

## Multiple Sign-Changing Solutions for Quasilinear Equations of Bounded Quasilinearity

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

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**Abstract.** The existence of an infinite sequence of sign-changing solutions are proved for a class of quasilinear elliptic equations under suitable conditions on the quasilinear coefficients and the nonlinearity

$$\begin{cases} \sum_{i,j=1}^N \left( b_{ij}(u) D_{ij} u + \frac{1}{2} D_z b_{ij}(u) D_i u D_j u \right) + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded domain with smooth boundary, and we use

$$D_i u = \frac{\partial u}{\partial x_i}, \quad D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \text{and} \quad D_z b_{ij}(z) = \frac{d}{dz} b_{ij}(z).$$

The main interest of this paper is for the case of bounded quasilinearity  $b_{ij}$ . The result is proved by an elliptic regularization method involving truncations of both  $u$  and the gradient of  $u$ .

**Key Words:** Quasilinear elliptic equations, sign-changing solution, an elliptic regularization method.

**AMS Subject Classifications:** 35J60, 35J20

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## 1 Introduction

In this paper, we study the existence of sign-changing solutions for the following quasi-linear elliptic equation

$$\begin{cases} \sum_{i,j=1}^N \left( b_{ij}(u) D_{ij}u + \frac{1}{2} D_z b_{ij}(u) D_i u D_j u \right) + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded domain with smooth boundary, and we use the notations

$$D_i u = \frac{\partial u}{\partial x_i}, \quad D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad D_z b_{ij}(z) = \frac{d}{dz} b_{ij}(z).$$

We assume the following conditions on  $b_{ij}$  and  $f$ . Denote the critical exponent by  $2^* = \frac{2N}{N-2}$  for  $N \geq 3$  and  $2^* = +\infty$  for  $N = 1, 2$ .

( $b_1$ ) Let  $b_{ij} = b_{ji} \in C^{1,1}(\mathbb{R}, \mathbb{R})$  for  $i, j = 1, \dots, N$ , satisfy that there exist positive constants  $b_-, b_+$  such that

$$b_- |\xi|^2 \leq \sum_{i,j=1}^N b_{ij}(z) \xi_i \xi_j \leq b_+ |\xi|^2 \quad \text{for } z \in \mathbb{R}, \quad \xi = (\xi_i) \in \mathbb{R}^N.$$

( $b_2$ ) There exist constants  $q > 2$ ,  $\delta > 0$  such that

$$\begin{aligned} \delta |\xi|^2 &\leq \sum_{i,j=1}^N \left( b_{ij}(z) + \frac{1}{2} z D_z b_{ij}(z) \right) \xi_i \xi_j \\ &\leq \frac{q}{2} \left( \sum_{i,j=1}^N b_{ij}(z) \xi_i \xi_j - \delta |\xi|^2 \right) \quad \text{for } z \in \mathbb{R}, \quad \xi \in \mathbb{R}^N. \end{aligned}$$

( $b_3$ ) There exists a positive constant  $c$  such that

$$|D_z b_{ij}(z) - D_z b_{ij}(w)| \leq c |z - w| \quad \text{for } z, w \in \mathbb{R}.$$

( $b_4$ )  $b_{ij}(z)$  is even in  $z$ .

( $f_1$ ) Let  $f \in C(\mathbb{R}, \mathbb{R})$  satisfy that there exist constants  $c > 0$  and  $r \in (2, 2^*)$  such that

$$|f(z)| \leq c(1 + |z|^{r-1}) \quad \text{for } z \in \mathbb{R}.$$

(f<sub>2</sub>) There exists a positive constant  $c$  such that

$$\frac{1}{q}zf(z) - F(z) \geq -c \quad \text{for } z \in \mathbb{R},$$

where  $F(z) = \int_0^z f(\tau)d\tau$ .

(f<sub>3</sub>)  $\lim_{z \rightarrow 0} \frac{f(z)}{z} = 0, \quad \lim_{|z| \rightarrow +\infty} \frac{f(z)}{z} = +\infty.$

(f<sub>4</sub>)  $f(z)$  is odd in  $z$ .

For simplicity of notations we use the same  $c$  to denote some constants in the above conditions.

We are looking for weak solutions for Eq. (1.1), namely a function  $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$  satisfying

$$\begin{aligned} & \int_{\Omega} \left( \sum_{i,j=1}^N b_{ij}(u)D_iuD_j\varphi + \frac{1}{2}D_z b_{ij}(u)D_iuD_ju\varphi \right) dx \\ &= \int_{\Omega} f(u)\varphi dx \quad \text{for } \varphi \in C_0^\infty(\Omega). \end{aligned} \tag{1.2}$$

For semilinear case, this is in the setting of the classical superlinear problems that goes back to the celebrated paper of Ambrosetti and Rabinowitz [1] (see also the book [24]) in which infinitely many solutions were obtained using the symmetric mountain pass theorem. Later a sequence of sign-changing solutions were assured to exist in the setting [4, 13].

