.11) and (5.7), we obtain the following bounds:

$$\begin{aligned}\langle \widetilde{B}_3^0, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \rangle &\le C \left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 \right), \\ \langle \widetilde{B}_4^0, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \rangle &\le C \left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 \right).\end{aligned}$$

It follows by (3.14) that

$$\left\langle \widehat{\boldsymbol{m}}_h^{\frac12} \times (\widehat{\boldsymbol{m}}_h^{\frac12} \times \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}), \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \right\rangle = -\frac{1}{2} \|\widehat{\boldsymbol{m}}_h^{\frac12} \times \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|^2 \le 0.$$

Thus, using (3.11), (5.7) and Lemma 5.6, we derive

$$\begin{aligned}\langle \widetilde{B}_5^0, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \rangle &\le C \left(\left\| g(\boldsymbol{m}_e^{\frac12}, r_e^{\frac12}) N(\boldsymbol{m}_e^{\frac12}) - g(\widehat{\boldsymbol{m}}_h^{\frac12}, \widehat{r}^{\frac12}) N(\widehat{\boldsymbol{m}}_h^{\frac12}) \right\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 \right) \\ &\le C \left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 + |\widehat{\boldsymbol{e}}_r^{\frac12}|^2 \right) + C(\tau^2 + h^2)^2.\end{aligned}$$

For the last term, it follows by (3.11) that

$$\langle \widetilde{B}_6^0, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \rangle \le C \left(\|\boldsymbol{R}_{\boldsymbol{m}}^{\frac12}\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 \right) \le C \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 + C(\tau^2 + h^2)^2.$$

Substituting all the estimates of $\langle \widetilde{B}_i^0, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \rangle$ for $i = 1, \dots, 6$ into (5.55), we arrive at

$$\langle \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \rangle \le C\tau \left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\|_{H_h^1}^2 + \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 + |\widehat{\boldsymbol{e}}_r^{\frac12}|^2 \right) + C\tau(\tau^2 + h^2)^2.$$

From (4.1) and (5.54), we obtain

$$\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 \le C(\tau^2 + h^2)^2 \tag{5.57}$$

under the condition $\tau \le \tau_4 \le \delta_1/(2C)$ for a positive constant $\tau_4 > 0$.

For the error function $\boldsymbol{e}_r^1$, we take $n = 0$, multiply (5.52) by $\boldsymbol{e}_r^1$ and obtain

$$\begin{aligned}|\boldsymbol{e}_r^1|^2 = \boldsymbol{e}_r^1 &\left\langle g(\boldsymbol{m}_e^{\frac12}, r_e^{\frac12}) N(\boldsymbol{m}_e^{\frac12}) - g(\widehat{\boldsymbol{m}}_h^{\frac12}, \widehat{r}^{\frac12}) N(\widehat{\boldsymbol{m}}_h^{\frac12}),\, \boldsymbol{m}_e^1 - \boldsymbol{m}_e^0 \right\rangle \\ &+ \boldsymbol{e}_r^1 \left\langle g(\widehat{\boldsymbol{m}}_h^{\frac12}, \widehat{r}^{\frac12}) N(\widehat{\boldsymbol{m}}_h^{\frac12}),\, \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1 \right\rangle + \tau\, \boldsymbol{e}_r^1 \boldsymbol{R}_r^{\frac12}.\end{aligned} \tag{5.58}$$

Using (3.8), (5.19) and the mean value theorem, the first term on the right-hand side of (5.58) can be estimated as

$$\begin{aligned}&\boldsymbol{e}_r^1 \left\langle g(\boldsymbol{m}_e^{\frac12}, r_e^{\frac12}) N(\boldsymbol{m}_e^{\frac12}) - g(\widehat{\boldsymbol{m}}_h^{\frac12}, \widehat{r}^{\frac12}) N(\widehat{\boldsymbol{m}}_h^{\frac12}),\, \boldsymbol{m}_e^1 - \boldsymbol{m}_e^0 \right\rangle \\ &\le |\boldsymbol{e}_r^1| \left\| g(\boldsymbol{m}_e^{\frac12}, r_e^{\frac12}) N(\boldsymbol{m}_e^{\frac12}) - g(\widehat{\boldsymbol{m}}_h^{\frac12}, \widehat{r}^{\frac12}) N(\widehat{\boldsymbol{m}}_h^{\frac12}) \right\| \|\boldsymbol{m}_e^1 - \boldsymbol{m}_e^0\| \\ &\le C\tau\, |\boldsymbol{e}_r^1| \left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\| + |\widehat{\boldsymbol{e}}_r^{\frac12}| \right) \le C\tau \left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac12}\|^2 + |\widehat{\boldsymbol{e}}_r^{\frac12}|^2 + |\boldsymbol{e}_r^1|^2 \right).\end{aligned}$$

According to the Hölder inequality and (5.18), the second term can be estimated as follows:

$$\begin{aligned}
&\boldsymbol{e}_r^1 g(\widehat{\boldsymbol{m}}_h^{\frac{1}{2}}, \widehat{r}^{\frac{1}{2}})\langle N(\widehat{\boldsymbol{m}}_h^{\frac{1}{2}}),\, \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\rangle\\
\le\ & |\Omega|^{\frac{1}{2}}\, g(\widehat{\boldsymbol{m}}_h^{\frac{1}{2}}, \widehat{r}^{\frac{1}{2}})\, |\boldsymbol{e}_r^1| \|N(\widehat{\boldsymbol{m}}_h^{\frac{1}{2}})\|_\infty \|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|\\
\le\ & \frac{1}{4}|\boldsymbol{e}_r^1|^2 + C\,\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|^2.
\end{aligned}$$

Using (3.8), the third term can be estimated as

$$\tau \boldsymbol{e}_r^1 \boldsymbol{R}_r^{\frac{1}{2}} \le C\tau |\boldsymbol{e}_r^1|^2 + C\tau(\tau^2+h^2)^2.$$

Substituting the estimates of the above three terms into (5.58), we obtain

$$\frac{3}{4}|\boldsymbol{e}_r^1|^2 - C\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|^2 \le C\tau\left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{\frac{1}{2}}\|^2 + |\widehat{\boldsymbol{e}}_r^{\frac{1}{2}}|^2 + |\boldsymbol{e}_r^1|^2\right) + C\tau(\tau^2+h^2)^2.$$

Applying (5.54), when $\tau \le \tau_5 \le 1/(2C)$ for a positive constant τ_5, we obtain

$$|\boldsymbol{e}_r^1|^2 \le C\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|^2 + C(\tau^2+h^2)^2. \tag{5.59}$$

According to (5.57) and (5.59), there exists a positive constant C_3 independent of τ and h such that the following inequality holds:

$$\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\|_{H_h^1}^2 + |\boldsymbol{e}_r^1|^2 \le C_3(\tau^2+h^2)^2.$$

Thus, (5.53) holds for $n=0$.

