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A First-Order Linear Energy Stable Scheme for the Cahn–Hilliard Equation with Dynamic Boundary Conditions Under the Effect of Hyperbolic Relaxation

Minghui Yu[1] and Rui Chen[1,2,*]

[1] *School of Mathematical Sciences, Beijing University of Posts and Telecommunications, Beijing 100876, China*
[2] *School of Mathematical Sciences, Key Laboratory of Mathematics and Information Networks (Ministry of Education), Beijing University of Posts and Telecommunications, Beijing 100876, China*

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Abstract. In this paper we focus on the Cahn–Hilliard equation with dynamic boundary conditions, by adding two hyperbolic relaxation terms to the system. We verify that the energy of the total system is decreasing with time. By adding two stabilization terms, we have constructed a first-order temporal accuracy numerical scheme, which is linear and energy stable. Then we prove that the scheme is of first-order in time by the error estimates. At last we present comprehensive numerical results to validate the the temporal convergence and the energy stability of such scheme. Moreover, we present the differences of the numerical results with and without the hyperbolic terms, which show that the hyperbolic terms can help the total energy decreasing slowly.

AMS subject classifications: 65N02, 65N12

Key words: Hyperbolic Cahn–Hilliard equation, dynamic boundary conditions, error estimates, linear numerical scheme, energy stability.

1 Introduction

The Cahn–Hilliard equation was first proposed by John W. Cahn and John E. Hilliard in 1958 to describe the phase separation phenomenon in binary mixtures (such as alloys and solutions) [1]. This equation has become a cornerstone in materials science, describing the phase separation process in binary alloys accurately, especially in the early stages of spinodal decomposition. The Cahn–Hilliard equation assumes that the material is isotropic and has been widely applied in theoretical studies of phase separation processes [2–4].

*Corresponding author.
Emails: ymh@bupt.edu.cn (M. Yu), ruichen@bupt.edu.cn (R. Chen)

For example, it not only simulates spontaneous heterogenization in binary mixtures such as spinodal decomposition, but also describes mechanisms of pattern formation such as nucleation and growth and coarsening [5–7].

As a representative of diffuse interface models, the Cahn–Hilliard equation avoids the explicit interface tracking issues of classical sharp-interface models by dividing the components of the mixture into thin layers, thus improving computational efficiency [8]. Moreover, this model can naturally handle complex geometries and topological changes of interfaces, significantly simplifying the computation process [9]. The Cahn–Hilliard equation and its variants have been widely applied in many fields, including block copolymers [10], image in painting [11, 12], tumor growth models [13–15], two-phase flow [16, 17], and moving contact line problems [18, 19].

The Cahn–Hilliard equation is usually equipped with periodic boundary conditions or homogeneous Neumann boundary conditions. Then Liu et al. [20] have proposed the Cahn–Hilliard type dynamic boundary condition for the Cahn–Hilliard equation. In their model, the system is energy-stable and conserves mass both in the bulk and on the boundary. Other variants of the Cahn–Hilliard equation, particularly those with dynamic boundary conditions, also exist in the literature (see references [21,22]). Numerous studies have investigated energy-stable numerical schemes for the Cahn–Hilliard equation under classical boundary conditions, particularly periodic and Neumann boundary conditions, such as the stabilization method [23], the convex splitting approach [24–26], the Lagrange multiplier approach [27–30], the Invariant Energy Quadratization (IEQ) approach [31–34], the Scalar Auxiliary Variable (SAV) approach [35, 36] and other approaches [37–40]. Meanwhile, several studies have also examined energy-stable numerical schemes for the Cahn–Hilliard equation with dynamic boundary conditions (see references [20, 41–49]).

Considering the delay in the separation of phases, Galenko et al. [50–54] have introduced the hyperbolic relaxation term to the Cahn–Hilliard system. Compared to the original equation, the equation with the inertial term is a hyperbolic equation with relaxation characteristics, which leads to different mathematical features in numerical solutions and introduces new challenges [54]. Additionally, the introduction of the hyperbolic term provides a deeper understanding of the dynamics of phase separation, especially in describing the delay of rapid phase transitions. There are also some works on designing the energy stable schemes for the hyperbolic Cahn–Hilliard model. Yang et al. [55, 56] have constructed energy stable schemes for the viscous Cahn–Hilliard equation with hyperbolic relaxation by the IEQ approach. Meanwhile, they show the error analysis for the second-order semi-discrete temporal discretization schemes. Wu et al. [57] have investigated the well-posedness and asymptotic behavior of solutions to the parabolic-hyperbolic phase field system with dynamic boundary conditions.

Inspired by the Cahn–Hilliard model [20] and hyperbolic effects, we incorporate hyperbolic terms into both the bulk equation and the dynamic boundary condition. We find that this hyperbolic model with the hyperbolic dynamic boundary condition simultaneously satisfies the energy dissipation law and preserves mass conservation in the bulk