The following is the main result of this paper which establish the above mentioned results for the case of bounded quasilinearity.

**Theorem 1.1.** *Assume  $(b_1)$ - $(b_4)$ ,  $(f_1)$ - $(f_4)$ . Then Eq. (1.1) has infinitely many sign-changing solutions.*

Formally the problem has a variational structure, given by the functional

$$I(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^N b_{ij}(u)D_iuD_ju \, dx - \int_{\Omega} F(u) \, dx \quad \text{for } u \in H_0^1(\Omega).$$

The functional  $I$  is continuous, but not differentiable on  $H_0^1(\Omega)$ . Historically there have been several approaches developed for handling this type of non-smooth variational problems. The critical point theory for nonsmooth functionals has been established, see [2, 3, 8, 10, 11, 15]. The authors of the present paper developed a new approach in [16, 17, 20, 21], by introducing a  $p$ -Laplacian perturbation. The theory in the above papers were used in [16] to treat the existence of an infinite sequence of sign-changing solutions

for a problem similar to Eq. (1.1) but having a quasilinear term  $b_{ij}$  of power growth in  $u$ . For some technical reasons, the  $p$ -Laplacian perturbation approach does not work well for the case we consider here, namely the quasilinear term  $b_{ij}$  is bounded. The existence of a nontrivial solution for (1.1) was given in [2, 8]. The question on the existence of sign-changing solutions in particular on whether there exist an infinite sequence of sign-changing solutions in this case was left open in [8, 16].

In this paper we will use a different approach which was used in [22] where localized solutions for semiclassical Schrödinger type equations were considered. We outline the approach here first. Instead of introducing a  $p$ -Laplacian perturbation, we truncate the quadratic form  $\sum_{i,j=1}^N b_{ij}(u)D_iuD_ju$ . Under suitable truncations, up to some level sets controlled by the truncation, the resulting problem share critical points with the original problem. More precisely, let  $\mu > 0$  be a parameter and define a family of functionals

$$I_\mu(u) = \frac{1}{2}\sigma \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \int_\Omega h_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u)D_iuD_ju \right) dx - \int_\Omega F(u)dx, \quad u \in H_0^1(\Omega), \tag{1.3}$$

where  $\sigma > 0$  is fixed satisfying

$$0 < \sigma < \min \left\{ \frac{2}{q}\delta, \frac{1}{2}b_- \right\}, \tag{1.4}$$

$h_\mu$  is a smooth function such that  $h_\mu(t) = t$  for  $t < \frac{1}{\mu}$  and  $h'_\mu(t) = 0$  for  $t \geq \frac{T}{\mu}$ , with  $T > 1$  being set as  $T = 2e^{\frac{6b_+}{\sigma}} + 1$  (see Section 2 for details), and  $\beta_{ij}(z) = b_{ij}(z) - \sigma\delta_{ij}$ ,  $i, j = 1, \dots, N$ .

We will verify that  $I_\mu$  is a  $C^1$ -functional on  $H_0^1(\Omega)$  and satisfies the Palais-Smale condition. The corresponding Euler-Lagrange equation for  $I_\mu$  is

$$\begin{aligned} & \sigma \int_\Omega \nabla u \nabla \varphi dx + \int_\Omega h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u)D_iuD_ju \right) \sum_{i,j=1}^N \left( \beta_{ij}(u)D_iuD_j\varphi \right. \\ & \quad \left. + \frac{1}{2}D_z b_{ij}(z)D_iuD_ju \varphi \right) dx \\ & = \int_\Omega f(u)\varphi dx \end{aligned} \tag{1.5}$$

for  $\varphi \in H_0^1(\Omega)$ . Now assume  $u$  is a solution of (1.5) (that is a critical point of (1.3)). Since the modification of the functional involves point-wisely the first order partial derivatives we will need some  $C^1$ -a priori estimates on the critical points of  $I_\mu$  under energy bounds. If  $u \in C^1(\overline{\Omega})$  and satisfies

$$\sum_{i,j=1}^N \beta_{ij}(u)D_iuD_ju < \frac{1}{\mu} \quad \text{for } x \in \overline{\Omega}, \tag{1.6}$$

then  $u$  will be a solution of the original equation (1.1). Note that no limit process  $\mu \rightarrow 0$  is needed for the existence of critical point of the original problem. By taking smaller  $\mu$  we obtain more solutions of the original problem. This is the main idea of the approach. In order to obtain sign-changing critical points of the functional  $I_\mu$  we shall apply the method of invariant sets for descending flow, as developed in [6, 18, 19]. The idea of using invariant sets for descending flow for studying sign-changing solutions goes back to some earlier work for semilinear elliptic boundary value problems, such as in [5,7,9,23] for the existence of sign-changing solutions and [4, 13] for an infinite sequence of sign-changing solutions.

