Monge-Ampère Equation with Bounded Periodic Data

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Abstract. We consider the Monge-Ampère equation $\det(D^2 u) = f$ in $\mathbb{R}^n$, where f is a positive bounded periodic function. We prove that u must be the sum of a quadratic polynomial and a periodic function. For $f \equiv 1$, this is the classic result by Jörgens, Calabi and Pogorelov. For $f \in C^\alpha$, this was proved by Caffarelli and the first named author.

Key Words: Monge-Ampère equation, Liouville theorem.

AMS Subject Classifications: 53C20, 53C21, 58J05, 35J60

1 Introduction

A classic theorem of Jörgens [17], Calabi [11] and Pogorelov [20] states that any classical convex solution of

$$\det(D^2 u) = 1 \quad \text{in } \mathbb{R}^n$$

must be a quadratic polynomial.

A simpler and more analytical proof, along the lines of affine geometry, was later given by Cheng and Yau [12]. The theorem was extended by Caffarelli [1] to viscosity solutions. Another proof of the theorem was given by Jost and Xin [18]. Trudinger and Wang [21] proved that if Ω is an open convex subset of $\mathbb{R}^n$ and u is a convex C^2 solution of $\det(D^2 u) = 1$ in Ω with $\lim_{x \to \partial\Omega} u(x) = \infty$, then $\Omega = \mathbb{R}^n$. Ferrer, Martínez and Milán [14, 15] extended the above Liouville type theorem in dimension two. Caffarelli and the first named author [8,9] made two extensions, and one of them includes periodic data.

More specifcly, assume for some $a_1, \cdots, a_n > 0$, f satisfies

$$f(x + a_i e_i) = f(x), \quad \forall x \in \mathbb{R}^n, \quad 1 \leq i \leq n, \tag{1.1}$$

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where $e_1 = (1, 0, \cdots, 0), \cdots, e_n = (0, \cdots, 0, 1)$.

Consider the Monge-Ampère equation

$$\det(D^2 u) = f \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

Theorem A ([9]). *Let $f \in C^\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$ with $f > 0$ satisfy* (1.1), *and let $u \in C^2(\mathbb{R}^n)$ be a convex solution of* (1.2). *Then there exist $b \in \mathbb{R}^n$ and a symmetric positive definite $n \times n$ matrix A with*

$$\det A = \fint_{\prod_{1 \le i \le n}[0, a_i]} f,$$

such that

$$v := u - \frac{1}{2} x^T A x - b \cdot x$$

is a_i-periodic in i-th variable, i.e.,

$$v(x + a_i e_i) = v(x), \quad \forall x \in \mathbb{R}^n, \quad 1 \le i \le n.$$

For applications, it is desirable to study the problem with less regularity assumption on f. It was conjectured in [9], see Remark 0.5 there, that Theorem A remains valid for $f \in L^\infty(\mathbb{R}^n)$ satisfying

$$0 < \inf_{\mathbb{R}^n} f \le \sup_{\mathbb{R}^n} f < \infty.$$

We confirm the conjecture in Theorem 1.2 below.

We first recall the definition of a solution of (1.2) in the Alexandrov sense.

Let u be a convex function in an open set Ω of $\mathbb{R}^n$. For $y \in \Omega$, denote

$$\nabla u(y) = \{p \in \mathbb{R}^n | u(x) \ge u(y) + p \cdot (x - y), \ \forall x \in \Omega\}$$

the generalized gradient of u at y.

For $f \in L^\infty(\Omega)$ with $f \ge 0$ a.e., u is called a solution of

$$\det(D^2 u) = f \quad \text{in } \Omega$$

in the Alexandrov sense if u is a convex function in Ω and $|\nabla u(O)| = \int_O f$, for every open set $O \subset \Omega$.

Similarly, for a symmetric $n \times n$ matrix A, we say that $v \in C^{0,1}(\Omega)$ is a solution

$$\det(A + D^2 v) = f \quad \text{in } \Omega$$

in the Alexandrov sense if $u := \frac{1}{2} x^T A x + v$ is convex in Ω and satisfies

$$\det(D^2 u) = f \quad \text{in } \Omega$$

in the Alexandrov sense.

Our first result is the existence and uniqueness of periodic solutions for $f \in L^\infty$.

Theorem 1.1. *Let $f \in L^\infty(\mathbb{R}^n)$ with*

$$0 < \inf_{\mathbb{R}^n} f \le \sup_{\mathbb{R}^n} f < \infty$$

satisfy (1.1) a.e., and let A be a symmetric positive definite $n \times n$ matrix satisfying

$$\det A = \fint_{\prod_{1 \le i \le n}[0,a_i]} f.$$

Then there exists a unique (up to addition of constants) $v \in C^{0,1}(\mathbb{R}^n)$ which is a_i-periodic in the i-th variable, such that

$$\det(A + D^2 v) = f \quad in \ \mathbb{R}^n \tag{1.3}$$

in the Alexandrov sense. Moreover, $v \in C^{1,\alpha}(\mathbb{R}^n)$ for some $0 < \alpha < 1$.

Remark 1.1. If $f \ge 0$, the existence part still holds by passing to limit.

Remark 1.2. If the smoothness assumption of f in Theorem 1.1 is strengthened to $f \in C^{k,\alpha}(\mathbb{R}^n)$, $k \ge 0$, $0 < \alpha < 1$, there exists a solution $v \in C^{k+2,\alpha}(\mathbb{R}^n)$. For $k \ge 4$, the method in [19] is applicable; for $0 \le k \le 3$, this can be established by a smooth approximation of f based on the $C^{2,\alpha}$ theory of Caffarelli in [3], together with the C^0 estimate of solutions in [19]. A different proof of these results under the assumption that $0 < f \in C^{k,\alpha}(\mathbb{R}^n)$, $k \ge 0$, $0 < \alpha < 1$, was given in [5]. Monge-Ampère equations on Hessian manifolds were studied in [13] and [10].