Step 2. To complete the induction, we assume that (5.53) holds for $n = 1, 2, \dots, k$ and aim to prove that it also holds for $n = k+1$. Then based on Lemma 5.7, (5.27) and the mesh ratio condition, we deduce the following bound:

$$\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_{H_h^1}^2 + |\widehat{\boldsymbol{e}}_r^{k+\frac{1}{2}}|^2 \le C(\tau^2+h^2)^2. \tag{5.60}$$

By using Lemma 3.3 and the mesh ratio condition, we have

$$\begin{aligned}
\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_\infty &\le h^{-1}\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\| \le C\tau \le \frac{1}{4},\\
\|\nabla_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_\infty &\le h^{-1}\|\nabla_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\| \le C\tau \le \frac{1}{4},
\end{aligned}$$

when $\tau \le \tau_6 \le 1/(4C)$ for a positive constant τ_6. Then it follows by the triangle inequality that

$$\begin{aligned}
\|\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\|_\infty &\le \|\boldsymbol{m}_e^{k+\frac{1}{2}}\|_\infty + \|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_\infty \le L^* + \frac{1}{4},\\
\|\nabla_h\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\|_\infty &\le \|\nabla_h\boldsymbol{m}_e^{k+\frac{1}{2}}\|_\infty + \|\nabla_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\|_\infty \le L^* + \frac{1}{4}.
\end{aligned}$$

Thus, in the remainder of the proof, we can apply Lemmas 5.2 and 5.6 directly.

We first consider the error function $\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}$. Setting $n = k$ in (5.51) and taking the ℓ^2-inner product on both sides of (5.51) with $2\tau\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1/2}$ yield

$$\langle\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}, \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}\rangle - \langle\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k}, \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k}\rangle = 2\tau\left\langle\sum_{i=1}^{6}\widetilde{B}_i^k, \mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\rangle.$$

The following estimates for the right-hand side of above equation can be derived analogously to the estimate for $\tau\langle\sum_{i=1}^{6}\widetilde{B}_i^0,\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^1\rangle$ in Step 1:

$$\begin{aligned}&\langle\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1},\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}\rangle-\langle\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k},\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k}\rangle\\&\le C\tau\left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac12}\|_{H_h^1}^2+\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac12}\|_{H_h^1}^2+|\widehat{\boldsymbol{e}}_r^{k+\frac12}|^2\right)+C\tau(\tau^2+h^2)^2.\end{aligned}\tag{5.61}$$

Multiplying (5.52) by $2\tau\boldsymbol{e}_r^{k+1}$, and using (5.30), we have

$$\begin{aligned}&|\boldsymbol{e}_r^{k+1}|^2-|\boldsymbol{e}_r^{k}|^2+|\boldsymbol{e}_r^{k+1}-\boldsymbol{e}_r^{k}|^2\\&=2\,\boldsymbol{e}_r^{k+1}\left\langle g\big(\boldsymbol{m}_e^{k+\frac12},r_e^{k+\frac12}\big)\,N\big(\boldsymbol{m}_e^{k+\frac12}\big)-g\big(\widehat{\boldsymbol{m}}_h^{k+\frac12},\widehat{r}^{k+\frac12}\big)\,N\big(\widehat{\boldsymbol{m}}_h^{k+\frac12}\big),\,\boldsymbol{m}_e^{k+1}-\boldsymbol{m}_e^{k}\right\rangle\\&\quad+2\,\boldsymbol{e}_r^{k+1}\left\langle g\big(\widehat{\boldsymbol{m}}_h^{k+\frac12},\widehat{r}^{k+\frac12}\big)\,N\big(\widehat{\boldsymbol{m}}_h^{k+\frac12}\big),\,\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k}\right\rangle+2\tau\,\boldsymbol{e}_r^{k+1}\boldsymbol{R}_r^{k+\frac12}.\end{aligned}\tag{5.62}$$

Similar to (5.32), the first term in the right-hand side of (5.62) can be estimated by

$$\begin{aligned}&\left|2\boldsymbol{e}_r^{k+1}\left\langle g\big(\boldsymbol{m}_e^{k+\frac12},r_e^{k+\frac12}\big)\,N\big(\boldsymbol{m}_e^{k+\frac12}\big)-g\big(\widehat{\boldsymbol{m}}_h^{k+\frac12},\widehat{r}^{k+\frac12}\big)\,N\big(\widehat{\boldsymbol{m}}_h^{k+\frac12}\big),\boldsymbol{m}_e^{k+1}-\boldsymbol{m}_e^{k}\right\rangle\right|\\&\le 2|\boldsymbol{e}_r^{k+1}|\left\|g\big(\boldsymbol{m}_e^{k+\frac12},r_e^{k+\frac12}\big)\,N\big(\boldsymbol{m}_e^{k+\frac12}\big)-g\big(\widehat{\boldsymbol{m}}_h^{k+\frac12},\widehat{r}^{k+\frac12}\big)\,N\big(\widehat{\boldsymbol{m}}_h^{k+\frac12}\big)\right\|\,\|\boldsymbol{m}_e^{k+1}-\boldsymbol{m}_e^{k}\|\\&\le C\tau|\boldsymbol{e}_r^{k+1}|\left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac12}\|+|\widehat{\boldsymbol{e}}_r^{k+\frac12}|\right)\le C\tau\left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac12}\|^2+|\widehat{\boldsymbol{e}}_r^{k+\frac12}|^2+|\boldsymbol{e}_r^{k+1}|^2\right).\end{aligned}\tag{5.63}$$

For the second term, there holds

$$\begin{aligned}&\left|2\boldsymbol{e}_r^{k+1}\left\langle g\big(\widehat{\boldsymbol{m}}_h^{k+\frac12},\widehat{r}^{k+\frac12}\big)\,N\big(\widehat{\boldsymbol{m}}_h^{k+\frac12}\big),\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k}\right\rangle\right|\\&=2\tau g\big(\widehat{\boldsymbol{m}}_h^{n+\frac12},\widehat{r}^{k+\frac12}\big)\,|\boldsymbol{e}_r^{k+1}|\left|\left\langle N\big(\widehat{\boldsymbol{m}}_h^{k+\frac12}\big),\frac{\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k}}{\tau}\right\rangle\right|\\&\le C\tau|\boldsymbol{e}_r^{k+1}|\left|\left\langle N\big(\widehat{\boldsymbol{m}}_h^{k+\frac12}\big),\sum_{i=1}^{6}B_i^k\right\rangle\right|.\end{aligned}\tag{5.64}$$

