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Singular Solutions to Monge-Ampère Equation

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Abstract. We construct merely Lipschitz and $C^{1,\alpha}$ with rational $\alpha \in (0, 1-2/n]$ viscosity solutions to the Monge-Ampère equation with constant right hand side.

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1 Introduction

In this note, we construct (convex) Lipschitz and $C^{1,\alpha}$ viscosity solutions to the Monge-Ampère equation with constant right hand side via Cauchy-Kovalevskaya, after integerizing fractional powers in the corresponding equation for those singular profiles from [8] and [3,5].

Theorem 1.1. *There exists a merely Lipschitz viscosity solution to* $\det D^2u = 1$ *in* $B_1 \subset \mathbb{R}^n$ *for* $n \geq 3$. *There also exist merely* $C^{1,\alpha-1}$ *with rational* $\alpha = \frac{q}{p} \in (1, 2-\frac{2}{n}]$ *viscosity solutions to* $\det D^2u = 1$ *in* $B_1 \subset \mathbb{R}^n$ *for* $n \geq 3$.

These $C^{1,\alpha}$ solutions to the Monge-Ampère equation $\det D^2u = 1$ illustrate a regularity wall phenomena: merely $C^{1,\alpha}$ with rational $\alpha \in (0, 1-2/n]$ solutions can never become better. This is in contrast with the regularity theory for Monge-Ampère equations [9] and [4]: once solutions are $C^{1,(1-2/n)+}$, they self-improve to smoothness.

Note that our singular solutions via Cauchy-Kovalevskaya to the Monge-Ampère equation $\det D^2u = 1$ are singular precisely along a segment of one axis, where the convex solutions are linear, or zero, to be precise. If one tries to produce higher dimensional

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subspace singular set, where the dimension S must be less than $n/2$ by the theorem in [3], a good start is the Pogorelov type profile there,

$$u(x) = |x'|^{2-2S/n} f(|x''|)$$

with $x = \left(x'_1, \cdots, x'_{n-S}, x''_1, \cdots, x''_S\right)$. The profile with

$$f(|x''|) = 1 + |x''|^2$$

satisfies the Monge-Ampère with the right hand side being a polynomial of $|x''|^2$, positive near the origin. The ODE for $f(|x''|)$ with singular term $f'(|x''|)/|x''|$ corresponding to $\det D^2 u = 1$ can be solved by the method in [2] and [1].

Alternatively, relying on the existence of solutions to the Dirichlet problem for Monge-Ampère equations, with S dimensional singular set profile $|x'|^{2-2S/n}(1+|x''|^2)$ as boundary value in a small ball, one obtains the following

Proposition 1.1. *There exist local merely $C^{1,1-S/n}$ viscosity solutions to* $\det D^2 u = 1$ *in* $\mathbb{R}^n$ *for* $n \geq 3$ *such that singular set of the solutions is the S dimensional set*

$$\mathbb{S} = \left\{(x', x'') : |x'| = 0\right\}$$

in a small ball for $1 \leq S < n/2$.

Let us sketch a proof for this proposition. Case $S = 1$ is also noted in the above. The Lipschitz limit of a family of (convex) smooth solutions to $\det D^2 u = 1$ with smooth boundary value approximations of subsolution

$$u_- = \gamma |x'|^{2-2S/n} \left(1 + |x''|^2\right)$$

for $\gamma = (1 - 2S/n)^{-1/n}$ on the boundary of a small ball is our viscosity solution. The convex solution $u(x)$ vanishes in subspace x'' with $|x'| = 0$, because it is between the convex combinations of zero boundary value and the subsolution u_- there. Surely $u(x)$ is singular in the S dimensional subspace $(0', x'')$.

We show that u is regular everywhere else. By [4,5], the other possible singular set of u outside $\mathbb{S}$, must contain a line segment, where u is linear. This singular segment intersects the boundary of the small ball or the set $\mathbb{S}$. The barrier argument in [9] and [4, 5] shows the two ends of the segment cannot be both on the boundary of the small ball, where u is smooth. The only other scenario that the segment has one end on $\mathbb{S}$, and the other end on the boundary of the small ball is not possible either. This is because the linear function, the restriction of u on the segment, equaling 0 and $u_- > 0$ respectively on the two ends, cannot be less than the supersolution

$$u^+ = 2\gamma |x'|^{2-2S/n}$$

with sublinear growth near $|x'|=0$.