Throughout the paper,  $c$  denotes various positive constants,  $c_\mu$  denotes constants depending on  $\mu$ . The paper is organized as follows. In Lemma 2.1 we define the truncation function  $h_\mu$ . In Lemma 2.2 and Lemma 2.3 we prove the functional  $I_\mu$  is a  $C^1$ -functional on  $H_0^1(\Omega)$  and satisfies the Palais-Smale condition. Lemma 2.4 to Lemma 2.8 and Proposition 2.1 are devoted to constructing critical values of the functional  $I_\mu$  by the method of invariant sets for descending flow. In Proposition 2.2 we prove the regularity result as the gradient estimate (1.6), which completes the proof of Theorem 1.1 consequently.

## 2 Proof of Theorem 1.1

We start with giving the definition of the function  $h_\mu$  used in the truncation for the functional  $I$  to get  $I_\mu$ .

**Lemma 2.1.** *Given  $\varepsilon > 0$  there exists a smooth function  $h \in C^\infty([0, \infty), [0, \infty))$  such that*

- (1)  $h(t) = t$  for  $t \leq 1$ .
- (2)  $h'(t) = 1$  for  $t \leq 1$ ,  $h'(t) = 0$  for  $t \geq T_\varepsilon = 2e^{\frac{1}{\varepsilon}} + 1$ ;  $h'(t) \in [0, 1]$ ,  $h'(t)$  is decreasing in  $t$ , hence  $h(t) \geq th'(t)$ .
- (3)  $|h''(t)|t \leq \varepsilon$ .

*Proof.* Choose a smooth function  $\psi \in C^\infty(\mathbb{R}, [0, 1])$  such that  $\psi(t) = 1$  for  $2 \leq t \leq 2e^{\frac{1}{\varepsilon}} = T_\varepsilon - 1$ ;  $\psi(t) = 0$  for  $t \leq 1$  or  $t \geq T_\varepsilon$ . We have

$$M = \int_{\mathbb{R}} \frac{\psi(t)}{t} dt \geq \int_2^{T_\varepsilon-1} \frac{dt}{t} = \ln \frac{T_\varepsilon - 1}{2} = \frac{1}{\varepsilon}.$$

Now define

$$q(t) = 1 - \frac{1}{M} \int_{-\infty}^t \frac{\psi(\tau)}{\tau} d\tau.$$

Then  $0 \leq q(t) \leq 1$ ,  $q(t) = 1$  for  $t \leq 1$ ;  $q(t) = 0$  for  $t \geq T_\varepsilon$ ;  $q(t)$  is decreasing in  $t$  and

$$|q'(t)|t = \frac{1}{M} \psi(t) \leq \frac{1}{M} \leq \varepsilon.$$

Now  $h(t) = \int_0^t q(\tau) d\tau$  is the desired function. □

In the future we choose and fix  $\varepsilon$  and  $T := T_\varepsilon$  in the definition of  $h$  above such that

$$\varepsilon = \frac{\sigma}{6b_+}. \quad (2.1)$$

Let

$$h_\mu(t) = \frac{1}{\mu}h(\mu t), \quad t \in [0, \infty), \quad \mu \in (0, 1].$$

We have

- (1)  $h_\mu(t) = t$  for  $t \leq \frac{1}{\mu}$ ,  $0 \leq h_\mu(t) \leq \frac{T_\varepsilon}{\mu}$ .
- (2)  $h'_\mu(t) = h'(\mu t) \in [0, 1]$ ,  $h'_\mu(t) = 1$  for  $t \leq \frac{1}{\mu}$ ,  $h'_\mu(t) = 0$  for  $t \geq \frac{T_\varepsilon}{\mu}$ ,  $h'_\mu$  is decreasing, hence  $h_\mu(t) \geq th'_\mu(t)$ .
- (3)  $|h''_\mu(t)|t = |h''(\mu t)|\mu t \leq \varepsilon$ .