Now we state our main theorem.

Theorem 1.2. *Let $f \in L^\infty(\mathbb{R}^n)$ with*

$$0 < \inf_{\mathbb{R}^n} f \le \sup_{\mathbb{R}^n} f < \infty$$

satisfy (1.1) a.e., and let u be a solution of (1.2) in the Alexandrov sense. Then there exist $b \in \mathbb{R}^n$ and a symmetric positive definite $n \times n$ matrix A with

$$\det A = \fint_{\prod_{1 \le i \le n}[0,a_i]} f,$$

such that

$$v := u - \frac{1}{2} x^T A x - b \cdot x$$

is a_i-periodic in the i-th variable. Moreover, $v \in C^{1,\alpha}(\mathbb{R}^n)$ for some $0 < \alpha < 1$.

Question 1.1. Does the conclusion of the theorem, except for the $C^{1,\alpha}$ regularity of v, still hold if $f \ge 0$?

The main difficulty in proving Theorem 1.2 is that C^2 estimates on u are no longer valid since f is only bounded, which can be seen from the counter examples in [22]. The proof in [9] for Theorem A makes use of the fact that D^2u is uniformly bounded in a non-trivial way, thus we can not carry out the same proof in the current setting. The key observation in our proof is that we can still prove the main propositions in [9] without the uniform bounds of D^2u, which also enables us to simplify the proof of Theorem A in several ways. The proof of Theorem 1.2 follows closely the main steps in [9].

The organization of the paper is as follows: in Section 2, we state two theorems on linearized Monge-Ampère equations established by Caffarelli and Gutiérrez [7] which play crucial roles in the proof of Theorem 1.2. In Section 3, we prove Theorem 1.1 about the existence and uniqueness of solutions on $\mathbb{T}^n$ which is used in the proof of Theorem 1.2. In Section 4, we give the proof of Theorem 1.2. We will mainly focus on the part that is different from [9].

2   Preliminary

In this section, we state two theorems on linearized Monge-Ampère equations.

Theorem B ([7]). *Let Ω be an open convex subset of $\mathbb{R}^n$ satifying $B_1 \subset \Omega \subset B_n$, $n \geq 2$, and let $\phi \in C^2(\bar{\Omega})$ be a convex function satisfying, for some constants λ and Λ,*

$$\begin{cases} 0 < \lambda \leq \det(D^2\phi) \leq \Lambda < \infty & in\ \Omega, \\ \phi = 0 & on\ \partial\Omega. \end{cases}$$

Let $a_{ij} = \det(D^2\phi)\phi^{ij}$ be the linearization of the Monge-Ampère operator at ϕ.

(1) Assume that $v \in C^2(\Omega)$ satisfies

$$a_{ij}v_{ij} \geq f, \quad v \geq 0 \quad in\ \Omega.$$

Then for any $p > 0$, $r > s > 0$, there exists some $C(n,\lambda,\Lambda,p,r,s) > 0$, such that

$$\sup_{x\in\Omega,\ dist(x,\partial\Omega)>r} v \leq C\left(\|v\|_{L^p(x\in\Omega,\ dist(x,\partial\Omega)>s)} + \|f\|_{L^n(x\in\Omega,\ dist(x,\ \partial\Omega)>s)} \right).$$

(2) Assume that $v \in C^2(\Omega)$ satisfies

$$a_{ij}v_{ij} \leq f, \quad v \geq 0 \quad in\ \Omega.$$

Then for $r > s > 0$, there exist $p_0 > 0$ and $C(n,\lambda,\Lambda,p_0,r,s) > 0$, such that

$$\|v\|_{L^{p_0}(x\in\Omega,\ dist(x,\partial\Omega)>s)} \leq C\left(\inf_{x\in\Omega,\ dist(x,\ \partial\Omega)>r} v + \|f\|_{L^n(x\in\Omega,\ dist(x,\ \partial\Omega)>s)} \right).$$

Proof. To prove the theorem, one needs to use sections of the convex function ϕ instead of cubes. More precisely, we notice that Theorem 1 and Theorem 4 in [7] hold for supersolutions, and thus the measure part of the proof of Lemma 4.1 in [7] holds for subsolutions, the rest follows exactly those of Theorem 4.8 in [6]. We remark that (1) is called local maximum principle and (2) is called weak Harnack inequality in literature. □

Theorem C ([7]). *Let Ω and $\tilde{\Omega}$ be open convex subsets of $\mathbb{R}^n$ satisfying $B_1 \subset \Omega, \tilde{\Omega} \subset B_n$, $n \geq 2$, and let $\phi \in C^2(\bar{\Omega})$ and $\tilde{\phi} \in C^2(\bar{\tilde{\Omega}})$ be convex functions satisfying, for some constants λ and Λ,*

$$\begin{cases} 0 < \lambda \leq \det(D^2\phi) \leq \Lambda < \infty & in \ \Omega, \\ \phi = 0 & on \ \partial\Omega, \end{cases}$$

$$\begin{cases} 0 < \lambda \leq \det(D^2\tilde{\phi}) \leq \Lambda < \infty & in \ \tilde{\Omega}, \\ \tilde{\phi} = 0 & on \ \partial\tilde{\Omega}. \end{cases}$$