Note that the solution u is trapped between the supersolution u^+ and the subsolution

$$u_- = \gamma \left|x'\right|^{2-2S/n}\left(1+\left|x''\right|^2\right).$$

We see that u is exactly $C^{1,1-2S/n}$. This finishes the sketch of the proof for Proposition 1.1.

In closing, we remark that Mooney [6] recently showed that the $n-1$ dimensional Hausdorff measure of the singular set of every subsolution to $\det D^2u=1$ is zero, and the collection of S-dimensional affine singular sets, on each of which the subsolution is linear, also has zero $n-S$ dimensional Hausdorff measure. In particular, the affine dimension S is less than $n/2$. This provides a new proof for the theorem in [3]. The no better than $C^{1,\beta}$ with $\beta\in[0,1/3]$ solutions in [6,7] have almost $n-1$ and exactly $n-1$ respectively Hausdorff dimensional singular sets, where each of the solutions is not a single linear function.

2 Proof of Theorem 1.1

Proof. **Lipschitz case.** We seek for solutions in the Lipschitz profile from [5]

$$u\left(x',x_n\right)=\rho+\rho^{n/2}f\left(\rho,x_n\right)$$

with

$$\rho=\left|x'\right|=\left|\left(x_1',\cdots,x_{n-1}'\right)\right|.$$

The upper half Hessian D^2u is

$$\begin{bmatrix} \frac{1+\frac{n}{2}\rho^{\frac{n}{2}-1}f+\rho^{\frac{n}{2}}f_\rho}{\rho} & & & \\ & \ddots & & \\ & & \frac{1+\frac{n}{2}\rho^{\frac{n}{2}-1}f+\rho^{\frac{n}{2}}f_\rho}{\rho} & & \\ & & & \begin{matrix}\frac{n}{2}\left(\frac{n}{2}-1\right)\rho^{\frac{n}{2}-2}f \\ +2\frac{n}{2}\rho^{\frac{n}{2}-1}f_\rho+\rho^{\frac{n}{2}}f_{\rho\rho}\end{matrix} & \begin{matrix}\frac{n}{2}\rho^{\frac{n}{2}-1}f_n \\ +\rho^{\frac{n}{2}}f_{\rho n}\end{matrix} \\ & & & & \rho^{\frac{n}{2}}f_{nn} \end{bmatrix},$$

and its determinant

$$\begin{aligned}\det D^2u &= \left[\frac{1+\frac{n}{2}\rho^{\frac{n}{2}-1}f+\rho^{\frac{n}{2}}f_\rho}{\rho}\right]^{n-2}\left\{\begin{array}{c}\left[\begin{array}{c}\frac{n}{2}\left(\frac{n}{2}-1\right)\rho^{\frac{n}{2}-2}f \\ +2\frac{n}{2}\rho^{\frac{n}{2}-1}f_\rho+\rho^{\frac{n}{2}}f_{\rho\rho}\end{array}\right]\rho^{\frac{n}{2}}f_{nn} \\ -\left(\frac{n}{2}\rho^{\frac{n}{2}-1}f_n+\rho^{\frac{n}{2}}f_{\rho n}\right)^2\end{array}\right\} \\ &= \left(1+\frac{n}{2}\rho^{\frac{n}{2}-1}f+\rho^{\frac{n}{2}}f_\rho\right)^{n-2}\left\{\begin{array}{c}\left[\frac{n}{2}\left(\frac{n}{2}-1\right)f+n\rho f_\rho+\rho^2f_{\rho\rho}\right]f_{nn} \\ -\left(\frac{n}{2}f_n+\rho f_{\rho n}\right)^2\end{array}\right\}.\end{aligned}$$

We make the following change of variable to move to an analytic equation.