**Lemma 2.2.** *The functional  $I_\mu$  is differentiable and*

$$\begin{aligned} & \langle DI_\mu(u), \varphi \rangle \\ &= \sigma \int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\Omega} h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) \\ & \quad \cdot \sum_{i,j=1}^N (\beta_{ij}(u) D_i u D_j \varphi + \frac{1}{2} D_z \beta_{ij}(u) D_i u D_j u \varphi) \, dx - \int_{\Omega} f(u) \varphi \, dx \quad \text{for } \varphi \in H_0^1(\Omega). \end{aligned}$$

*Proof.* Let  $H_\mu$  be the truncated quadratic term in  $I_\mu$

$$H_\mu(u) = \frac{1}{2} \int_{\Omega} h_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) \, dx.$$

The other two terms in  $I_\mu$  are “harmless” to the smoothness. We compute the Gateaux derivative of  $H_\mu$ . The Gateaux derivative of  $H_\mu$  at  $u$  in the direction  $\varphi$  is defined as

$$\langle D_G H_\mu(u), \varphi \rangle = \lim_{t \rightarrow 0^+} \frac{1}{t} (H_\mu(u + t\varphi) - H_\mu(u)).$$

Denote  $u_t = u + t\varphi$ ,  $t \in [0, 1]$ . Then

$$\begin{aligned} & \frac{1}{t} (H_\mu(u + t\varphi) - H_\mu(u)) \\ &= \int_{\Omega} dx \frac{1}{t} \int_0^t \frac{d}{dt} \frac{1}{2} h_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u_t) D_i u_t D_j u_t \right) dt \\ &= \int_{\Omega} dx \frac{1}{t} \int_0^t h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u_t) D_i u_t D_j u_t \right) \sum_{i,j=1}^N (\beta_{ij}(u_t) D_i u_t D_j \varphi + \frac{1}{2} D_z \beta_{ij}(u_t) D_i u_t D_j u_t \varphi) \, dt. \end{aligned}$$

Since

$$\sum_{i,j=1}^N \beta_{ij}(u_t) D_i u_t D_j u_t \geq (b_- - \sigma) |\nabla u_t|^2 \geq \frac{1}{2} b_- |\nabla u_t|^2,$$

we have

$$h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u_t) D_i u_t D_j u_t \right) = 0 \quad \text{for } \frac{1}{2} b_- |\nabla u_t|^2 \geq \frac{T_\varepsilon}{\mu}.$$

Hence

$$\left| \frac{1}{t} \int_0^t h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u_t) D_i u_t D_j u_t \right) \sum_{i,j=1}^N \left( \beta_{ij}(u_t) D_i u_t D_j \varphi + \frac{1}{2} D_z \beta_{ij}(u_t) D_i u_t D_j u_t \varphi \right) dt \right| \leq c_\mu (|\nabla \varphi| + |\varphi|).$$

By Lebesgue’s dominated convergence theorem we have

$$\begin{aligned} \langle D_G H_\mu(u), \varphi \rangle &= \lim_{t \rightarrow 0^+} \frac{1}{t} (H_\mu(u + t\varphi) - H_\mu(u)) \\ &= \int_\Omega h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) \sum_{i,j=1}^N \left( \beta_{ij}(u) D_i u D_j \varphi + \frac{1}{2} D_z \beta_{ij}(u) D_i u D_j u \varphi \right) dx, \\ \langle D_G H_\mu(u), \varphi \rangle &\leq c_\mu \int_\Omega (|\nabla \varphi| + |\varphi|) dx \leq c_\mu \|\varphi\|, \quad \varphi \in H_0^1(\Omega). \end{aligned}$$

The Gateaux derivative  $D_G H_\mu(u)$  is a bounded linear functional on  $H_0^1(\Omega)$ . Moreover  $D_G H_\mu(u)$  continuously depends on  $u$  and there exists a positive constant  $c_\mu$  such that

$$\|D_G H_\mu(u) - D_G H_\mu(v)\| \leq c_\mu \|u - v\| \quad \text{for } u, v \in H_0^1(\Omega). \tag{2.2}$$

In fact denoting  $w_t = tu + (1 - t)v$ ,  $t \in [0, 1]$  for  $\varphi \in H_0^1(\Omega)$ , we have

$$\begin{aligned} &\langle D_G H_\mu(u) - D_G H_\mu(v), \varphi \rangle \\ &= \int_\Omega h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j \varphi + \frac{1}{2} D_z \beta_{ij}(u) D_i u D_j u \varphi \right) dx \\ &\quad - \int_\Omega h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(v) D_i v D_j v \right) \left( \sum_{i,j=1}^N \beta_{ij}(v) D_i v D_j \varphi + \frac{1}{2} D_z \beta_{ij}(v) D_i v D_j v \varphi \right) dx \\ &= \int_\Omega dx \int_0^1 dt \left\{ h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \right) \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j \varphi + \frac{1}{2} D_z \beta_{ij}(w_t) D_i w_t D_j w_t \varphi \right) \right\} \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} dx \int_0^1 h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \right) \sum_{i,j=1}^N \left( \beta_{ij}(w_t) D_i (u-v) D_j \varphi \right. \\
&\quad \left. + D_z \beta_{ij}(w_t) (u-v) D_i w_t D_j \varphi + D_z \beta_{ij}(w_t) D_i (u-v) D_j w_t \varphi \right) dt \\
&\quad + \int_{\Omega} dx \int_0^1 h''_\mu \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \right) \sum_{i,j=1}^N \frac{1}{2} D_i w_t D_j w_t \varphi dt D_z \beta_{ij}(w_t) \\
&\quad + \int_{\Omega} dx \int_0^1 h''_\mu \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \right) \left( \sum_{i,j=1}^N 2 \beta_{ij}(w_t) D_i (u-v) D_j w_t \right. \\
&\quad \left. + D_z \beta_{ij}(w_t) (u-v) D_i w_t D_j w_t \right) \cdot \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j \varphi + \frac{1}{2} D_z \beta_{ij}(w_t) D_i w_t D_j w_t \varphi \right) dt \\
&\leq c_\mu \int_{\Omega} (|\nabla(u-v)| + |u-v|) (|\nabla \varphi| + |\varphi|) dx \\
&\leq c_\mu \|u-v\| \|\varphi\|.
\end{aligned}$$

