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Error Analyses of the Single Hidden Layer Neural Network Based on the Extreme Learning Machine for Solving Differential Equations

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Abstract. The differential equations, especially the stiff differential equations with the large eigenvalue and instability of solutions, impose strict limitations on step size for traditional numerical methods. The single hidden layer neural network method based on the extreme learning machine is widely used to solve various differential equations due to its advantages of few parameters and high efficiency, and the optimal estimation of parameters is obtained. In practice, this does not mean that the solutions of differential equations converge well. This paper aims to establish theoretical results for solving non-stiff and stiff differential equations using the single hidden layer neural network based on the extreme learning machine. Under the classical Lipschitz condition and the one-sided Lipschitz condition, we derive convergence results for the approximate solutions of these equations. Numerical experiments validate the theoretical findings, and the results indicate that computational speed can be improved by converting deep neural networks into shallow ones when solving differential equations.

AMS subject classifications: 68T07, 65L05

Key words: Stiff, approximate solution models, single hidden layer neural network, extreme learning machine, error analyses.

1 Introduction

In 1943, neurophysiologist and neuroanatomist Warren McCulloch and mathematician Walter Pitts [1] proposed the artificial neuron model, which was the enlightenment of neural networks. Since then, the research of artificial neural networks has gone through

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multiple stages of development and improvement [2, 3]. Until 1986, David E. Rumelhart et al. [4] introduced a sigmoid activation function into neural networks, which increased the nonlinear ability and overcame the difficulty of single-layer perceptron. At the same time, a backpropagation algorithm also appeared and neural networks are therefore widely used for approximating functions. And their theoretical analyses also emerged. In 1989, Kurt Hornik et al. [5] gave the universal approximation theorem, which promoted the development and application of neural network methods.

According to the number of layers, the networks can be divided into shallow and deep neural networks. The transition from shallow neural networks to deep neural networks was in 2006, when Geoffrey Hinton et al. proposed deep belief networks [6], one of the early models of deep learning, which made it possible to train deep neural networks through the method of pre-training layer by layer. Until 2010, the emergence of ReLU function [7] overcame the problem of gradient disappearance and gradient explosion caused by sigmoid function, which promoted the further development of neural networks, and set off a boom in deep learning. At the same time, there has been a lot of work on solving differential equations by deep neural networks [8, 9]. Therefore, many scholars have begun to study the approximation ability and effect of deep neural networks and deep ReLU neural networks.

In 2017, Dmitry Yarotsky [10] studied the expressibility of shallow and deep neural networks based on the ReLU activation functions, demonstrating that deep ReLU networks approximate smooth functions more efficiently than shallow neural networks. In the same year, many scholars [11–13] proved the approximation ability of deep neural networks from different views. In 2018, E Weinan et al. [14] showed that the convergence rate of deep neural networks is approximately exponential for low-dimensional analytic functions. In the same year, many scholars also analyzed the approximation ability of deep networks based on the ReLU activation function [15, 16]. In 2021, Zuowei Shen and Haizhao Yang et al. [17] quantitatively characterized the approximation ability of deep feedforward neural networks in terms of the number of neurons. In 2022, Sean Hona and Haizhao Yang [18] were inspired by the development of numerical solvers using deep neural networks to solve partial differential equations, and established approximate results of deep neural networks for smooth functions measurable in Sobolev space.

At the same time, many scholars have proved the approximation ability of shallow neural networks. In 2018, Namig J. Guliyev et al. [19] proved the approximation ability of two hidden layer neural networks based on fixed weights. In 2021, Zuowei Shen and Haizhao Yang et al. [20] introduced three hidden layer neural networks with super-approximation capability, and classified the errors into three categories: approximation error, generalization error, and optimization error. In the same year, Tim De Ryck et al. [21] gave the error bound of solving the Sobolev regular function by tanh neural network under high order Sobolev norm, and proved that tanh neural networks with only two hidden layers were sufficient to approximate the function at a better rate, not the deep ReLU neural networks.

All of the above work directly explores the approximation abilities of shallow and