We denote

$$K_i:=|\boldsymbol{e}_r^{k+1}|\left|\left\langle N\big(\widehat{\boldsymbol{m}}_h^{k+\frac12}\big),B_i^k\right\rangle\right|.$$

and estimate K_i term by term below. For K_1, it follows by (5.3) that

$$\begin{aligned}K_1&\le|\Omega|^{\frac12}|\boldsymbol{e}_r^{k+1}|\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac12}\|\left\|\mathcal{L}_h\widetilde{\boldsymbol{m}}_e^{k+\frac12}+g\big(\boldsymbol{m}_e^{k+\frac12},r_e^{k+\frac12}\big)\,N\big(\boldsymbol{m}_e^{k+\frac12}\big)\right\|_\infty\|N\big(\widehat{\boldsymbol{m}}_h^{k+\frac12}\big)\|_\infty\\&\le C\left(\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac12}\|^2+|\boldsymbol{e}_r^{k+1}|^2\right).\end{aligned}$$

By applying (3.12) and (5.5), we obtain

$$\begin{aligned}&|\boldsymbol{e}_r^{k+1}|\left|\left\langle\widehat{\boldsymbol{m}}_h^{k+\frac12}\times\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac12},N\big(\widehat{\boldsymbol{m}}_h^{k+\frac12}\big)\right\rangle\right|\\&=|\boldsymbol{e}_r^{k+1}|\left|\left\langle N\big(\widehat{\boldsymbol{m}}_h^{k+\frac12}\big)\times\widehat{\boldsymbol{m}}_h^{k+\frac12},\mathcal{L}_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac12}\right\rangle\right|\\&\le C|\boldsymbol{e}_r^{k+1}|\left(\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac12}\|+\|\nabla_h\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac12}\|\right)\\&\le C\left(\|\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac12}\|_{H_h^1}^2+|\boldsymbol{e}_r^{k+1}|^2\right).\end{aligned}$$

Hence for K_2, we obtain by (5.3) and (5.19) that

$$\begin{aligned}K_2 &\leq |\Omega|^{\frac{1}{2}}\left|\boldsymbol{e}_r^{k+1}\right|\left\|\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\right\|_\infty\left\|g\left(\boldsymbol{m}_e^{k+\frac{1}{2}},r_e^{k+\frac{1}{2}}\right)N\left(\boldsymbol{m}_e^{k+\frac{1}{2}}\right)-g\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}},\widehat{r}^{k+\frac{1}{2}}\right)N\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\right)\right\|\\ &\quad\times\left\|N\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\right)\right\|_\infty + C\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2+\left|\boldsymbol{e}_r^{k+1}\right|^2\right)\\ &\leq C\left|\boldsymbol{e}_r^{k+1}\right|\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|+\left|\widehat{\boldsymbol{e}}_r^{k+\frac{1}{2}}\right|\right)+C\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2+\left|\boldsymbol{e}_r^{k+1}\right|^2\right)\\ &\leq C\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|^2+\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2+\left|\widehat{\boldsymbol{e}}_r^{k+\frac{1}{2}}\right|^2+\left|\boldsymbol{e}_r^{k+1}\right|^2\right).\end{aligned}$$

For K_3 and K_4, it follows by (3.10) and (5.3) that

$$\begin{aligned}K_3,K_4 &\leq \alpha|\Omega|^{\frac{1}{2}}\left|\boldsymbol{e}_r^{k+1}\right|\left\|N\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\right)\right\|_\infty\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|\left\|\boldsymbol{m}_e^{k+\frac{1}{2}}\right\|_\infty\\ &\quad\times\left\|\mathcal{L}_h\widetilde{\boldsymbol{m}}_e^{k+\frac{1}{2}}+g\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}},\widehat{r}^{k+\frac{1}{2}}\right)N\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\right)\right\|_\infty\\ &\leq C\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|^2+\left|\boldsymbol{e}_r^{k+1}\right|^2\right).\end{aligned}$$

Using (3.13) and (5.6), we have

$$\begin{aligned}&\left|\boldsymbol{e}_r^{k+1}\right|\left|\left\langle\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\times\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\times\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right),N\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\right)\right\rangle\right|\\ &=\left|\boldsymbol{e}_r^{k+1}\right|\left|\left\langle\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\times\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\times N\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\right)\right),\mathcal{L}_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\rangle\right|\\ &\leq C\left|\boldsymbol{e}_r^{k+1}\right|\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|+\left\|\nabla_h\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|\right)\leq C\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2+\left|\boldsymbol{e}_r^{k+1}\right|^2\right).\end{aligned}$$

Hence for K_5, we obtain by (5.3), (5.1), and (5.19) that

$$\begin{aligned}K_5 &\leq \alpha|\Omega|^{\frac{1}{2}}\left|\boldsymbol{e}_r^{k+1}\right|\left\|N\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\right)\right\|_\infty\left\|\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\right\|_\infty^2\\ &\quad\times\left\|g\left(\boldsymbol{m}_e^{k+\frac{1}{2}},r_e^{k+\frac{1}{2}}\right)N\left(\boldsymbol{m}_e^{k+\frac{1}{2}}\right)-g\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}},\widehat{r}^{k+\frac{1}{2}}\right)N\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\right)\right\|\\ &\quad+C\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2+\left|\boldsymbol{e}_r^{k+1}\right|^2\right)\\ &\leq C\left|\boldsymbol{e}_r^{k+1}\right|\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|+\left|\widehat{\boldsymbol{e}}_r^{k+\frac{1}{2}}\right|\right)+C\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2+\left|\boldsymbol{e}_r^{k+1}\right|^2\right)\\ &\leq C\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|^2+\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2+\left|\widehat{\boldsymbol{e}}_r^{k+\frac{1}{2}}\right|^2+\left|\boldsymbol{e}_r^{k+1}\right|^2\right).\end{aligned}$$

