Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **18**, No. 5, pp. 1602-1615

DOI: 10.4208/aamm.OA-2025-0179
October 2026

A Spectral Split-Step Padé Method for Guided Wave Propagation

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Received 15 August 2025; Accepted (in revised version) 5 January 2026

Abstract. In this study, a Fourier-based split-step Padé (SSP) method for solving the parabolic wave equation with applications in guided wave propagation in ocean acoustics is presented. Traditional SSP implementations rely on finite-difference discretizations of the depth-dependent differential operator. This approach limits accuracy in coarse discretizations as well as computational efficiency in dense discretizations, since it is does not significantly benefit from parallelization. In contrast, our proposed method replaces finite differences with a spectral representation using the discrete sine transform (DST). This enables an exact treatment of the vertical operator under homogeneous boundary conditions. For non-constant sound speed profiles, we use a Neumann series expansion to treat inhomogeneities as perturbations. Numerical experiments demonstrate the method's accuracy in range-independent and range-dependent scenarios, including propagation in deep ocean with Munk profile and in the presence of a parameterized synoptic eddy. Compared to finite-difference SSP methods, the Fourier-based approach achieves higher accuracy with fewer depth discretization points and avoids the resolution bottleneck associated with sharp field features, making it well-suited for large-scale, high-frequency wave propagation problems in ocean environments.

AMS subject classifications: 35Q60, 78A45, 42A38, 65T50

Key words: Split Step Padé method, parabolic wave equation, Fourier transform, discrete Sine transform, guided wave propagation, Isovelocity approximation, spectral method.

1 Introduction

The parabolic equation method originates from the work of Leontovich and Fock [16,32] where it was proposed as a practical tool for radiowave propagation over the Earth's

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surface. This formulation reduced the computational burden by transforming the full-wave elliptic equation into an evolutionary equation that can be efficiently solved by a one-way marching numerical scheme (with back scattering neglected). In geophysics, Claerbout applied similar principles to seismic waves simulation [7], while Tappert later on introduced the parabolic equation in ocean acoustics [30].

Paraxial (narrow-angle) parabolic equations from early papers were not capable of handling the waves propagating at large grazing angles to the waveguide axis. For instance, in ocean acoustics this manifests in errors when modelling sound reflection from the seabottom. This issue was resolved by introducing the so-called wide-angle parabolic equations [7, 26] (WAPEs) which are currently widely used in optics [17], radio waves theory [18] and atmospheric and underwater acoustics [1, 10, 20].

Subsequent developments led to the creation of mode parabolic equations describing the evolution of amplitudes of normal modes in a three-dimensional waveguide. The adiabatic mode parabolic equation (PE) [9,22,25] assumes weak mode coupling, whereas the coupled mode PEs [1, 14, 23, 31] take mode interactions into account.

The WAPE solution technique known as *split-step Padé* (SSP) method, independently proposed by Collins [10] and Avilov [5] has become an essential step in the development of the parabolic equation theory. In this approach, the Padé series is used to approximate the exponential of the square-root of a differential operator (rather than the square root itself) in the transverse direction to the waveguide axis. A finite-difference discretization of this differential operator is commonly used to compute this Padé approximation numerically. Such discretization imposes certain restrictions on the meshsize and introduces a truncation error for the derivatives in this direction.

Modern spectral methods overcome this limitation by precisely representing the vertical differential operator in spectral space, allowing for accurate simulations on coarser grids. The first such method, the *split-step Fourier* (SSF) method, introduced independently by both Hardin [13] and Tappert [29], is suitable for narrow-angle parabolic equations, that is, for equations obtained from first-order Taylor expansion. The aim of this study is to combine the high accuracy at high grazing angles of the SSP method with the exact representation of the differential operator in the SSF method.

Compared to solving the full wave equation numerically (e.g., [21]), the PE approach requires far less computational effort because direct discretizations must finely resolve each wavelength. Helmholtz equation solvers based on separation of variables and finite differences (e.g., [19,27]) also incur a high computational cost due to coupled eigenvalue problems. In both cases, convergence depends on the spatial discretization because the depth operator is approximated by finite differences. In constant sound-speed (isovelocity) environments, however, Fourier transforms apply the operator exactly in spectral space, avoiding this restriction entirely. This improves both accuracy and also efficiency by making the propagation step inherently parallelizable.

The remainder of this paper is organized as follows: In Section 2, we discuss the split step Padé (SSP) method in its original form (hereafter abbreviation SSP is reserved for its traditional implementation based the on finite-difference discretization). In Section 3,