Let

$$a_{ij} = \det(D^2\phi)\phi^{ij} \quad and \quad \tilde{a}_{ij} = \det(D^2\tilde{\phi})\tilde{\phi}^{ij}$$

be the linearizations of the Monge-Ampère operator at ϕ and $\tilde{\phi}$ respectively.
    Assume that $v \in C^2(\Omega)$ with $v \geq 0$ satisfies

$$\begin{cases} a_{ij}v_{ij} \geq 0 & in \ \Omega, \\ \tilde{a}_{ij}v_{ij} \leq 0 & in \ \tilde{\Omega}. \end{cases}$$

Let $O \subset \bar{O} \subset \Omega \cap \tilde{\Omega}$ be an open set, then there exist constants $\alpha(n, \lambda, \Lambda, O)$ and $C(n, \lambda, \Lambda, O)$ such that

$$\sup_O v \leq C \inf_O v,$$

$$\|v\|_{C^\alpha(O)} \leq C.$$

Proof. Apply (1) and (2) of Theorem B to v with $f = 0$, we obtain

$$\sup_{\hat{O}} v \leq C \inf_{\hat{O}} v$$

for an open set $\hat{O}$ satisfying

$$O \subset \bar{O} \subset \hat{O} \subset \bar{\hat{O}} \subset \Omega \cap \tilde{\Omega}.$$

It follows that

$$\|v\|_{C^\alpha(O)} \leq C.$$

Thus, we complete the proof. □

3   Proof of Theorem 1.1

We now prove Theorem 1.1. This is based on the result in [19], together with the regularity theory of Caffarelli [4].

Proof. Since Monge-Ampère equations are affine invariant, we may assume without loss of generality that $a_i = 1$ for all i, and f satisfies

$$\fint_{[0,1]^n} f = 1.$$

For convenience, we identify periodic functions as functions on $\mathbb{T}^n$.
   We first establish the existence part. Let

$$\rho \in C_c^\infty(B_1), \quad \int_{B_1} \rho = 1.$$

For $\epsilon > 0$, $\rho_\epsilon(x) = \epsilon^{-n}\rho(\epsilon x)$, let

$$f_\epsilon(x) = \int_{\mathbb{R}^n} \rho_\epsilon(x - y)f(y)dy \tag{3.1}$$

be the mollification of f. It is clear that f_ϵ is periodic. Define

$$\tilde{f}_\epsilon = f_\epsilon - \fint_{\mathbb{T}^n} f_\epsilon + \det A.$$

It follows that

$$\fint_{\mathbb{T}^n} \tilde{f}_\epsilon = \det A.$$

By Theorem 2.2 in [19], there exists a unique function $\tilde{v}_\epsilon \in C^\infty(\mathbb{T}^n)$ with

$$(A + D^2\tilde{v}_\epsilon) > 0, \quad \int_{\mathbb{T}^n} \tilde{v}_\epsilon = 0,$$

satisfying

$$\det(A + D^2\tilde{v}_\epsilon) = \tilde{f}_\epsilon \quad \text{on } \mathbb{T}^n$$

and $|\tilde{v}_\epsilon| + |\nabla\tilde{v}_\epsilon| \le C(A)$ on $\mathbb{T}^n$. Passing to a subsequence, $\tilde{v}_\epsilon \to v$ in $C^0(\mathbb{T}^n)$ and v is a solution of (1.3) in the Alexandrov sense, see e.g., Proposition 2.6 in [16]. The $C^{1,\alpha}$ regularity of v for some $\alpha \in (0,1)$ follows from Theorem 2 in [4].
   Now we establish the uniqueness part. Suppose that there exist two solutions v and $\hat{v}$. Without loss of generality, assume

$$\min_{\mathbb{T}^n}(v - \hat{v}) = 0.$$

Then

$$u(x) := \frac{1}{2}x^T A x + v(x) \quad \text{and} \quad \hat{u}(x) := \frac{1}{2}x^T A x + \hat{v}(x)$$

are solutions of (1.2) in the Alexandrov sense.

Since v is bounded, we can find $M > 0$ large enough such that

$$\Omega_M = \{x \in \mathbb{R}^n | u(x) < M\}$$

contains $[-2,2]^n$.

Let

$$u_\epsilon \in C^0(\bar{\Omega}_M) \cap C^\infty(\Omega_M)$$

be the solution of the following Dirichlet problem (see e.g., Proposition 2.4 in [8])

$$\begin{cases} \det(D^2 u_\epsilon(x)) = f_\epsilon(x) & \text{in } \Omega_M, \\ u_\epsilon(x) = M & \text{on } \partial\Omega_M. \end{cases}$$

By a barrier argument,

$$u_\epsilon(x) - M \geq -C dist(x, \partial\Omega_M)^{\frac{2}{n}},$$

if $n \geq 3$ and

$$u_\epsilon(x) - M \geq -C dist(x, \partial\Omega_M)^\alpha$$

for some $0 < \alpha < 1$ if $n = 2$, see e.g., [2] or Lemma A.1 in [8]. Since $f_\epsilon \to f$ in $L^1(\Omega_M)$ as $\epsilon \to 0$, it follows that $u_\epsilon \to \tilde{u}$ in $C^0(\bar{\Omega}_M)$ along a subsequence as $\epsilon \to 0$. As mentioned earlier $\tilde{u}$ satisfies $\det(D^2\tilde{u}) = f$ in the Alexandrov sense. By the uniqueness of solution to Dirichlet problem in the Alexandrov sense, e.g., Corollary 2.11 in [16], we have $\tilde{u} = u$.