For K_6, applying Hölder's inequality and (3.8), we have

$$K_6\leq C\left|\boldsymbol{e}_r^{k+1}\right|^2+C(\tau^2+h^2)^2.$$

Substituting all these estimates of K_i into (5.64), we obtain

$$\begin{aligned}&2\boldsymbol{e}_r^{k+1}\left\langle g\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}},\widehat{r}^{k+\frac{1}{2}}\right)N\left(\widehat{\boldsymbol{m}}_h^{k+\frac{1}{2}}\right),\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1}-\widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k}\right\rangle\\ &\leq C\tau\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|^2+\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2+\left|\widehat{\boldsymbol{e}}_r^{k+\frac{1}{2}}\right|^2+\left|\boldsymbol{e}_r^{k+1}\right|^2\right)+C\tau(\tau^2+h^2)^2.\end{aligned}\tag{5.65}$$

For the third term on the right hand side of (5.62), there holds

$$\left\|2\tau\boldsymbol{e}_r^{k+1}\boldsymbol{R}_r^{k+\frac{1}{2}}\right\|\leq\tau\left|\boldsymbol{e}_r^{k+1}\right|^2+C\tau(\tau^2+h^2)^2.\tag{5.66}$$

Putting (5.63), (5.65) and (5.66) into (5.62), we conclude that

$$\left|\boldsymbol{e}_r^{k+1}\right|^2-\left|\boldsymbol{e}_r^{k}\right|^2\leq C\tau\left(\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|_{H_h^1}^2+\left\|\widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}}\right\|^2+\left|\widehat{\boldsymbol{e}}_r^{k+\frac{1}{2}}\right|^2+\left|\boldsymbol{e}_r^{k+1}\right|^2\right)+C\tau(\tau^2+h^2)^2.\tag{5.67}$$

Similar as in the proof of Theorem 3.3, we define

$$Z^k = \langle \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k, \mathcal{L}_h \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k \rangle + \left| \boldsymbol{e}_r^k \right|^2,$$

and by using Lemma 5.7, (5.61), (5.67) and (5.36), we obtain that

$$Z^{k+1} - Z^k \le C\tau \left(\left\| \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^k \right\|_{H_h^1}^2 + \left\| \widehat{\boldsymbol{e}}_{\boldsymbol{m}}^{k+\frac{1}{2}} \right\|_{H_h^1}^2 + \left\| \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1} \right\|_{H_h^1}^2 + \left| \widehat{\boldsymbol{e}}_r^{k+\frac{1}{2}} \right|^2 + \left| \boldsymbol{e}_r^{k+1} \right|^2 \right) + C\tau(\tau^2 + h^2)^2.$$

It follows by (5.37) and (5.60) that

$$Z^{k+1} - Z^k \le C\tau (Z^{k+1} + Z^k) + C\tau(\tau^2 + h^2)^2.$$

When $\tau \le \tau_7 \le 1/(2C)$ holds for a positive constant τ_7, similar to the analysis of (5.38), by applying Lemma 3.4 and (5.37), we obtain

$$\left\| \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^{k+1} \right\|_{H_h^1}^2 + \left| \boldsymbol{e}_r^{k+1} \right|^2 \le \frac{\exp(4CT)(C_3 + 1/2)}{\delta_3} (\tau^2 + h^2)^2,$$

where $\delta_3 = \min\{\delta_1, 1\}$. Thus, by mathematical induction, (5.53) holds for all $1 \le n \le N$.

Finally, it follows by (5.53) and (5.27) that

$$\left\| \boldsymbol{e}_{\boldsymbol{m}}^n \right\|_{H_h^1}^2 + \left| \boldsymbol{e}_r^n \right|^2 \le C_2 \left(\left\| \widetilde{\boldsymbol{e}}_{\boldsymbol{m}}^n \right\|_{H_h^1}^2 + \left| \boldsymbol{e}_r^n \right|^2 \right) \le \widehat{C}_0 C_2 (\tau^2 + h^2)^2.$$

This completes the proof of Theorem 3.4 for $\widehat{C}^* = \widehat{C}_0 C_2$ and $\tau^* = \min\{\tau_0, \min_{3 \le i \le 7} \tau_i\}$.

6. Numerical Experiments

In this section, several numerical experiments are presented to evaluate the ESAV-1 and ESAV-2 schemes developed in the previous section. These examples illustrate the schemes' convergence rates, accuracy, stability, and overall numerical performance, particularly in the context of magnetic skyrmions.

6.1. Convergence rate with a known exact solution

We aim to evaluate the convergence rate of the LL equation on the rectangular 2D domain $\Omega = (0, 1]^2$. In the absence of a known exact solution to the LL equation (1.1) for general parameters, we validate our results using the method of manufactured solutions. A suitable forcing term $\boldsymbol{f}$ is determined to ensure that the exact solution is given by

$$\boldsymbol{m}_e = \left(\cos\left(x^2(1-x)^2 y^2 (1-y)^2\right) \sin t, \sin\left(x^2(1-x)^2 y^2 (1-y)^2\right) \sin t, \cos t \right).$$

It is noteworthy that in the ESAV-2 scheme, $\boldsymbol{f}$ is evaluated at $t^{n+1/2}$, whereas in the ESAV-1 scheme, it is evaluated at t^{n+1}. We test the numerical solutions for two different damping coefficients: $\alpha = 0.5$ and $\alpha = 0$. The physical parameters in the LL equation are set as follows: $\kappa = 0.6, A_1 = -0.1, A_2 = 0.1, \gamma = 0.1$, and the stabilizing constant $\lambda = 1$. The mesh ratio condition is specified as $h/\tau = 2$, and the time step sizes are chosen as $\tau = 2^{-k}$, where $k = 3, 4, \ldots, 9$. We compute the numerical solutions at $T = 1$ using both the ESAV-1 and ESAV-2 schemes.

Fig. 6.1 shows the H^1-norm error as a function of the time step size τ, with the condition $h/\tau = 2$. In Table 6.1, we present the temporal convergence rates in the ℓ^2-norm for $\alpha = 0.5$.