In the above we have used the fact that

$$\begin{aligned}
&|d_t D_z \beta_{ij}(w_t)| \leq c |dw_t| \leq c |u-v| |dt|, \\
&h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \right) = h''_\mu \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \right) \\
&= 0 \quad \text{for } \frac{1}{2} b_- |\nabla w_t|^2 \geq \frac{T_\varepsilon}{\mu}.
\end{aligned}$$

Since the Gateaux derivative  $D_G H_\mu(u)$  is a bounded linear functional on  $H_0^1(\Omega)$  and continuously depends on  $u$ ,  $H_\mu(u)$  is Frechet differentiable and the Frechet derivative  $DH_\mu(u) = D_G H_\mu(u)$ . Finally

$$\begin{aligned}
&\langle DI_\mu(u), \varphi \rangle \\
&= \sigma \int_{\Omega} \nabla u \nabla \varphi dx + \langle DH_\mu(u), \varphi \rangle - \int_{\Omega} f(u) \varphi dx \\
&= \sigma \int_{\Omega} \nabla u \nabla \varphi dx + \int_{\Omega} h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) \\
&\quad \cdot \sum_{i,j=1}^N \left( \beta_{ij}(u) D_i u D_j \varphi + \frac{1}{2} D_z \beta_{ij}(u) D_i u D_j u \varphi \right) dx - \int_{\Omega} f(u) \varphi dx
\end{aligned}$$

for  $\varphi \in H_0^1(\Omega)$ . □

**Lemma 2.3.**  $I_\mu$  satisfies the Palais-Smale condition.



*Proof.* We have

$$\begin{aligned}
 & I_\mu(u) - \frac{1}{q} \langle DI_\mu(u), u \rangle \\
 &= \left(\frac{1}{2} - \frac{1}{q}\right) \sigma \int_\Omega |\nabla u|^2 dx + \int_\Omega \left\{ \frac{1}{2} h_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) - \frac{1}{q} h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) \right. \\
 &\quad \cdot \sum_{i,j=1}^N \left( \beta_{ij}(u) + \frac{1}{2} u D_z \beta_{ij}(u) \right) D_i u D_j u \left. \right\} dx + \int_\Omega \left( \frac{1}{q} u f(u) - F(u) \right) dx \\
 &\geq \left(\frac{1}{2} - \frac{1}{q}\right) \sigma \int_\Omega |\nabla u|^2 dx + \int_\Omega \left\{ \frac{1}{2} h_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) - \frac{1}{q} h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) \right. \\
 &\quad \cdot \left( \frac{q}{2} \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u + \left( \frac{q}{2} (\sigma - \delta) - \sigma \right) |\nabla u|^2 \right) \left. \right\} dx + \int_\Omega \left( \frac{1}{q} u f(u) - F(u) \right) dx \\
 &\geq \left(\frac{1}{2} - \frac{1}{q}\right) \sigma \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \int_\Omega \left\{ h_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) - h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) \right. \\
 &\quad \cdot \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \left. \right\} dx - c \\
 &\geq \left(\frac{1}{2} - \frac{1}{q}\right) \sigma \int_\Omega |\nabla u|^2 dx - c. \tag{2.3}
 \end{aligned}$$

In the above we have used the fact that

$$\sigma < \frac{2}{q} \delta, \quad h'(t) \geq 0, \quad \frac{1}{q} z f(z) - F(z) \geq -c \quad \text{and} \quad h_\mu(t) \geq t h'_\mu(t).$$