Set

$$s=\rho^{1/2}\quad\text{and}\quad h\left(s,x_n\right)=f\left(s^2,x_n\right),$$

then

$$\partial_s=2s\partial_\rho\quad\text{or}\quad\partial_\rho=\frac{1}{2s}\partial_s,\quad\text{and}\quad\partial_\rho^2=\frac{1}{4}\left(\frac{-1}{s^3}\partial_s+\frac{1}{s^2}\partial_s^2\right).$$

The determinant becomes

$$\det D^2u=\left(\begin{array}{c}1+\frac{n}{2}s^{n-2}h\\+\frac{1}{2}s^{n-1}h_s\end{array}\right)^{n-2}\left\{\begin{array}{c}\left[\begin{array}{c}\frac{n}{2}\left(\frac{n}{2}-1\right)h\\+\frac{n}{2}sh_s+\frac{1}{4}\left(-sh_s+s^2h_{ss}\right)\end{array}\right]h_{nn}\\-\left(\frac{n}{2}h_n+\frac{1}{2}sh_{sn}\right)^2\end{array}\right\}.$$

Now we solve the reduced Monge-Ampère equation

$$\begin{cases}h_{nn}=\dfrac{\left(1+\frac{n}{2}s^{n-2}h+\frac{1}{2}s^{n-1}f_s\right)^{2-n}+\left(\frac{n}{2}h_n+\frac{1}{2}sh_{sn}\right)^2}{\frac{n}{2}\left(\frac{n}{2}-1\right)h+\frac{(2n-1)}{4}sh_s+\frac{1}{4}s^2h_{ss}},\\h_n\left(s,0\right)=0,\\h\left(s,0\right)=1.\end{cases}$$

Cauchy-Kovalevskaya gives the analytic solution in $B_{r_1}\left(0\right)\subset\mathbb{R}^2$

$$h\left(s,x_n\right)=1+\frac{2}{n\left(n-2\right)}x_n^2+\cdots.$$

Thus we have a Lipschitz solution to $\det D^2u=1$

$$u\left(x\right)=\left|x'\right|+\left|x'\right|^{\frac{n}{2}}h\left(\left|x'\right|^{\frac{1}{2}},x_n\right)$$

in $B_1\left(0\right)\subset\mathbb{R}^n$ by scaling.

Lastly, let us check our u is a viscosity to $\det D^2u=1$. For any convex quadratic $Q\left(x\right)$ touching $u\left(x\right)$ from above, observe that the touching point can never be a singular Lipschitz point of $u\left(x\right)$, and in turn, $\det D^2Q\geq 1$ at the smooth touching point of $u\left(x\right)$. On the lower side, for any quadratic $Q\left(x\right)$ touching $u\left(x\right)$ from below, when the touching point is at $x'=0$, observe that convex $Q\left(x\right)$ must vanish along $x'=0$ as $u\left(x',x_n\right)$ vanishes, then $\det D^2Q=0<1$; when the touching point is at $x'\neq 0$, immediately $\det D^2Q\leq 1$ as $u\left(x\right)$ is smooth nearby.

$C^{1,\frac{q}{p}-1}$ **case.** We search for solutions in the form

$$u\left(x',x_n\right)=\rho^\alpha f\left(\rho,x_n\right)=\rho^\alpha\left[1+\rho^\beta g\left(\rho,x_n\right)\right]\quad\text{with}\quad\beta=2\left(n-1\right)-n\alpha.$$

The upper half Hessian D^2u is

$$\begin{bmatrix} \frac{\alpha\rho^{\alpha-1}f+\rho^{\alpha}f_\rho}{\rho} & & & & \\ & \cdots & & & \\ & & \frac{\alpha\rho^{\alpha-1}f+\rho^{\alpha}f_\rho}{\rho} & & \\ & & & \begin{matrix}\alpha(\alpha-1)\rho^{\alpha-2}f \\ +2\alpha\rho^{\alpha-1}f_\rho+\rho^{\alpha}f_{\rho\rho}\end{matrix} & \alpha\rho^{\alpha-1}f_n+\rho^{\alpha}f_{\rho n} \\ & & & & \rho^{\alpha}f_{nn} \end{bmatrix},$$