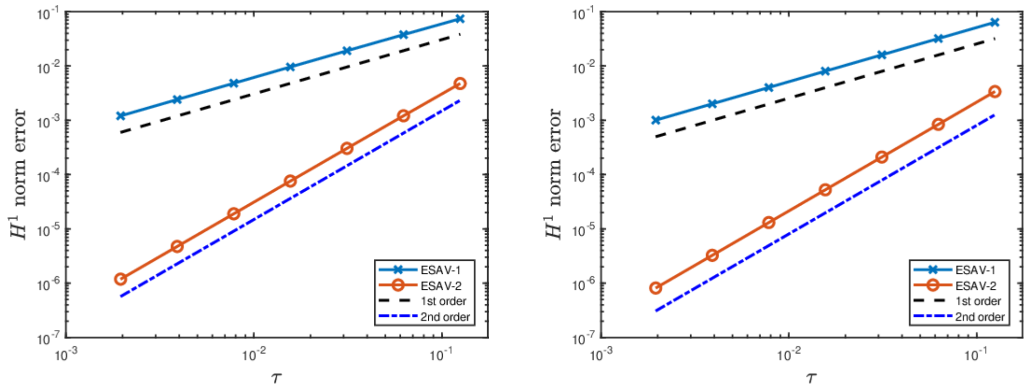

Fig. 6.1. The H^1-norm errors of the proposed ESAV-1 and ESAV-2 schemes for $\alpha = 0.5$ (left) and $\alpha = 0$ (right).

Table 6.1: Accuracy comparison in the ℓ^2-norm among ESAV-1, backward Euler, ESAV-2, and improved Euler schemes when $\alpha = 0.5$.

τ	ESAV-1		Backward Euler		ESAV-2		Improved Euler	
	Error	Order	Error	Order	Error	Order	Error	Order
1/8	7.4053e−02	–	6.7082e−02	–	3.7844e−03	–	3.2013e−03	–
1/16	3.7737e−02	0.97	3.3939e−02	0.98	9.4571e−04	2.00	8.0568e−04	1.99
1/32	1.9070e−02	0.98	1.6996e−02	1.00	2.3576e−04	2.00	2.0174e−04	2.00
1/64	9.5885e−03	0.99	8.4982e−03	1.00	5.8804e−05	2.00	5.0319e−05	2.00
1/128	4.8081e−03	1.00	4.2467e−03	1.00	1.4681e−05	2.00	1.2584e−05	2.00
1/256	2.4076e−03	1.00	2.1227e−03	1.00	3.6674e−06	2.00	3.1478e−06	2.00
1/512	1.2047e−03	1.00	1.0615e−03	1.00	9.1647e−07	2.00	7.8825e−07	2.00

The results show that the ESAV-1 and backward Euler schemes achieve first-order accuracy, while the ESAV-2 and improved Euler schemes attain second-order accuracy. Overall, the numerical errors are very close among the methods of the same order. The example indicate that, for both $\alpha = 0$ and $\alpha = 0.5$, the ESAV-1 scheme consistently achieves first-order accuracy, while the ESAV-2 scheme demonstrates the expected second-order accuracy.

6.2. Convergence rate with a unknown exact solution

Next, we test the temporal convergence rate for the LL equation in a scenario without a pre-constructed exact solution. We choose $\Omega = (0, \pi]^2$ and the initial condition as

$$\boldsymbol{m}_0 = \big(\sin(2x)\cos(2y), \cos(2x)\cos(2y), \sin(2y)\big).$$

The stabilizing constant λ, final time T, and other physical parameters are set the same as in Section 6.1. A uniform mesh size of $h = \pi/100$ is used throughout the simulations, while the time step sizes are chosen as $\tau = 10^{-2} \times 2^{-k}$, where $k = 3, 4, \ldots, 8$. Since the exact solution is not available, a reference solution is computed using a fourth-order Runge-Kutta method with a small time step $\tau = 10^{-6}$.

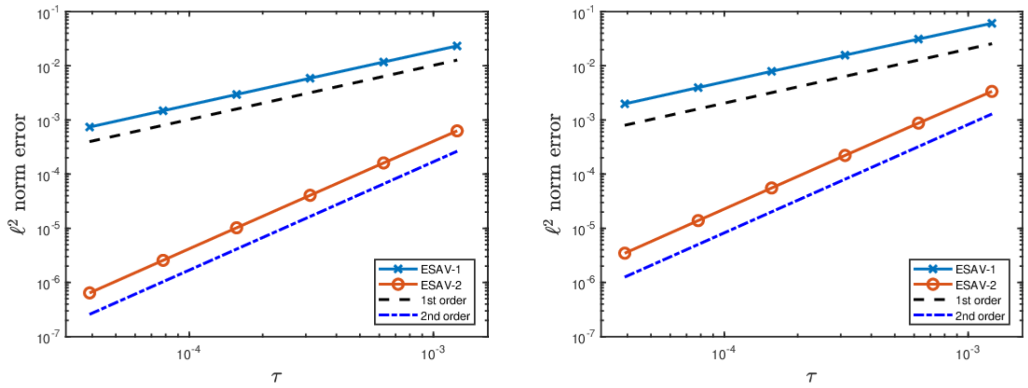

Fig. 6.2. The ℓ^2-norm errors of the proposed ESAV-1 and ESAV-2 schemes for $\alpha = 0.5$ (left) and $\alpha = 0$ (right).

Fig. 6.2 presents the ℓ^2-norm error as a function of the time step size for $\alpha = 0.5$ and $\alpha = 0$. Consistent with theoretical expectations, the ESAV-1 scheme and ESAV-2 scheme achieve first-order and second-order convergence rates, respectively. This ensures the proposed methods are robust and applicable to more general problems where the exact solution is unknown, which is often the case in practical applications of the LL equation.

6.3. Micromagnetic simulations without damping

In this subsection, we conduct micromagnetic simulations for the LL equation without damping ($\alpha = 0$) on the domain $\Omega = (0, \pi/2]^2$, using the initial condition

$$\boldsymbol{m}_0 = \big(\sin(4x)\cos(4y), \cos(4x)\cos(4y), \sin(4y)\big).$$

The primary objective is to evaluate the performance of the ESAV-2 scheme under undamped conditions. Additionally, inspired by the work in [35], we construct a second-order semi-implicit scheme based on BDF-2 (Algorithm 6.1) to serve as a reference for comparison. The simulation parameters are set as follows: $\kappa = 0.4, A_1 = -0.1, A_2 = 0.1, \gamma = 0.1, \lambda = 1$, and $T = 7$.

In Table 6.2, we compare the numerical accuracy of the improved Euler, SI BDF-2, and ESAV-2 schemes in the ℓ^2-norm. A uniform mesh size of $h = \pi/100$ is used throughout the simulations, and the time step sizes are chosen as $\tau = 1/32, 1/64, \ldots, 1/512$. Since the exact solution is not available, a reference solution is computed using a fourth-order Runge-Kutta method with a sufficiently small time step $\tau = 10^{-6}$. The results show that all three schemes achieve second-order convergence in time, and their overall accuracy is comparable.

