

Singular Solutions to Monge-Ampère Equation

Luis A. Caffarelli¹ and Yu Yuan^{2,*}

¹ *Department of Mathematics, The University of Texas at Austin, Austin, TX 78712, USA*

² *University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195, USA*

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Abstract. We construct merely Lipschitz and $C^{1,\alpha}$ with rational $\alpha \in (0, 1 - 2/n]$ viscosity solutions to the Monge-Ampère equation with constant right hand side.

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1 Introduction

In this note, we construct (convex) Lipschitz and $C^{1,\alpha}$ viscosity solutions to the Monge-Ampère equation with constant right hand side via Cauchy-Kovalevskaya, after integerizing fractional powers in the corresponding equation for those singular profiles from [8] and [3,5].

Theorem 1.1. *There exists a merely Lipschitz viscosity solution to $\det D^2u = 1$ in $B_1 \subset \mathbb{R}^n$ for $n \geq 3$. There also exist merely $C^{1,\alpha-1}$ with rational $\alpha = \frac{q}{p} \in (1, 2 - \frac{2}{n}]$ viscosity solutions to $\det D^2u = 1$ in $B_1 \subset \mathbb{R}^n$ for $n \geq 3$.*

These $C^{1,\alpha}$ solutions to the Monge-Ampère equation $\det D^2u = 1$ illustrate a regularity wall phenomena: merely $C^{1,\alpha}$ with rational $\alpha \in (0, 1 - 2/n]$ solutions can never become better. This is in contrast with the regularity theory for Monge-Ampère equations [9] and [4]: once solutions are $C^{1,(1-2/n)+}$, they self-improve to smoothness.

Note that our singular solutions via Cauchy-Kovalevskaya to the Monge-Ampère equation $\det D^2u = 1$ are singular precisely along a segment of one axis, where the convex solutions are linear, or zero, to be precise. If one tries to produce higher dimensional

*Corresponding author. Email addresses: caffarel@math.utexas.edu (L. Caffarelli), yuan@math.washington.edu (Y. Yuan)

subspace singular set, where the dimension S must be less than $n/2$ by the theorem in [3], a good start is the Pogorelov type profile there,

$$u(x) = |x'|^{2-2S/n} f(|x''|)$$

with $x = (x'_1, \dots, x'_{n-S}, x''_1, \dots, x''_S)$. The profile with

$$f(|x''|) = 1 + |x''|^2$$

satisfies the Monge-Ampère with the right hand side being a polynomial of $|x''|^2$, positive near the origin. The ODE for $f(|x''|)$ with singular term $f'(|x''|)/|x''|$ corresponding to $\det D^2 u = 1$ can be solved by the method in [2] and [1].

Alternatively, relying on the existence of solutions to the Dirichlet problem for Monge-Ampère equations, with S dimensional singular set profile $|x'|^{2-2S/n} (1 + |x''|^2)$ as boundary value in a small ball, one obtains the following

Proposition 1.1. *There exist local merely $C^{1,1-S/n}$ viscosity solutions to $\det D^2 u = 1$ in \mathbb{R}^n for $n \geq 3$ such that singular set of the solutions is the S dimensional set*

$$S = \{(x', x'') : |x'| = 0\}$$

in a small ball for $1 \leq S < n/2$.

Let us sketch a proof for this proposition. Case $S = 1$ is also noted in the above. The Lipschitz limit of a family of (convex) smooth solutions to $\det D^2 u = 1$ with smooth boundary value approximations of subsolution

$$u_- = \gamma |x'|^{2-2S/n} (1 + |x''|^2)$$

for $\gamma = (1 - 2S/n)^{-1/n}$ on the boundary of a small ball is our viscosity solution. The convex solution $u(x)$ vanishes in subspace x'' with $|x'| = 0$, because it is between the convex combinations of zero boundary value and the subsolution u_- there. Surely $u(x)$ is singular in the S dimensional subspace $(0', x'')$.

We show that u is regular everywhere else. By [4,5], the other possible singular set of u outside S , must contain a line segment, where u is linear. This singular segment intersects the boundary of the small ball or the set S . The barrier argument in [9] and [4, 5] shows the two ends of the segment cannot be both on the boundary of the small ball, where u is smooth. The only other scenario that the segment has one end on S , and the other end on the boundary of the small ball is not possible either. This is because the linear function, the restriction of u on the segment, equaling 0 and $u_- > 0$ respectively on the two ends, cannot be less than the supersolution

$$u^+ = 2\gamma |x'|^{2-2S/n}$$

with sublinear growth near $|x'| = 0$.