Similarly there exists a convex solution $\hat{u}_\epsilon \in C^\infty([-2,2]^n)$ satisfying

$$\det(D^2 \hat{u}_\epsilon(x)) = f_\epsilon(x) \quad \text{in } [-2,2]^n$$

with $\hat{u}_\epsilon \to \hat{u}$ in $C^0([-2,2]^n)$.

For any function w, denote

$$F(D^2 w) = \det^{\frac{1}{n}}(D^2 w) \quad \text{and} \quad F_{ij}(D^2 w) = \frac{\partial F}{\partial w_{ij}},$$

since F is concave, we have

$$F(D^2 \hat{u}_\epsilon) \leq F(D^2 u_\epsilon) + F_{ij}(D^2 u_\epsilon)\partial_{ij}(\hat{u}_\epsilon - u_\epsilon),$$

i.e.,

$$F_{ij}(D^2 u_\epsilon)\partial_{ij}(u_\epsilon - \hat{u}_\epsilon) \leq 0.$$

Similarly,

$$F(D^2 u_\epsilon) \le F(D^2 \hat{u}_\epsilon) + F_{ij}(D^2 \hat{u}_\epsilon) \partial_{ij}(u_\epsilon - \hat{u}_\epsilon),$$

i.e.,

$$F_{ij}(D^2 \hat{u}_\epsilon) \partial_{ij}(u_\epsilon - \hat{u}_\epsilon) \ge 0.$$

Let

$$\delta_\epsilon = \min_{[-2,2]^n}(u_\epsilon - \hat{u}_\epsilon),$$

then

$$u_\epsilon - \hat{u}_\epsilon - \delta_\epsilon \ge 0 \quad \text{on } [-2,2]^n.$$

Now by Theorem B and Theorem C, we have

$$\max_{[-1,1]^n}(u_\epsilon - \hat{u}_\epsilon - \delta_\epsilon) \le C \min_{[-1,1]^n}(u_\epsilon - \hat{u}_\epsilon - \delta_\epsilon).$$

Let $\epsilon \to 0$, we have

$$\lim_{\epsilon \to 0} \delta_\epsilon = 0 \quad \text{on } \mathbb{T}^n \quad \text{as} \quad \min_{\mathbb{T}^n}(v - \hat{v}) = 0.$$

It follows that

$$\max_{[-1,1]^n}(u - \hat{u}) \le C \min_{[-1,1]^n}(u - \hat{u}) = 0.$$

Thus $u = \hat{u}$ on $[-1,1]^n$. It follows that $v = \hat{v}$ on $\mathbb{T}^n$. The theorem is now proved. $\square$

4 Proof of Theorem 1.2

We now start to prove Theorem 1.2. As mentioned in the introduction, we will follow the main steps in [9].

By the affine invariance of the problem, we may assume without loss of generality that $a_i = 1$ for all i, and f satisfies

$$\int_{[0,1]^n} f = 1.$$

We first note that Proposition 2.1 and its proof in [9] still hold in the current setting.

Proposition 4.1. *There exist a symmetric positive definite $n \times n$ matrix A with $\det A = 1$ and postive constants δ and C_1, such that*

$$\left| u(x) - \frac{1}{2} x^T A x \right| \le C_1 |x|^{2-\delta}, \quad \forall |x| \ge 1.$$

For nonzero $e \in \mathbb{R}^n$, as in [9], we define the second incremental quotient,

$$\Delta_e^2 u(x) = \frac{u(x+e) + u(x-e) - 2u(x)}{\|e\|^2}$$

where $\|e\|$ denotes the Euclidean norm of e.

Let

$$E = \{k_1 e_1 + \cdots + k_n e_n; \ k_1, \cdots, k_n \text{ are integers}, \ k_1^2 + \cdots + k_n^2 > 0\}.$$

The following is analogous to Lemma 2.4 in [9].

Proposition 4.2.

$$\gamma := \sup_{e \in E} \sup_{y \in \mathbb{R}^n} \Delta_e^2 u(y) < \infty.$$

Proof. We will follow the main steps as in [9], with some modifications.

For any large $M > 0$, define

$$\Omega_M = \{x \in \mathbb{R}^n | u(x) < M\}.$$

By John's lemma, there exists an affine transformation

$$A_M = a_M x + b_M,$$

such that

$$B_R \subset A_M(\Omega_M) \subset B_{nR}$$

with $\det a_M = 1$. Denote

$$O_M = \frac{1}{R} a_M(\Omega_M).$$

Define

$$u_M(x) = \frac{1}{R^2} u(a_M^{-1}(Rx)), \quad x \in O_M.$$

Now for $e \in E$ and $y \in \mathbb{R}^n$, let $x = \frac{1}{R} a_M(y)$. Take M large so that $y \in \Omega_{\frac{M}{2}}$. It follows from Proposition 4.1 that

$$dist(x, \partial O_M) \geq \frac{1}{C_0},$$

where C_0 depends only on $n, \inf f$ and $\sup f$.

Let $\tilde{e} = \frac{1}{R} a_M(e)$, then

$$
\begin{aligned}
\Delta_e^2 u(y) &= \frac{u(y+e) + u(y-e) - 2u(y)}{\|e\|^2} \\
&= \frac{u(a_M^{-1}(R(x+\tilde{e}))) + u(a_M^{-1}(R(x-\tilde{e}))) - 2u(a_M^{-1}(Rx))}{\|e\|^2} \\
&= \frac{R^2 \|\tilde{e}\|^2}{\|e\|^2} \Delta_{\tilde{e}}^2 u_M(x) \\
&= \frac{\|a_M(e)\|^2}{\|e\|^2} \Delta_{\tilde{e}}^2 u_M(x).
\end{aligned}
$$

In the rest of the proof, we use C to denote various positive constants depending only on n, $\inf f$, $\sup f$ and the constants δ and C_1 in Proposition 4.1.