Fig. 6.3(left) shows the time evolution of the modified energy computed by the ESAV-2 scheme using a fixed time step size of $\tau = 10^{-4}$ and a spatial step size of $h = \pi/200$, alongside the original energy calculated by the semi-implicit BDF-2 scheme. In the absence of damping, the ESAV-2 scheme successfully preserves energy conservation, as evidenced by the stability of the modified energy over time. In contrast, the semi-implicit BDF-2 scheme exhibits noticeable energy fluctuations, indicating less consistent energy preservation. Fig. 6.3(right) shows the time evolution of the relative error of the modified energy with respect to the original energy, calculated as $|(E(\boldsymbol{m}_h^n) - E_h^n)/E(\boldsymbol{m}_h^n)|$. Additionally, Fig. 6.4 presents the time evolution of the m_3-component of $\boldsymbol{m}_h$, obtained using the ESAV-2 scheme, where the periodic nature of the evolution is clearly observed.

Algorithm 6.1: Semi-implicit BDF-2 Scheme (SI BDF-2).

Input: $\boldsymbol{m}_h^0 = \widetilde{\boldsymbol{m}}_h^0 = \boldsymbol{m}_0$, $r^0 = E(\boldsymbol{m}_0)$.

Loop: For $n = 0, \ldots, N-1$ iterate the following two steps:

Step 1: Compute $\widetilde{\boldsymbol{m}}_h^{n+1}$ by solving the following linear system:

if $n = 0$ **then**

$$\frac{\widetilde{\boldsymbol{m}}_h^1 - \boldsymbol{m}_h^0}{\tau} = \boldsymbol{m}_h^0 \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_h^1 + N(\boldsymbol{m}_h^0)\right) + \alpha \boldsymbol{m}_h^0 \times \Big(\boldsymbol{m}_h^0 \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_h^1 + N(\boldsymbol{m}_h^0)\right)\Big).$$

else

$$\begin{aligned} &\frac{3\widetilde{\boldsymbol{m}}_h^{n+1} - 4\widetilde{\boldsymbol{m}}_h^n + \widetilde{\boldsymbol{m}}_h^{n-1}}{2\tau} \\ &= \widehat{\boldsymbol{m}}_h^{n+1} \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_h^{n+1} + N\left(\widehat{\boldsymbol{m}}_h^{n+1}\right)\right) \\ &\quad + \alpha \widehat{\boldsymbol{m}}_h^{n+1} \times \Big(\widehat{\boldsymbol{m}}_h^{n+1} \times \left(\mathcal{L}_h \widetilde{\boldsymbol{m}}_h^{n+1} + N\left(\widehat{\boldsymbol{m}}_h^{n+1}\right)\right)\Big), \end{aligned}$$

where $\widehat{\boldsymbol{m}}_h^{n+1} = 2\boldsymbol{m}_h^n - \boldsymbol{m}_h^{n-1}$.

end

Step 2: Normalize the magnetic field $\widetilde{\boldsymbol{m}}_h^{n+1}$ to obtain $\boldsymbol{m}_h^{n+1}$

$$\boldsymbol{m}_h^{n+1} = \frac{\widetilde{\boldsymbol{m}}_h^{n+1}}{|\widetilde{\boldsymbol{m}}_h^{n+1}|}.$$

Output: $\boldsymbol{m}_h^n$ for all $n = 1, \ldots, N$.

Table 6.2: Accuracy comparison between improved Euler, SI BDF-2, and ESAV-2 schemes in the ℓ^2-norm.

τ	Improved Euler		SI BDF-2		ESAV-2	
	Error	Order	Error	Order	Error	Order
1/32	3.3517e−4	–	3.5281e−4	–	4.1956e−4	–
1/64	8.3518e−5	2.00	8.9335e−5	1.98	1.0411e−4	2.01
1/128	2.0910e−5	2.00	2.2411e−5	1.99	2.6224e−5	1.99
1/256	5.1852e−6	2.02	5.6158e−6	2.00	6.6898e−6	1.92
1/512	1.3184e−6	1.98	1.4031e−6	2.00	1.6851e−6	1.99

6.4. Micromagnetic simulations with damping

In this subsection, we perform micromagnetic simulations for the LL equation with damping in the context of magnetic skyrmions research. Due to the particle-like nature of skyrmions, there are interactions between different skyrmion entities, leading to their condensation into a regular lattice. We consider the simplest method for generating skyrmion lattices: within $\Omega = (0, 2\pi)^2$, define four square regions centered at $(\pi/2, \pi/2), (\pi/2, 3\pi/2), (3\pi/2, \pi/2)$, and $(3\pi/2, 3\pi/2)$, each with a side length of $\pi/4$. In these regions, the initial magnetization $\boldsymbol{m}_0$ is set to $[0, 0, -1]$, while outside these regions, $\boldsymbol{m}_0$ is set to $[0, 0, 1]$. The simulation parameters are configured as follows: $\alpha = 0.6, \kappa = 0.8, A_1 = -0.1, A_2 = 0.1, \gamma = 0.1, \lambda = 1.5$, and $T = 5$. The spatial step size is set to $h = 2\pi/151$, and temporal step size is set to $\tau = 10^{-3}/4$.

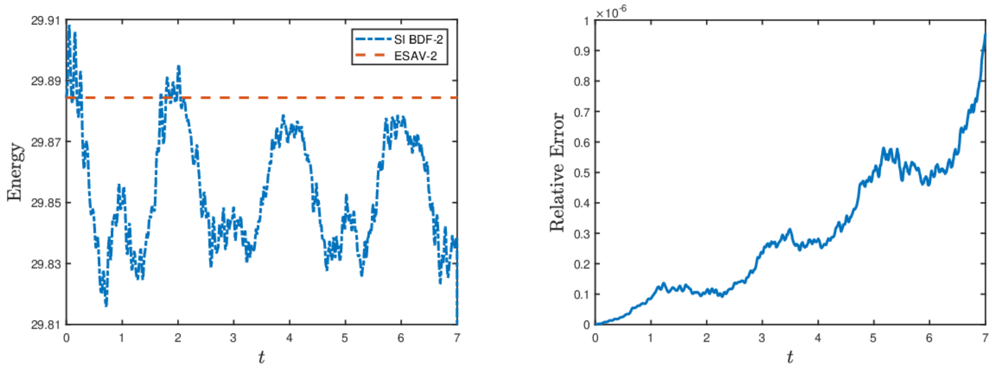

Fig. 6.3. Time evolution of the energy comparing the results obtained using the SI BDF-2 scheme and the ESAV-2 scheme (left), and the relative error of energy of the modified energy with respect to the original energy $|(E(\boldsymbol{m}_h^n) - E_h^n)/E(\boldsymbol{m}_h^n)|$ (right).