Let  $\{u_n\} \subset H_0^1(\Omega)$  be a Palais-Smale sequence of  $I_\mu$ . By (2.3),  $u_n$  is bounded in  $H_0^1(\Omega)$ . Assume  $u_n \rightharpoonup u$  in  $H_0^1(\Omega)$ ,  $u_n \rightarrow u$  in  $L^s(\Omega)$  ( $1 \leq s < 2^*$ ),  $u_n(x) \rightarrow u(x)$  a.e.  $x \in \Omega$ . We have

$$\begin{aligned}
 o(1) &= \langle DI_\mu(u_n) - DI_\mu(u_m), u_n - u_m \rangle \\
 &= \sigma \int_\Omega |\nabla(u_n - u_m)|^2 dx + \langle DH_\mu(u_n) - DH_\mu(u_m), u_n - u_m \rangle \\
 &\quad - \int_\Omega (f(u_n) - f(u_m))(u_n - u_m) dx \\
 &= \sigma \int_\Omega |\nabla(u_n - u_m)|^2 dx + \langle DH_\mu(u_n) - DH_\mu(u_m), u_n - u_m \rangle + o(1). \tag{2.4}
 \end{aligned}$$

We estimate the term  $\langle DH_\mu(u_n) - DH_\mu(u_m), u_n - u_m \rangle$ . Assume  $u, v \in H_0^1(\Omega)$ . Denote

$w_t = tu + (1-t)v$ ,  $t \in [0, 1]$ . We have

$$\begin{aligned}
& \langle DH_\mu(u) - DH_\mu(v), u - v \rangle \\
&= \int_\Omega h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) \sum_{i,j=1}^N \left( \beta_{ij}(u) D_i u D_j (u - v) + \frac{1}{2} D_z \beta_{ij}(u) D_i u D_j u (u - v) \right) dx \\
&\quad - \int_\Omega h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(v) D_i v D_j v \right) \sum_{i,j=1}^N \left( \beta_{ij}(v) D_i v D_j (u - v) + \frac{1}{2} D_z \beta_{ij}(v) D_i v D_j v (u - v) \right) dx \\
&= \int_\Omega dx \int_0^1 dt \left\{ h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \right) \sum_{i,j=1}^N \left( \beta_{ij}(w_t) D_i w_t D_j (u - v) \right. \right. \\
&\quad \left. \left. + \frac{1}{2} D_z \beta_{ij}(w_t) D_i w_t D_j w_t (u - v) \right) \right\} \\
&= \int_\Omega dx \int_0^1 h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \right) \sum_{i,j=1}^N \left( \beta_{ij}(w_t) D_i (u - v) D_j (u - v) \right. \\
&\quad \left. + 2 D_z \beta_{ij}(w_t) D_i w_t (u - v) D_j (u - v) \right) dt \\
&\quad + \int_\Omega dx \int_0^1 h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \right) \sum_{i,j=1}^N \frac{1}{2} D_i w_t D_j w_t (u - v) dt D_z \beta_{ij}(w_t) \\
&\quad + \int_\Omega dx \int_0^1 2 h''_\mu \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \right) \left( \sum_{i,j=1}^N \left( \beta_{ij}(w_t) D_i w_t D_j (u - v) \right. \right. \\
&\quad \left. \left. + \frac{1}{2} D_z \beta_{ij}(w_t) D_i w_t D_j w_t (u - v) \right) \right)^2 dt \\
&\geq -c_\mu \int_\Omega (|\nabla(u - v)| + |u - v|) |u - v| dx + \int_\Omega dx \int_0^1 2 h''_\mu \left( \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \right) \\
&\quad \cdot \sum_{i,j=1}^N \beta_{ij}(w_t) D_i w_t D_j w_t \cdot \sum_{i,j=1}^N \beta_{ij}(w_t) D_i (u - v) D_j (u - v) dt \\
&\geq -c_\mu \int_\Omega (|\nabla(u - v)| + |u - v|) |u - v| dx - 2\epsilon b_+ \int_\Omega |\nabla(u - v)|^2 dx \\
&\geq -c_\mu^* \int_\Omega (u - v)^2 dx - 3\epsilon b_+ \int_\Omega |\nabla(u - v)|^2 dx. \tag{2.5}
\end{aligned}$$

Here  $\epsilon$  is from (2.1) so that  $\epsilon b_+ = \frac{\sigma}{6}$ . It follows (2.4), (2.5) that

$$\begin{aligned}
o(1) &\geq \sigma \|u_n - u_m\|^2 - 3\epsilon b_+ \|u_n - u_m\|^2 - c_\mu^* \|u_n - u_m\|_{L^2(\Omega)}^2 + o(1) \\
&\geq \frac{1}{2} \sigma \|u_n - u_m\|^2 + o(1),
\end{aligned}$$

$u_n$  is a Cauchy sequence, hence a convergent sequence in  $H_0^1(\Omega)$ .  $\square$

From the proof of Lemma 2.2 and Lemma 2.3 (see (2.2), (2.5)), we have the following lemma, which is useful in our proofs later.