By Proposition 4.1,

$$
C^{-1} \le \frac{M}{R^2} \le C, \quad \|a_M\| \le C.
$$

The proposition will follow as long as $\Delta_{\tilde{e}}^2 u_M(x) \le C$ for $dist(x, \partial O_M) \ge \frac{1}{C_0}$.

We now prove $\Delta_{\tilde{e}}^2 u_M(x) \le C$.

Note that $u_M(x)$ satisfies

$$
\begin{cases}
\det(D^2 u_M(x)) = f(a_M^{-1}(Rx)) & \text{in } O_M, \\
u_M(x) = \dfrac{M}{R^2} & \text{on } \partial O_M.
\end{cases}
$$

Let f_ϵ be the mollification of f given by (3.1) and let $u_{M,\epsilon}(x)$ be the solution of the following Dirichlet problem

$$
\begin{cases}
\det(D^2 u_{M,\epsilon}(x)) = f_\epsilon(a_M^{-1}(Rx)) & \text{in } O_M, \\
u_{M,\epsilon}(x) = \dfrac{M}{R^2} & \text{on } \partial O_M.
\end{cases}
$$

As in the proof of Theorem 1.1, we have $u_{M,\epsilon} \to u_M$ in $C^0(\bar{O}_M)$ as $\epsilon \to 0$.

By Lemma 2.2 in [9], $u_{M,\epsilon}$ satisfies

$$
F_{ij}(D^2 u_{M,\epsilon}(x)) \partial_{ij}\left(\Delta_{\tilde{e}}^2 u_{M,\epsilon}(x)\right) \ge 0, \quad x \in \Omega_M, \quad dist(x, \partial O_M) \ge \frac{1}{8C_0}. \tag{4.1}
$$

By Lemma A.1 in [9], we have

$$
\int_{x \in O_M, dist(x, \partial O_M) \ge \frac{1}{4C_0}} \Delta_{\tilde{e}}^2 u_{M,\epsilon}(x) \le C.
$$

Together with Theorem B, we have

$$\Delta^2_{\bar{e}} u_{M,\,\epsilon}(x) \leq C$$

for $x \in O_M$ with

$$dist(x,\, \partial O_M) \geq \frac{1}{2C_0}.$$

Let $\epsilon \to 0$, we have

$$\Delta^2_{\bar{e}} u_M(x) \leq C$$

for $x \in O_M$ with

$$dist(x,\partial O_M) \geq \frac{1}{C_0}.$$

The proposition is now proved. $\square$

For $\lambda \geq 1$ and any function v, let

$$v^\lambda(x) = \frac{v(\lambda x)}{\lambda^2}, \quad x \in \mathbb{R}^n.$$

Denote

$$Q(x) = \frac{1}{2}x^T A x.$$

Lemma 4.1. *There exists a constant $\mu \in (0,1)$ such that,*

$$u^\lambda \to Q \quad in \ \ C^{1,\mu}_{loc}(\mathbb{R}^n) \quad as \ \ \lambda \to \infty.$$

Proof. By Proposition 4.1, $u^\lambda \to Q$ in $C^0_{loc}(\mathbb{R}^n)$ as $\lambda \to \infty$. For $r > 0$, denote $D_r = \{x \in \mathbb{R}^n | Q(x) < r^2\}$. There exists $\lambda_1 > 0$ such that for $\lambda \geq \lambda_1$,

$$D_{\frac{3}{2}} \subset \{u^\lambda < 4\} =: \Omega_{4,\lambda} \subset D_{\frac{5}{2}}.$$

We know that

$$\det(D^2 u^\lambda) = f(\lambda x)$$

in $\Omega_{4,\lambda}$ in the Alexandrov sense. By Theorem 2 in [4], there exist $\mu' \in (0,1)$ and $C \geq 1$ depending only on n, $\inf f$, $\sup f$ and A such that

$$\|u^\lambda\|_{C^{1,\mu'}(D_{\frac{4}{3}})} \leq C.$$

Thus we have $u^\lambda \to Q$ in $C^{1,\mu}(D_1)$ for $0 < \mu < \mu' < 1$. The lemma follows given the fact that

$$u^\lambda(x) = a^2 u^{\lambda a}\left(\frac{x}{a}\right)$$

for all $a, \lambda > 0$ and $x \in \mathbb{R}^n$. $\square$

The following proposition is Proposition 2.3 in [9].

Proposition 4.3.

$$\sup_{\mathbb{R}^n} \Delta_e^2 u = \frac{e'Ae}{\|e\|^2}, \quad \forall e \in E.$$

Proof. Denote

$$\alpha = \sup_{\mathbb{R}^n} \Delta_e^2 u, \quad \beta = \frac{e'Ae}{\|e\|^2}.$$

For $\lambda > 0$, $\hat{e} = \frac{e}{\lambda}$, by strict convexity (see e.g., [2]) and Proposition 4.2, we have

$$0 < \Delta_{\hat{e}}^2 u^\lambda(x) = \Delta_e^2 u(\lambda x) \le \alpha < \infty, \quad x \in \mathbb{R}^n.$$

By Lemma A.2 in [9] and Lemma 4.1, we have

$$\lim_{\lambda \to \infty} \int_{B_1} \Delta_{\hat{e}}^2 u^\lambda dx = \int_{B_1} \beta dx = \beta |B_1|.$$