Note that the solution u is trapped between the supersolution u^+ and the subsolution

$$u_- = \gamma |x'|^{2-2S/n} (1 + |x''|^2).$$

We see that u is exactly $C^{1,1-2S/n}$. This finishes the sketch of the proof for Proposition 1.1.

In closing, we remark that Mooney [6] recently showed that the $n - 1$ dimensional Hausdorff measure of the singular set of every subsolution to $\det D^2 u = 1$ is zero, and the collection of S -dimensional affine singular sets, on each of which the subsolution is linear, also has zero $n - S$ dimensional Hausdorff measure. In particular, the affine dimension S is less than $n/2$. This provides a new proof for the theorem in [3]. The no better than $C^{1,\beta}$ with $\beta \in [0, 1/3]$ solutions in [6, 7] have almost $n - 1$ and exactly $n - 1$ respectively Hausdorff dimensional singular sets, where each of the solutions is not a single linear function.

2 Proof of Theorem 1.1

Proof. Lipschitz case. We seek for solutions in the Lipschitz profile from [5]

$$u(x', x_n) = \rho + \rho^{n/2} f(\rho, x_n)$$

with

$$\rho = |x'| = |(x'_1, \dots, x'_{n-1})|.$$

The upper half Hessian $D^2 u$ is

$$\begin{bmatrix} \frac{1 + \frac{n}{2} \rho^{\frac{n}{2}-1} f + \rho^{\frac{n}{2}} f_\rho}{\rho} & \dots & \dots & \dots \\ & \dots & \frac{1 + \frac{n}{2} \rho^{\frac{n}{2}-1} f + \rho^{\frac{n}{2}} f_\rho}{\rho} & \dots \\ & & & \frac{n}{2} \left(\frac{n}{2} - 1 \right) \rho^{\frac{n}{2}-2} f \\ & & & + 2 \frac{n}{2} \rho^{\frac{n}{2}-1} f_\rho + \rho^{\frac{n}{2}} f_{\rho\rho} \\ & & & \frac{n}{2} \rho^{\frac{n}{2}-1} f_n \\ & & & + \rho^{\frac{n}{2}} f_{\rho n} \\ & & & \rho^{\frac{n}{2}} f_{nn} \end{bmatrix},$$

and its determinant

$$\begin{aligned} \det D^2 u &= \left[\frac{1 + \frac{n}{2} \rho^{\frac{n}{2}-1} f + \rho^{\frac{n}{2}} f_\rho}{\rho} \right]^{n-2} \left\{ \begin{bmatrix} \frac{n}{2} \left(\frac{n}{2} - 1 \right) \rho^{\frac{n}{2}-2} f \\ + 2 \frac{n}{2} \rho^{\frac{n}{2}-1} f_\rho + \rho^{\frac{n}{2}} f_{\rho\rho} \end{bmatrix} \rho^{\frac{n}{2}} f_{nn} \right\} \\ &= \left(1 + \frac{n}{2} \rho^{\frac{n}{2}-1} f + \rho^{\frac{n}{2}} f_\rho \right)^{n-2} \left\{ \begin{bmatrix} \frac{n}{2} \left(\frac{n}{2} - 1 \right) f + n \rho f_\rho + \rho^2 f_{\rho\rho} \\ - \left(\frac{n}{2} f_n + \rho f_{\rho n} \right)^2 \end{bmatrix} f_{nn} \right\}. \end{aligned}$$

We make the following change of variable to move to an analytic equation.

Set

$$s = \rho^{1/2} \quad \text{and} \quad h(s, x_n) = f(s^2, x_n),$$

then

$$\partial_s = 2s\partial_\rho \quad \text{or} \quad \partial_\rho = \frac{1}{2s}\partial_s, \quad \text{and} \quad \partial_\rho^2 = \frac{1}{4} \left(\frac{-1}{s^3}\partial_s + \frac{1}{s^2}\partial_s^2 \right).$$

The determinant becomes

$$\det D^2 u = \left(1 + \frac{n}{2}s^{n-2}h + \frac{1}{2}s^{n-1}h_s \right)^{n-2} \left\{ \begin{aligned} & \left[+\frac{n}{2}sh_s + \frac{1}{4} \left(-sh_s + s^2h_{ss} \right) \right] h_{nn} \\ & - \left(\frac{n}{2}h_n + \frac{1}{2}sh_{sn} \right)^2 \end{aligned} \right\}.$$