**Lemma 2.4.** *There exist positive constants  $c_\mu$  and  $c_\mu^*$  such that*

- (1)  $\|DH_\mu(u) - DH_\mu(v)\| \leq c_\mu \|u - v\|.$
- (2)  $\langle DH_\mu(u) - DH_\mu(v), u - v \rangle \geq -\frac{\sigma}{2} \|u - v\|^2 - c_\mu^* \|u - v\|_{L^2(\Omega)}^2.$

We construct multiple solutions of the approximate problem (1.5), in particular multiple sign-changing solutions by the method of invariant sets for descending flow. The abstract framework is established in [16, 18, 19, 23] generalizing the classical mountain pass theorems without the setting of invariant sets [1, 24].

**Proposition 2.1.** *Let  $X$  be a Banach space,  $f$  be an even  $C^1$ -functional on  $X$  and satisfy the Palais-Smale condition. Let  $P, Q$  be open convex subsets of  $X$ ,  $Q = -P$ ,  $W = P \cup Q$ ,  $W \neq \emptyset$ . Assume there exists an odd map  $A : X \rightarrow X$  satisfying*

- (A<sub>1</sub>) *Given  $c_0, b_0 > 0$  there exists  $b = b(c_0, b_0)$  such that if  $\|Df(x)\| \geq b_0$ ,  $|f(x)| \leq c_0$  then*

$$\langle Df(x), x - Ax \rangle \geq b \|x - Ax\| > 0.$$

- (A<sub>2</sub>)  $A(\partial P) \subset P$ ,  $A(\partial Q) \subset Q$ .

*Let  $E$  be a finite-dimensional subspace of  $X$  and  $B$  be a ball in  $E$  centered at the origin. Define*

$$c = \inf_{\varphi \in \Gamma} \sup_{u \in \varphi(B) \setminus W} f(u),$$

$$\Gamma = \{\varphi \mid \varphi \in C(B, X), \varphi \text{ is odd, } \varphi|_{\partial B} = Id\}.$$

*Assume*

- (C)  $c > 0 \geq \sup_{x \in \partial B} f(x).$

*Then  $c$  is a critical value of  $f$  and*

$$K_c^* = \{x \mid x \in X \setminus W, Df(x) = 0, f(x) = c\} \neq \emptyset.$$

In the following we verify that  $I_\mu$  satisfies all the assumptions of Proposition 2.1 for a suitable operator  $A$  and subsets  $P, Q$  and  $B$ . In Lemma 2.2 and Lemma 2.3 we have proved that  $I_\mu$  is a  $C^1$ -functional on  $H_0^1(\Omega)$  and satisfies the Palais-Smale condition. Now we define the operator  $A$  and the subsets  $P, Q$  and verify the assumptions (A<sub>1</sub>), (A<sub>2</sub>).

**Definition 2.1.** Given  $u \in H_0^1(\Omega)$ , define  $v = Au \in H_0^1(\Omega)$  by the following equation

$$\begin{aligned} \langle DJ_\mu(v), \varphi \rangle &= \sigma \int_\Omega \nabla v \nabla \varphi \, dx + \langle DH_\mu(v), \varphi \rangle + c_\mu^* \int_\Omega v \varphi \, dx \\ &= \int_\Omega f(u) \varphi \, dx + c_\mu^* \int_\Omega u \varphi \, dx \quad \text{for } \varphi \in H_0^1(\Omega), \end{aligned} \tag{2.6}$$

where  $c_\mu^*$  is the positive constant which appears in Lemma 2.4 and

$$J_\mu(v) = \frac{1}{2} \sigma \int_\Omega |\nabla v|^2 \, dx + H_\mu(v) + \frac{1}{2} c_\mu^* \int_\Omega v^2 \, dx.$$

Without loss of generality we assume

$$(f(z) + c_\mu^* z)z \geq 0 \quad \text{for } z \in \mathbb{R}.$$

**Lemma 2.5.** The operator  $A$  is well-defined and continuous.

*Proof.* By Lemma 2.4 we have for  $v, \bar{v} \in H_0^1(\Omega)$

$$\begin{aligned} &\langle DJ_\mu(v) - DJ_\mu(\bar{v}), v - \bar{v} \rangle \\ &= \sigma \|v - \bar{v}\|^2 + \langle DH_\mu(v) - DH_\mu(\bar{v}), v - \bar{v} \rangle + c_\mu^* \|v - \bar{v}\|_{L^2(\Omega)}^2 \\ &\geq \sigma \|v - \bar{v}\|^2 - \frac{\sigma}{2} \|v - \bar{v}\|^2 \geq \frac{1}{2} \sigma \|v - \bar{v}\|^2. \end{aligned} \tag{2.7}$$