Thus $\alpha \ge \beta$. Now suppose $\alpha > \beta$, let $\beta < \beta' < \alpha' < \alpha'' < \alpha$, we have

$$\limsup_{\lambda \to \infty} \left(\alpha' |\{\Delta_{\hat{e}}^2 u^\lambda \ge \alpha'\} \cap B_1| \right) \le \lim_{\lambda \to \infty} \int_{B_1} \Delta_{\hat{e}}^2 u^\lambda dx = \beta |B_1|.$$

Thus for all large λ, we have

$$\alpha' |\{\Delta_{\hat{e}}^2 u^\lambda \ge \alpha'\} \cap B_1| \le \beta' |B_1|,$$

i.e.,

$$\frac{|\{\Delta_{\hat{e}}^2 u^\lambda \le \alpha'\} \cap B_1|}{|B_1|} \ge \frac{\alpha' - \beta'}{\alpha'}.$$

For $M > 0$, denote

$$\Omega_{M,\lambda} = \{x \in \mathbb{R}^n | u^\lambda(x) < M\}.$$

By Lemma 4.1, there exist M, λ_1 such that for $\lambda > \lambda_1$, we have $B_2 \subset \Omega_{M,\lambda}$.

As in the proof of Proposition 4.2, let f_ϵ be the mollification of f given by (3.1), let $u_{M,\epsilon}^\lambda(x)$ be the solution of the following Dirichlet problem

$$\begin{cases} \det(D^2 u_{M,\epsilon}^\lambda(x)) = f_\epsilon(\lambda x) & \text{in } \Omega_{M,\lambda}, \\ u_{M,\epsilon}^\lambda(x) = M & \text{on } \partial\Omega_{M,\lambda}. \end{cases}$$

Then we have $u^\lambda_{M,\,\epsilon} \to u^\lambda$ in $C^0(\bar{\Omega}_{M,\,\lambda})$ as $\epsilon \to 0$, see in the proof of Theorem 1.1.

For ϵ small enough, we have

$$\frac{|\{\Delta^2_{\hat{e}} u^\lambda_{M,\,\epsilon} \leq \alpha''\} \cap B_1|}{|B_1|} \geq \frac{\alpha' - \beta'}{\alpha'}.$$

By (4.1), $\Delta^2_{\hat{e}} u^\lambda_{M,\,\epsilon}$ is a subsolution of the linearized Monge-Ampère equation at $u^\lambda_{M,\epsilon}$.

Apply Theorem B, we have, for some $p_0 > 0$ and $C > 0$,

$$\|\alpha - \Delta^2_{\hat{e}} u^\lambda_{M,\epsilon}\|_{L^{p_0}(B_1 \cap \{\Delta^2_{\hat{e}} u^\lambda_{M,\epsilon} \leq \alpha''\})} \leq \|\alpha - \Delta^2_{\hat{e}} u^\lambda_{M,\epsilon}\|_{L^{p_0}(B_1)} \leq C \inf_{B_{\frac{3}{4}}} \left(\alpha - \Delta^2_{\hat{e}} u^\lambda_{M,\epsilon}\right).$$

Consequently,

$$(\alpha - \alpha'')|B_1 \cap \{\Delta^2_{\hat{e}} u^\lambda_{M,\epsilon} \leq \alpha''\}|^{\frac{1}{p_0}} \leq C \inf_{B_{\frac{3}{4}}} \left(\alpha - \Delta^2_{\hat{e}} u^\lambda_{M,\epsilon}\right).$$

Therefore,

$$\sup_{B_{\frac{3}{4}}} \Delta^2_{\hat{e}} u^\lambda_{M,\epsilon} \leq \alpha - C^{-1}$$

for all $\lambda > \lambda_1$. Let $\epsilon \to 0$, then

$$\sup_{B_{\frac{\lambda}{2}}} \Delta^2_e u = \sup_{B_{\frac{1}{2}}} \Delta^2_{\hat{e}} u^\lambda \leq \alpha - C^{-1}$$

for all $\lambda > \lambda_1$.

This contradicts the definition of α. Thus we have

$$\sup_{\mathbb{R}^n} \Delta^2_e u = \frac{e'Ae}{\|e\|^2}.$$

This completes the proof. $\square$

To proceed, we choose $b \in \mathbb{R}^n$ such that

$$w(e_k) = w(-e_k), \quad 1 \leq k \leq n,$$

where

$$w(x) := u(x) - \frac{1}{2}x^T A x - b \cdot x.$$

By Theorem 1.1, there exists $v \in C^{0,1}(\mathbb{R}^n)$ which is 1-periodic satisfying $\det(A + D^2v) = f$ in the Alexandrov sense. Choose v such that $v(0) = w(0)$.

Define

$$h = w - v. \tag{4.2}$$

Then we have $h(0) = 0$. We now prove that h is bounded from above.

Lemma 4.2.

$$\sup_{\mathbb{R}^n} h < \infty.$$

Proof. We follow the proof of Lemma 2.9 in [9]. On the other hand, since uniform C^2 estimates are not available for $f \in L^\infty$, we need to provide new arguments in several places.