Now we solve the reduced Monge-Ampère equation

$$\begin{cases} h_{nn} = \frac{\left(1 + \frac{n}{2}s^{n-2}h + \frac{1}{2}s^{n-1}h_s \right)^{2-n} + \left(\frac{n}{2}h_n + \frac{1}{2}sh_{sn} \right)^2}{\frac{n}{2} \left(\frac{n}{2} - 1 \right) h + \frac{(2n-1)}{4}sh_s + \frac{1}{4}s^2h_{ss}}, \\ h_n(s, 0) = 0, \\ h(s, 0) = 1. \end{cases}$$

Cauchy-Kovalevskaya gives the analytic solution in $B_{r_1}(0) \subset \mathbb{R}^2$

$$h(s, x_n) = 1 + \frac{2}{n(n-2)}x_n^2 + \dots$$

Thus we have a Lipschitz solution to $\det D^2 u = 1$

$$u(x) = |x'| + |x'|^{\frac{n}{2}} h\left(|x'|^{\frac{1}{2}}, x_n\right)$$

in $B_1(0) \subset \mathbb{R}^n$ by scaling.

Lastly, let us check our u is a viscosity to $\det D^2 u = 1$. For any convex quadratic $Q(x)$ touching $u(x)$ from above, observe that the touching point can never be a singular Lipschitz point of $u(x)$, and in turn, $\det D^2 Q \geq 1$ at the smooth touching point of $u(x)$. On the lower side, for any quadratic $Q(x)$ touching $u(x)$ from below, when the touching point is at $x' = 0$, observe that convex $Q(x)$ must vanish along $x' = 0$ as $u(x', x_n)$ vanishes, then $\det D^2 Q = 0 < 1$; when the touching point is at $x' \neq 0$, immediately $\det D^2 Q \leq 1$ as $u(x)$ is smooth nearby.

$C^{1, \frac{q}{p}-1}$ case. We search for solutions in the form

$$u(x', x_n) = \rho^\alpha f(\rho, x_n) = \rho^\alpha \left[1 + \rho^\beta g(\rho, x_n) \right] \quad \text{with} \quad \beta = 2(n-1) - n\alpha.$$

The upper half Hessian D^2u is

$$\begin{bmatrix} \frac{\alpha\rho^{\alpha-1}f+\rho^\alpha f_\rho}{\rho} & \dots & \frac{\alpha\rho^{\alpha-1}f+\rho^\alpha f_\rho}{\rho} & \alpha(\alpha-1)\rho^{\alpha-2}f + 2\alpha\rho^{\alpha-1}f_\rho + \rho^\alpha f_{\rho\rho} & \alpha\rho^{\alpha-1}f_n + \rho^\alpha f_{\rho n} \\ & & & & \rho^\alpha f_{nn} \end{bmatrix},$$

and its determinant

$$\begin{aligned} \det D^2u &= [\rho^{\alpha-2}(\alpha f + \rho f_\rho)]^{n-2} \rho^{2\alpha-2} \left\{ \begin{aligned} &[\alpha(\alpha-1)f + 2\alpha\rho f_\rho + \rho^2 f_{\rho\rho}] f_{nn} \\ &- (\alpha f_n + \rho f_{\rho n})^2 \end{aligned} \right\} \\ &= (\alpha f + \rho f_\rho)^{n-2} \left\{ \begin{aligned} &[\alpha(\alpha-1)f + 2\alpha\rho f_\rho + \rho^2 f_{\rho\rho}] f_{nn} \\ &- (\alpha f_n + \rho f_{\rho n})^2 \end{aligned} \right\} \rho^{n\alpha-2(n-1)}. \end{aligned}$$

Note that

$$\begin{aligned} f &= 1 + \rho^\beta g(\rho, x_n), \\ f_\rho &= \beta\rho^{\beta-1}g + \rho^\beta g_\rho, \\ f_{\rho\rho} &= \beta(\beta-1)\rho^{\beta-2}g + 2\beta\rho^{\beta-1}g_\rho + \rho^\beta g_{\rho\rho}, \end{aligned}$$