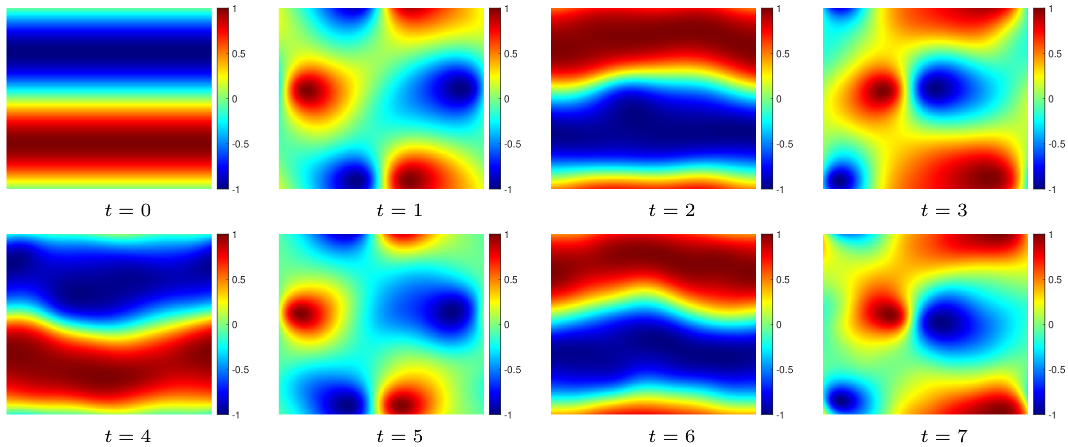

Fig. 6.4. Time evolution of the m_3-component of the numerical magnetization $\boldsymbol{m}_h$.

Fig. 6.5(left) shows that the energy computed by both schemes is nearly identical and both exhibit energy dissipation as expected. Fig. 6.5(right) depicts the time evolution of the relative error of the modified energy with respect to the original energy. In Fig. 6.6, it is observed that under the specified initial conditions, the magnetization $\boldsymbol{m}_h$ reaches a steady state at $t = 5$ and forms four skyrmions. Fig. 6.7 provides a 3D visualization of the numerical magnetization $\boldsymbol{m}_h$ at $t = 0.4$ and $t = 5$.

6.5. Simulation of blow-up phenomena

In this subsection, we study the potential blow-up behavior of the LL equation under a smooth initial configuration, inspired by [3]. The computational domain is taken as $\Omega = [-1/2, 1/2)^2$, and the initial magnetization field is defined by

$$\boldsymbol{m}_0(\boldsymbol{x}) = \begin{cases} (0, 0, -1)^\top, & \text{if} \quad |\boldsymbol{x}| \geq \frac{1}{2}, \\ \left(\frac{2x_1 A}{A^2 + |\boldsymbol{x}|^2}, \frac{2x_2 A}{A^2 + |\boldsymbol{x}|^2}, \frac{A^2 - |\boldsymbol{x}|^2}{A^2 + |\boldsymbol{x}|^2} \right)^\top, & \text{if} \quad |\boldsymbol{x}| < \frac{1}{2}, \end{cases} \tag{6.1}$$

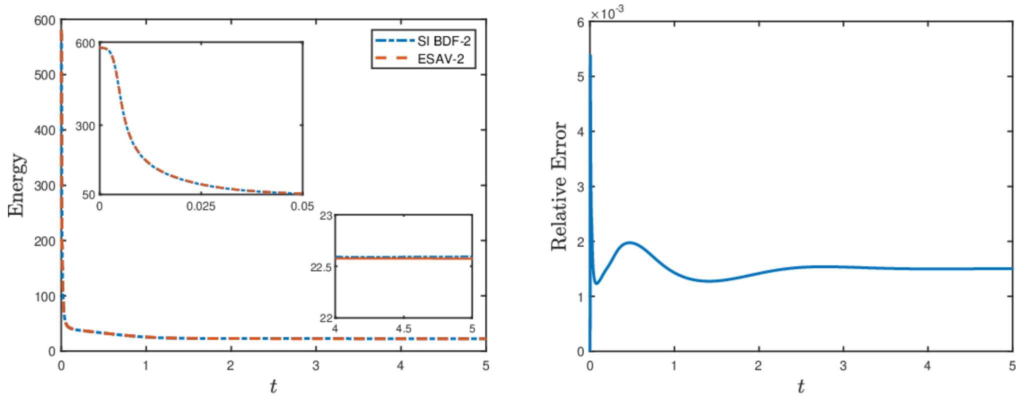

Fig. 6.5. Time evolution of the energy comparing the results obtained using the SI BDF-2 scheme and the ESAV-2 scheme (left), and relative error of the modified energy with respect to the original energy (right).

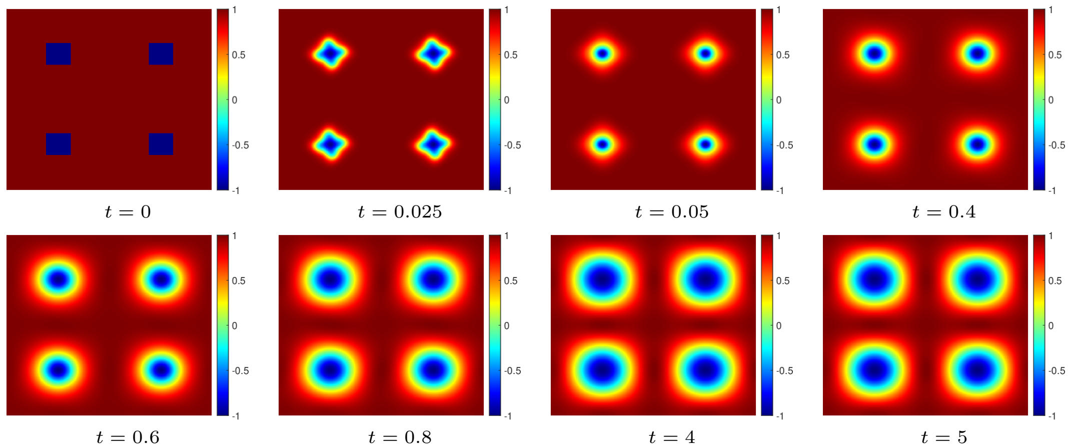

Fig. 6.6. Time evolution of the m_3-component of the numerical magnetization $\boldsymbol{m}_h$.