and its determinant

$$\begin{aligned} \det D^2u &= \left[\rho^{\alpha-2}\left(\alpha f+\rho f_\rho\right)\right]^{n-2}\rho^{2\alpha-2}\left\{\begin{matrix}\left[\alpha(\alpha-1)f+2\alpha\rho f_\rho+\rho^2 f_{\rho\rho}\right]f_{nn} \\ -\left(\alpha f_n+\rho f_{\rho n}\right)^2\end{matrix}\right\} \\ &= \left(\alpha f+\rho f_\rho\right)^{n-2}\left\{\begin{matrix}\left[\alpha(\alpha-1)f+2\alpha\rho f_\rho+\rho^2 f_{\rho\rho}\right]f_{nn} \\ -\left(\alpha f_n+\rho f_{\rho n}\right)^2\end{matrix}\right\}\rho^{n\alpha-2(n-1)}. \end{aligned}$$

Note that

$$\begin{aligned} &f=1+\rho^{\beta}g\left(\rho,x_n\right), \\ &f_\rho=\beta\rho^{\beta-1}g+\rho^{\beta}g_\rho, \\ &f_{\rho\rho}=\beta(\beta-1)\rho^{\beta-2}g+2\beta\rho^{\beta-1}g_\rho+\rho^{\beta}g_{\rho\rho}, \end{aligned}$$

then

$$\begin{aligned} \det D^2u =& \left[\alpha+\alpha\rho^{\beta}g+\beta\rho^{\beta}g+\rho^{\beta+1}g_\rho\right]^{n-2}\rho^{n\alpha-2(n-1)} \\ &\left\{\begin{matrix}\left[\begin{matrix}\alpha(\alpha-1)\left(1+\rho^{\beta}g\right)+2\alpha\left(\beta\rho^{\beta}g+\rho^{\beta+1}g_\rho\right) \\ +\beta(\beta-1)\rho^{\beta}g+2\beta\rho^{\beta+1}g_\rho+\rho^{\beta+2}g_{\rho\rho}\end{matrix}\right]\rho^{\beta}g_{nn} \\ -\left(\alpha\rho^{\beta}g_n+\beta\rho^{\beta}g_n+\rho^{\beta+1}g_{\rho_n}\right)^2\end{matrix}\right\} \\ =& \left[\alpha+(\alpha+\beta)\rho^{\beta}g+\rho^{\beta+1}g_\rho\right]^{n-2}\rho^{n\alpha-2(n-1)+\beta} \\ &\left\{\begin{matrix}\left[\begin{matrix}\alpha(\alpha-1)+(\alpha+\beta)(\alpha+\beta-1)\rho^{\beta}g \\ +2(\alpha+\beta)\rho^{\beta+1}g_\rho+\rho^{\beta+2}g_{\rho\rho}\end{matrix}\right]g_{nn} \\ -\rho^{\beta}\left[(\alpha+\beta)g_n+\rho g_{\rho_n}\right]^2\end{matrix}\right\} \\ =& \left[\alpha+(\alpha+\beta)\rho^{\beta}g+\rho^{\beta+1}g_\rho\right]^{n-2} \\ &\left\{\begin{matrix}\left[\begin{matrix}\alpha(\alpha-1)+(\alpha+\beta)(\alpha+\beta-1)\rho^{\beta}g \\ +2(\alpha+\beta)\rho^{\beta+1}g_\rho+\rho^{\beta+2}g_{\rho\rho}\end{matrix}\right]g_{nn} \\ -\rho^{\beta}\left[(\alpha+\beta)g_n+\rho g_{\rho_n}\right]^2\end{matrix}\right\}, \end{aligned}$$

where we used $n\alpha-2(n-1)+\beta=0$.

We make the following change of variable for $\alpha = q/p$ to move to an analytic equation.

Set

$$s = \rho^{1/p} \quad \text{and} \quad h(s, x_n) = g(s^p, x_n),$$

then

$$\partial_s = ps^{p-1}\partial_\rho \quad \text{or} \quad \partial_\rho = \frac{1}{ps^{p-1}}\partial_s, \quad \text{and} \quad \partial_\rho^2 = \frac{1}{p^2}\left(\frac{1-p}{s^{2p-1}}\partial_s + \frac{1}{s^{2p-2}}\partial_s^2\right).$$