then

$$\begin{aligned} \det D^2u &= [\alpha + \alpha\rho^\beta g + \beta\rho^\beta g + \rho^{\beta+1}g_\rho]^{n-2} \rho^{n\alpha-2(n-1)} \\ &\quad \left\{ \begin{aligned} &[\alpha(\alpha-1)(1+\rho^\beta g) + 2\alpha(\beta\rho^\beta g + \rho^{\beta+1}g_\rho) \\ &+ \beta(\beta-1)\rho^\beta g + 2\beta\rho^{\beta+1}g_\rho + \rho^{\beta+2}g_{\rho\rho} \\ &- (\alpha\rho^\beta g_n + \beta\rho^\beta g_n + \rho^{\beta+1}g_{\rho n})^2 \end{aligned} \right\} \rho^\beta g_{nn} \\ &= [\alpha + (\alpha + \beta)\rho^\beta g + \rho^{\beta+1}g_\rho]^{n-2} \rho^{n\alpha-2(n-1)+\beta} \\ &\quad \left\{ \begin{aligned} &[\alpha(\alpha-1) + (\alpha + \beta)(\alpha + \beta - 1)\rho^\beta g \\ &+ 2(\alpha + \beta)\rho^{\beta+1}g_\rho + \rho^{\beta+2}g_{\rho\rho} \\ &- \rho^\beta[(\alpha + \beta)g_n + \rho g_{\rho n}]^2 \end{aligned} \right\} g_{nn} \\ &= [\alpha + (\alpha + \beta)\rho^\beta g + \rho^{\beta+1}g_\rho]^{n-2} \\ &\quad \left\{ \begin{aligned} &[\alpha(\alpha-1) + (\alpha + \beta)(\alpha + \beta - 1)\rho^\beta g \\ &+ 2(\alpha + \beta)\rho^{\beta+1}g_\rho + \rho^{\beta+2}g_{\rho\rho} \\ &- \rho^\beta[(\alpha + \beta)g_n + \rho g_{\rho n}]^2 \end{aligned} \right\} g_{nn}, \end{aligned}$$

where we used $n\alpha - 2(n-1) + \beta = 0$.

We make the following change of variable for $\alpha = q/p$ to move to an analytic equation.

Set

$$s = \rho^{1/p} \quad \text{and} \quad h(s, x_n) = g(s^p, x_n),$$

then

$$\partial_s = p s^{p-1} \partial_\rho \quad \text{or} \quad \partial_\rho = \frac{1}{p s^{p-1}} \partial_s, \quad \text{and} \quad \partial_\rho^2 = \frac{1}{p^2} \left(\frac{1-p}{s^{2p-1}} \partial_s + \frac{1}{s^{2p-2}} \partial_s^2 \right).$$

The determinant becomes

$$\det D^2 u = \left[\alpha + (\alpha + \beta) s^{p\beta} h + s^{p\beta} \frac{1}{p} s h_s \right]^{n-2} \left\{ \begin{array}{l} \left\{ \begin{array}{l} \alpha(\alpha-1) + (\alpha+\beta)(\alpha+\beta-1) s^{p\beta} h \\ + 2(\alpha+\beta) s^{p\beta} \frac{1}{p} s h_s + s^{p\beta} \frac{1}{p^2} [(1-p) s h_s + s^2 h_{ss}] \end{array} \right\} h_{nn} \\ - s^{p\beta} \left[(\alpha+\beta) h_n + \frac{1}{p} s h_{sn} \right]^2 \end{array} \right\}.$$

Now we solve the reduced Monge-Ampère equation

$$\begin{cases} h_{nn} = \frac{\left[\alpha + (\alpha + \beta) s^{p\beta} h + s^{p\beta} \frac{1}{p} s h_s \right]^{2-n} + s^{p\beta} \left[(\alpha + \beta) h_n + \frac{1}{p} s h_{sn} \right]^2}{\alpha(\alpha-1) + (\alpha+\beta)(\alpha+\beta-1) s^{p\beta} h + 2(\alpha+\beta) s^{p\beta} \frac{1}{p} s h_s + s^{p\beta} \frac{1}{p^2} [(1-p) s h_s + s^2 h_{ss}]}, \\ h_n(s, 0) = 0, \\ h(s, 0) = 1, \end{cases}$$

where integer $p\beta = 2p(n-1) - nq \geq 0$. Cauchy-Kovalevskaya gives the analytic solution in $B_{r_{q/p}}(0) \subset \mathbb{R}^2$

$$h(s, x_n) = 1 + \frac{1}{2} \frac{\alpha^{2-n}}{\alpha(\alpha-1)} x_n^2 + \dots$$

Thus we have a $C^{1, \frac{q}{p}-1}$ solution to $\det D^2 u = 1$

$$u(x', x_n) = |x'|^{\frac{q}{p}} \left[1 + |x'|^{2(n-1)-n\frac{q}{p}} h\left(|x'|^{\frac{1}{p}}, x_n\right) \right]$$

in $B_1(0) \subset \mathbb{R}^n$ by scaling.

Exactly as in the Lipschitz case, we verify that our $u(x)$ is a viscosity solution to the Monge-Ampère equation $\det D^2 u = 1$. \square

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