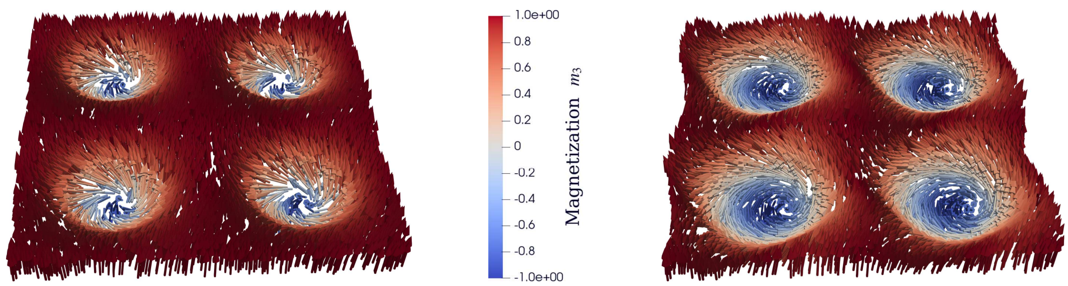

Fig. 6.7. 3D view of the numerical magnetization $\boldsymbol{m}_h$ computed with the ESAV-2 scheme at t=0.4 (left) and t=5 (right).

where $A = (1 - 2|\boldsymbol{x}|)^4$. This profile is smooth and satisfies the unit-length condition $|\boldsymbol{m}_0| = 1$. The physical parameters are set as $\alpha = 1, \kappa = 0, A_1 = -0.1, A_2 = 0.1, \gamma = 0, \lambda = 1$, with final time $T = 0.7$. All simulations are performed with the proposed ESAV-2 scheme, using mesh size $h = 1/100$ and time step $\tau = 10^{-3}$.

Snapshots of the numerical magnetization $\boldsymbol{m}_h^n$, projected onto the xy-plane, are shown in Fig. 6.8. To further illustrate the behavior near the origin, magnified 3D vector plots are presented in Fig. 6.9. These figures reveal that $\boldsymbol{m}_h$ near the origin gradually rotates downward toward $(0,0,-1)^\top$, while the spin vector at the origin itself remains fixed at $(0,0,1)^\top$. This suggests the formation of a large gradient near the origin, potentially indicating finite-time blow-up of the LL dynamics under this smooth initial data [3].

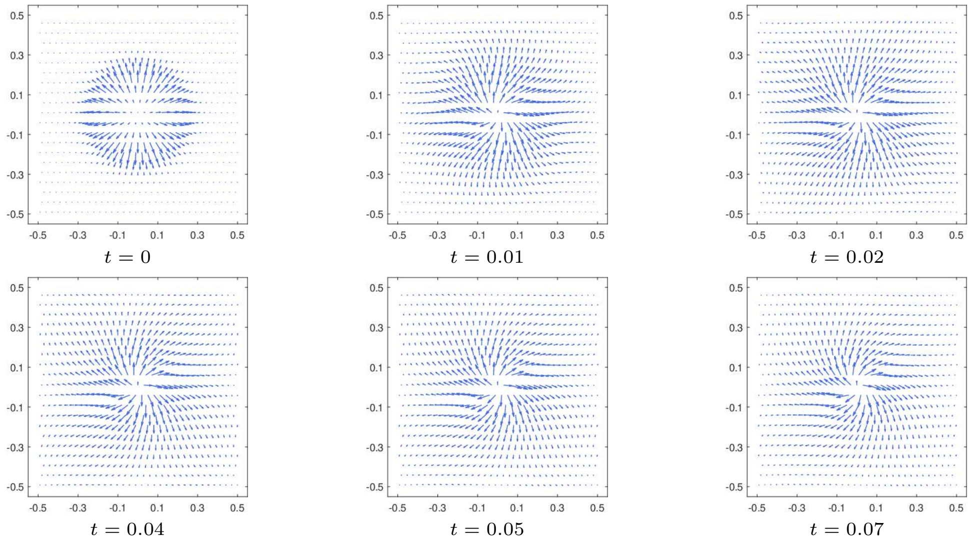

Fig. 6.8. Numerical magnetization $\boldsymbol{m}_h$ projected on xy-plane.

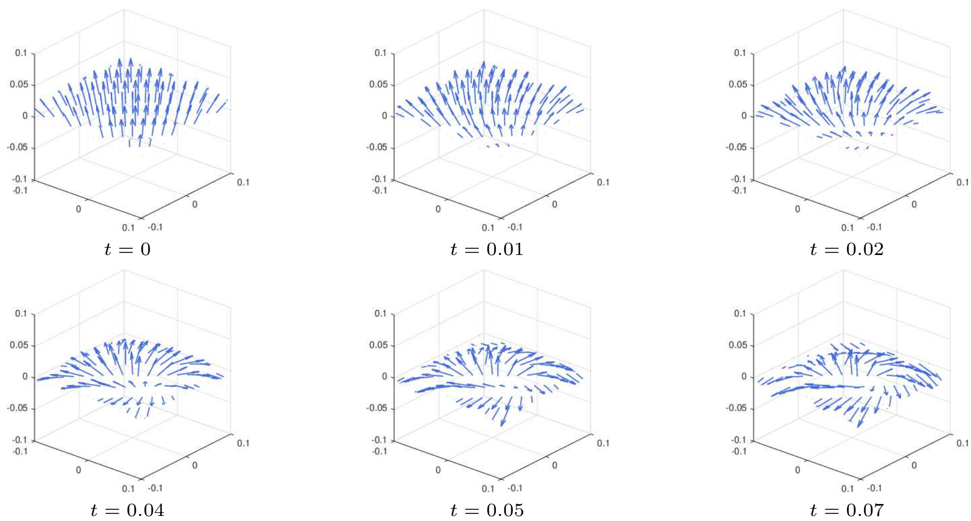

Fig. 6.9. Numerical magnetization $\boldsymbol{m}_h$ around the origin.

7. Conclusion

In this paper, we have proposed two numerical schemes that combine the ESAV method with the projection method to solve the LL equation with periodic boundary conditions. These schemes simultaneously preserve the length constraint and the modified energy dissipation law, ensuring numerical stability and physical accuracy. Introducing a stabilizing term into the linear energy functional makes the linear operator positive definite, guaranteeing energy stability. Numerical experiments in the 2D case were conducted to validate the accuracy and robustness of the proposed schemes. These results demonstrate the potential of the proposed ESAV-based schemes for solving the LL equation, particularly in the study of magnetic skyrmions involving the DMI and other higher-order interaction terms.

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