The determinant becomes

$$\det D^2 u = \left[\alpha + (\alpha+\beta)\, s^{p\beta} h + s^{p\beta}\frac{1}{p} s h_s\right]^{n-2} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \alpha(\alpha-1) + (\alpha+\beta)(\alpha+\beta-1)\, s^{p\beta} h \\ +2(\alpha+\beta)\, s^{p\beta}\frac{1}{p} s h_s + s^{p\beta}\frac{1}{p^2}\left[(1-p)\, s h_s + s^2 h_{ss}\right] \end{array} \right\} h_{nn} \\ -s^{p\beta}\left[(\alpha+\beta)\, h_n + \frac{1}{p} s h_{sn}\right]^2 \end{array} \right\}.$$

Now we solve the reduced Monge-Ampère equation

$$\begin{cases} h_{nn} = \dfrac{\left[\alpha + (\alpha+\beta)\, s^{p\beta} h + s^{p\beta}\frac{1}{p} s h_s\right]^{2-n} + s^{p\beta}\left[(\alpha+\beta)\, h_n + \frac{1}{p} s h_{sn}\right]^2}{\alpha(\alpha-1) + (\alpha+\beta)(\alpha+\beta-1)\, s^{p\beta} h + 2(\alpha+\beta)\, s^{p\beta}\frac{1}{p} s h_s + s^{p\beta}\frac{1}{p^2}\left[(1-p)\, s h_s + s^2 h_{ss}\right]}, \\ h_n(s,0) = 0, \\ h(s,0) = 1, \end{cases}$$

where integer $p\beta = 2p(n-1) - nq \geq 0$. Cauchy-Kovalevskaya gives the analytic solution in $B_{r_{q/p}}(0) \subset \mathbb{R}^2$

$$h(s, x_n) = 1 + \frac{1}{2}\frac{\alpha^{2-n}}{\alpha(\alpha-1)} x_n^2 + \cdots.$$

Thus we have a $C^{1,\frac{q}{p}-1}$ solution to $\det D^2 u = 1$

$$u(x', x_n) = |x'|^{\frac{q}{p}}\left[1 + |x'|^{2(n-1)-n\frac{q}{p}}\, h\left(|x'|^{\frac{1}{p}}, x_n\right)\right]$$

in $B_1(0) \subset \mathbb{R}^n$ by scaling.

Exactly as in the Lipschitz case, we verify that our $u(x)$ is a viscosity solution to the Monge-Ampère equation $\det D^2 u = 1$. □

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References

[1] D. B. Bacani, J. E. C. Lope, and H. Tahara, On the unique solvability of nonlinear Fuchsian partial differential equations, Tokyo J. Math., 41 (2018), 225–239.

[2] M. S. Baouendi, and C. Goulaouic, Singular nonlinear Cauchy problems, J. Differential Equations, 22 (1976), 268–291.

[3] L. A. Caffarelli, A note on the degeneracy of convex solutions to Monge Ampère equation, Commun. Partial Differential Equations, 18 (1993), 1213–1217.

[4] L. A. Caffarelli, A priori estimates and the geometry of the Monge Ampère equation, Nonlinear partial differential equations in differential geometry (Park City, UT, 1992), 5–63, IAS/Park City Math. Ser., 2, Amer. Math. Soc., Providence, RI, 1996.

[5] L. A. Caffarelli, Monge Ampère equation div-curl theorems in Lagrangian coordinates compression and rotation, NSF-CBMS Lecture, Florida Atlantic University, 1997.

[6] C. Mooney, Partial regularity for singular solutions to the Monge-Ampère equation, Commun. Pure Appl. Math., 68 (2015), 1066–1084.

[7] C. Mooney, $W^{2,1}$ estimate for singular solutions to the Monge-Ampère equation, Ann. Sc. Norm. Super. Pisa Cl. Sci., 14 (2015), 1283–1303.

[8] A. V. Pogorelov, The Minkowski Multidimensional Problem, Translated from the Russian by Vladimir Oliker. Introduction by Louis Nirenberg. Scripta Series in Mathematics. V. H. Winston & Sons, Washington, D. C.; Halsted Press [John Wiley & Sons], New York-Toronto-London, 1978.

[9] J. I. E. Urbas, Regularity of generalized solutions of Monge-Ampère equations, Math. Z., 197 (1988), 365–393.