Let

$$M_i = \sup_{x \in [-i,i]^n} h(x), \quad i = 1, 2, \cdots.$$

Suppose h is not bounded above, then we have

$$\lim_{i \to \infty} M_i = \infty.$$

We claim that for some constant C independent of i, we have

$$M_{2^i} \leq 4M_{2^{i-1}} + C, \quad \forall i = 1, 2, \cdots. \tag{4.3}$$

First of all, since both w and v are locally Lipschitz and $h(0) = 0$, we have

$$|h(x)| \leq C, \quad \forall x \in [-1, 1]^n.$$

Now for $x = (x_1, \cdots, x_n) \in [-m, m]^n$, where m is an integer, let $[x_k]$ be the integer part of x_k. Define

$$\epsilon_k = \begin{cases} 1, & \text{if } [x_k] \text{ is odd,} \\ 0, & \text{if } [x_k] \text{ is even.} \end{cases}$$

Then by Proposition 4.3, we have

$$\Delta_e^2 h = \Delta_e^2 w \leq 0 \quad \text{in } \mathbb{R}^n, \quad e \in E. \tag{4.4}$$

Thus

$$h(x) + h\Big(x - \sum_{k=1}^n ([x_k] + \epsilon_k)e_k\Big) \leq 2h\Big(x - \sum_{k=1}^n \frac{[x_k] + \epsilon_k}{2}e_k\Big).$$

Since

$$x - \sum_{k=1}^n ([x_k] + \epsilon_k)e_k \in [-1, 1]^n,$$

$$x - \sum_{k=1}^n \frac{[x_k] + \epsilon_k}{2}e_k \in \Big[-\Big[\frac{m+1}{2}\Big] - 1, \Big[\frac{m+1}{2}\Big] + 1\Big]^n,$$

we have

$$h(x) \le 2M_{[\frac{m+1}{2}]+1} + C.$$

It follows that

$$M_m \le 2M_{[\frac{m+1}{2}]+1} + C.$$

Taking $m = 2^i$, we have proved (4.3). Let

$$H_i(x) = \frac{h(2^i x)}{M_{2^i}}, \quad x \in [-1,1]^n.$$

By Lemma A.3 in [9], (4.4) and the fact that $h(0) = 0$, $h(e_k) = h(-e_k)$, we have

$$H_i(\pm \frac{1}{2} e_k) = \frac{h(\pm 2^{i-1} e_k)}{M_{2^i}} \le 0, \quad 1 \le k \le n, \quad i = 1, 2, \cdots. \tag{4.5}$$

By (4.3), we have

$$\max_{[-\frac{1}{2},\frac{1}{2}]^n} H_i = \frac{M_{2^{i-1}}}{M_{2^i}} \ge \frac{M_{2^i} - C}{4M_{2^i}} \ge \frac{1}{8} \tag{4.6}$$

for large i.

By the definition of H_i,

$$H_i \le 1 \quad \text{on } [-1,1]^n$$

and

$$H_i(0) = \frac{h(0)}{M_{2^i}} = 0.$$

Claim: Let $0 < b' < b \le 1$, if $l(x) - H_i \ge 0$ in $[-b, b]^n$ for a linear function $l(x)$, then for some positive constants α and C independent of i and $l(x)$, we have

$$\max_{[-b',b']^n} (l - H_i) \le C \min_{[-b',b']^n} (l - H_i),$$

and

$$\|H_i\|_{C^\alpha([-b',b']^n)} \le C.$$

We now prove the claim.

Proof. Recall that v in (4.2) is the unique solution of $\det(A + D^2v) = f$ in $\mathbb{T}^n$ satisfying $v(0) = w(0)$.

As in the proof of Theorem 1.1, denote

$$\tilde{f}_\epsilon = f_\epsilon - \fint_{\mathbb{T}^n} f_\epsilon + \det A.$$

Let $\tilde{v}_\epsilon$ be the unique function with $(A + D^2\tilde{v}_\epsilon) > 0$ satisfying

$$\begin{cases} \det(A + D^2\tilde{v}_\epsilon) = \tilde{f}_\epsilon & \text{in } \mathbb{T}^n, \\ \tilde{v}_\epsilon(0) = v(0) = w(0). \end{cases}$$

Since $|\nabla v_\epsilon| \leq C(A)$ and $\tilde{f}_\epsilon \to f$ in $C^0(\mathbb{T}^n)$ as $\epsilon \to 0$, by the uniqueness of solution of $\det(A + D^2v) = f$ on $\mathbb{T}^n$ in the Alexandrov sense, we have $\tilde{v}_\epsilon \to v$ in $C^0(\mathbb{T}^n)$ as $\epsilon \to 0$.

For i fixed, denote

$$\Omega_i = \{x \in \mathbb{R}^n | u(x) < C2^{2i-1}\},$$

where C is a fixed constant greater than the largest eigenvalue of A. By Proposition 4.1, we have $[-2^i, 2^i]^n \subset \Omega_i \subset [-C2^i, C2^i]^n$, where C is another constant depending only on A.

Let $\tilde{u}_\epsilon$ be the solution of the following Dirichlet problem

$$\begin{cases} \det(D^2\tilde{u}_\epsilon(x)) = \tilde{f}_\epsilon(x) & \text{in } \Omega_i, \\ \tilde{u}_\epsilon(x) = M & \text{on } \partial\Omega_i. \end{cases}$$

As before, $\tilde{f}_\epsilon \to f$ in $C^0(\mathbb{T}^n)$ as $\epsilon \to 0$, and we have $\tilde{u}_\epsilon \to u$ in $C^0(\bar{\Omega}_i)$ as $\epsilon \to 0$.

Denote

$$\tilde{h}_\epsilon(x) = \tilde{u}_\epsilon(x) - \frac{1}{2}x^T Ax - bx - \tilde{v}_\epsilon(x).$$

It follows that $\tilde{h}_\epsilon \to h$ in $C^0(\bar{\Omega}_i)$ as $\epsilon \to 0$.

