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Convergence Analysis of a Weak Galerkin Finite Element Method on a Bakhvalov-Type Mesh for a Singularly Perturbed Convection-Diffusion Equation in 2D

Shicheng Liu[1], Xiangyun Meng[2] and Qilong Zhai[1,*]

[1] *School of Mathematics, Jilin University, Changchun, Jilin 130012, China*
[2] *School of Mathematics and Statistics, Beijing Jiaotong University, Beijing 100044, China*

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Abstract. In this paper, we propose a weak Galerkin finite element method (WG) for solving singularly perturbed convection-diffusion problems on a Bakhvalov-type mesh in 2D. Our method is flexible and allows the use of discontinuous approximation functions on the mesh. An error estimate is developed in a suitable norm, and the optimal convergence order is obtained. Finally, numerical experiments are conducted to support the theory and to demonstrate the efficiency of the proposed method.

AMS subject classifications: 65N15, 65N30, 35B25

Key words: Weak Galerkin finite element method, convection-diffusion, singularly perturbed, Bakhvalov-type mesh.

1 Introduction

Consider the following singularly perturbed convection-diffusion problem

$$-\varepsilon\Delta u - \mathbf{b}\cdot\nabla u + cu = f \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.1b}$$

with a positive parameter ε satisfying $0<\varepsilon\ll 1$, and $\mathbf{b}\in[W^{1,\infty}(\Omega)]^2$. The functions $\mathbf{b}$, c, and f are assumed to be smooth on Ω. For any $(x,y)\in\overline{\Omega}$, assume that

$$b_1(x,y)\geq\beta_1>0, \quad b_2(x,y)\geq\beta_2>0, \quad c(x,y)+\frac{1}{2}\nabla\cdot\mathbf{b}(x,y)\geq\gamma>0, \tag{1.2}$$

*Corresponding author.
Emails: lsc22@mails.jlu.edu.cn (S. Liu), xymeng1@bjtu.edu.cn (X. Meng), zhaiql@jlu.edu.cn (Q. Zhai)

where β_1, β_2, and γ are some positive constants. Assumption (1.2) ensures that problem (1.1a)-(1.1b) has a unique solution in $H^2(\Omega)\cup H_0^1(\Omega)$ for all $f\in L^2(\Omega)$; see details in [17].

Singularly perturbed problems are an important topic in scientific computation. It is well known that the solution of a boundary value problem usually has layers, which are thin regions where the solution or its derivatives change rapidly due to a very small diffusion coefficient. To resolve the difficulty, numerical stabilization techniques have been developed, which can be divided into fitted operator methods and fitted mesh methods. One effective method for solving singularly perturbed problems is to use layer-adapted meshes. Boundary layers can be resolved by designing layer-adapted meshes if we have prior knowledge of the layer structure. Commonly used layer-adapted meshes for solving singularly perturbed problems include Bakhvalov-type and Shishkin-type meshes. Bakhvalov mesh was proposed for the first time in [2]; its application requires a nonlinear equation that cannot be solved explicitly. To avoid this difficulty, meshes that arise from an approximation of Bakhvalov's mesh generating function are called Bakhvalov-type meshes. These are among the most popular layer-adapted meshes; see details in [12]. Another piecewise equidistant mesh was proposed by Shishkin in [19], but a logarithmic factor will be present in the error bounds when using Shishkin-type mesh. Therefore, Bakhvalov-type meshes have better numerical performance than Shishkin-type meshes in general. Even with the use of layer-adapted meshes, the numerical solution of convection-dominated problems can still exhibit some oscillation, as detailed in [4]. Additional stabilization is added to the numerical scheme to address these oscillatory behaviors. Examples for singularly perturbed onvection-diffusion problems include the streamline-diffusion finite element method [13, 14], the classical finite difference method of up-winding flavor [1, 6, 11, 20] and the discontinuous Galerkin methods [7, 8, 18, 30, 35, 37].

In this paper, we consider the WG method to solve the singularly perturbed convection-diffusion boundary value problem on a Bakhvalov-type mesh. The WG method has proven to be an effective numerical technique for the partial differential equations (PDEs). The main idea of this method is to replace the classical derivative with a weak derivative, allowing the use of discontinuous functions in numerical schemes with parameter-independent stabilizers. The initial proposal for its application in solving second-order elliptic problems was made by Junping Wang and Xiu Ye in [25]. The WG method has been applied to various problems, including Stokes equations [26, 27, 29], Maxwell's equations [16], Brinkman equations [15, 28, 31], fractional time convection-diffusion problems [21] and others. For singular perturbed value problems, the WG method has also yielded results, as shown in [3, 9, 22–24, 33, 34, 36]. The main purpose of this paper is to present optimal order uniform convergence in the energy norm on a Bakhvalov-type mesh for convection-dominated problems in 2D.

This paper is organized as follows. In Section 2, we describe the assumptions and introduce a Bakhvalov-type mesh. In Section 3, we introduce the definition of the weak operator, the WG scheme, and some properties of the projection operator involved. In Section 4, we provide the convergence analysis. In Section 5, numerical results verify the