Recall that

$$F(A + D^2\tilde{v}_\epsilon) \leq F(D^2\tilde{u}_\epsilon) + F_{ij}(D^2\tilde{u}_\epsilon)(A_{ij} + \partial_{ij}\tilde{v}_\epsilon - \partial_{ij}\tilde{u}_\epsilon),$$

i.e.,

$$F_{ij}(D^2\tilde{u}_\epsilon)\partial_{ij}\tilde{h}_\epsilon \leq 0. \tag{4.7}$$

Similarly,

$$F(D^2\tilde{u}_\epsilon) \leq F(A + D^2\tilde{v}_\epsilon) + F_{ij}(A + D^2\tilde{v}_\epsilon)(\partial_{ij}\tilde{u}_\epsilon - A_{ij} - \partial_{ij}\tilde{v}_\epsilon),$$

i.e.,

$$F_{ij}(A + D^2 \tilde{v}_\epsilon) \partial_{ij} \tilde{h}_\epsilon \geq 0. \tag{4.8}$$

Define

$$\tilde{H}_{\epsilon i}(x) = \frac{\tilde{h}_\epsilon(2^i x)}{M_{2^i}}, \quad x \in [-1,1]^n.$$

Then $\tilde{H}_{\epsilon i} \to H_i$ in $C^0([-1,1]^n)$ as $\epsilon \to 0$. For any $\delta > 0$, we have $l + \delta - \tilde{H}_{\epsilon i}$ is nonnegative in $[-b,b]^n$ for all ϵ small enough.

By (4.7) and (4.8), we have

$$F_{ij}(A + D^2 \tilde{v}_\epsilon) \partial_{ij} \left(l + \delta - \tilde{H}_{\epsilon i}\right) \geq 0 \quad \text{in } \frac{1}{2^i} \Omega_i,$$

$$F_{ij}(D^2 \tilde{u}_\epsilon) \partial_{ij} \left(l + \delta - \tilde{H}_{\epsilon i}\right) \leq 0 \quad \text{in } \frac{1}{2^i} \Omega_i.$$

By our choice of Ω_i, we have

$$[-1,1]^n \subset \frac{1}{2^i} \Omega_i \subset [-C,C]^n.$$

By Theorem C, we have

$$\max_{[-b',b']^n} (l + \delta - \tilde{H}_{\epsilon i}) \leq C(l + \delta - \tilde{H}_{\epsilon i}(0)) \leq 2C,$$

$$\|l + \delta - \tilde{H}_{\epsilon i}\|_{C^\alpha([-b',b']^n)} \leq C,$$

where α, C only depends on n, λ, Λ and A, in particular, α, C does not depend on ϵ and i. The claim is now proved after sending ϵ to 0. $\qquad \square$

It follows that there exist some $0 < \alpha' < \alpha < 1$ and H such that

$$H_i \to H \quad \text{in } C^{\alpha'}\left(\left[-\tfrac{3}{4}, \tfrac{3}{4}\right]^n\right) \quad \text{along a subsequence } i \to \infty.$$

By (4.6), we have

$$\max_{[-\frac{1}{2},\frac{1}{2}]^n} H \geq \frac{1}{8}. \tag{4.9}$$

By (4.5), we have

$$H\left(\pm \frac{1}{2} e_k\right) \leq 0, \quad 1 \leq k \leq n. \tag{4.10}$$

We also know that

$$H(0) = \lim_{i \to \infty} H_i(0) = 0.$$

By (4.4),

$$\Delta^2_{2^{-i}e} H_i = \frac{\Delta^2_e h}{M_{2_i}} \le 0, \quad \forall e \in E.$$

It follows that H is concave. We can then find a linear function l such that $l - H \ge 0$ in $[-\frac{3}{4}, \frac{3}{4}]^n$ with $l(0) = 0$. By the convergence of H_i to H, there exist constants $\delta_i \to 0$ such that $l_i(x) = l(x) + \delta_i$ satisfies $l_i - H_i \ge 0$ in $[-\frac{3}{4}, \frac{3}{4}]^n$. Applying the earlier claim to $l_i - H_i$ with $b = \frac{3}{4}$ and $b' = \frac{1}{2}$, and then sending i to ∞, we conclude that

$$\max_{[-\frac{1}{2}, \frac{1}{2}]^n} (l - H) \le C(l(0) - H(0)) = 0.$$

Thus

$$H = \sum_{k=1}^{n} c_k x_k \quad \text{on} \quad \left[-\frac{1}{2}, \frac{1}{2} \right]^n.$$

Now by (4.10), we conclude that $c_k = 0$, i.e., $H \equiv 0$. However, this violates (4.9). The lemma is now proved. $\qquad \square$

Proof of Theorem 1.2. By Lemma 4.2, there exists some constant a such that

$$\inf_{\mathbb{R}^n}(a - h) = 0.$$

Since

$$\frac{a}{M_{2^i}} - H_i = \frac{a - h(2^i x)}{M_{2^i}} \ge 0,$$

by the earlier claim, there exists some constant C such that

$$\max_{[-\frac{1}{2}, \frac{1}{2}]^n} \left(\frac{a}{M_{2^i}} - H_i \right) \le C \min_{[-\frac{1}{2}, \frac{1}{2}]^n} \left(\frac{a}{M_{2^i}} - H_i \right)$$

for all large i.

Namely,

$$\max_{[-2^{i-1}, 2^{i-1}]^n} (a - h) \le C \min_{[-2^{i-1}, 2^{i-1}]^n} (a - h)$$

for all large i.

It follows that

$$\sup_{\mathbb{R}^n}(a - h) \le C \inf_{\mathbb{R}^n}(a - h) = 0.$$

Thus $h \equiv a$, i.e.,

$$u \equiv \frac{1}{2} x^T A x + b \cdot x + a + v.$$

Thus, we complete the proof. $\qquad \square$

Acknowledgements

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