$DJ_\mu(v)$  is a strongly monotone operator, hence given  $u \in H_0^1(\Omega)$ , Eq. (2.6) has a unique solution  $v = Au \in H_0^1(\Omega)$ . Moreover assume  $u, \bar{u} \in H_0^1(\Omega)$ ,  $v = Au$ ,  $\bar{v} = A\bar{u}$ , then

$$\begin{aligned} \frac{1}{2} \sigma \|v - \bar{v}\|^2 &\leq \langle DJ_\mu(v) - DJ_\mu(\bar{v}), v - \bar{v} \rangle \\ &= \int_\Omega (f(u) - f(\bar{u})) (v - \bar{v}) \, dx + c_\mu^* \int_\Omega (u - \bar{u})(v - \bar{v}) \, dx \\ &\leq |f(u) - f(\bar{u})|_{L^r(\Omega)} \|v - \bar{v}\|_{L^r(\Omega)} + c_\mu^* \|u - \bar{u}\|_{L^2(\Omega)} \|v - \bar{v}\|_{L^2(\Omega)} \\ &\leq c_\mu \left( |f(u) - f(\bar{u})|_{L^r(\Omega)} + \|u - \bar{u}\|_{L^2(\Omega)} \right) \|v - \bar{v}\|, \end{aligned}$$

where  $\frac{1}{r} + \frac{1}{r'} = 1$ . Hence

$$\|v - \bar{v}\| \leq c_\mu \left( |f(u) - f(\bar{u})|_{L^r(\Omega)} + \|u - \bar{u}\|_{L^2(\Omega)} \right) \rightarrow 0 \quad \text{as } \bar{u} \rightarrow u \quad \text{in } H_0^1(\Omega).$$

We complete the proof. □

**Lemma 2.6.** There exists a constant  $c_\mu$  such that

$$\|DI_\mu(u)\| \leq c_\mu \|u - Au\|.$$

Moreover

$$\langle DI_\mu(u), u - Au \rangle \geq \frac{1}{2} \sigma \|u - Au\|^2.$$

Consequently the assumption  $(A_1)$  holds.

*Proof.* Assume  $u \in H_0^1(\Omega)$ ,  $v = Au$ . By the definition of the operator  $A$ ,

$$\begin{aligned} & \sigma \int_{\Omega} \nabla v \nabla \varphi \, dx + \langle DH_{\mu}(v), \varphi \rangle + c_{\mu}^* \int_{\Omega} v \varphi \, dx \\ &= \int_{\Omega} f(u) \varphi \, dx + c_{\mu}^* \int_{\Omega} u \varphi \, dx \quad \text{for } \varphi \in H_0^1(\Omega). \end{aligned}$$

We have

$$\begin{aligned} \langle DI_{\mu}(u), \varphi \rangle &= \sigma \int_{\Omega} \nabla u \nabla \varphi \, dx + \langle DH_{\mu}(u), \varphi \rangle - \int_{\Omega} f(u) \varphi \, dx \\ &= \sigma \int_{\Omega} \nabla(u - v) \nabla \varphi \, dx + \langle DH_{\mu}(u) - DH_{\mu}(v), \varphi \rangle + c_{\mu}^* \int_{\Omega} (u - v) \varphi \, dx. \end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned} \|DI_{\mu}(u)\| &\leq \sigma \|u - v\| + \|DH_{\mu}(u) - DH_{\mu}(v)\| + c_{\mu} \|u - v\| \leq c_{\mu} \|u - v\|, \\ \langle DI_{\mu}(u), u - v \rangle &= \sigma \|u - v\|^2 + \langle DH_{\mu}(u) - DH_{\mu}(v), u - v \rangle + c_{\mu}^* \|u - v\|_{L^2(\Omega)}^2 \\ &\geq \sigma \|u - v\|^2 - 3\epsilon b_+ \|u - v\|^2 = \frac{1}{2} \sigma \|u - v\|^2. \end{aligned}$$

Thus, we complete the proof. □

We define the open convex sets  $P$  and  $Q$ , and verify the assumption  $(A_2)$  of Proposition 2.1.

**Definition 2.2.** Define the open convex sets  $P$  and  $Q$  as

$$P = P_{\nu} = \{u \mid u \in H_0^1(\Omega), \sigma S |u_-|_{L^r(\Omega)}^2 + c_{\mu}^* |u_-|_{L^2(\Omega)}^2 < \nu^2\}, \quad Q = -P,$$

where  $u_{\pm} = \max\{\pm u, 0\}$ ,  $c_{\mu}^*$  is the constant in Lemma 2.4,  $S = S_p(\Omega)$  is the Sobolev constant for the embedding  $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ ,  $\nu > 0$  is a small constant.

**Lemma 2.7.** There exists a positive constant  $\nu_0$  such that for  $\nu < \nu_0$ , it holds that  $A(\partial P) \subset P$ ,  $A(\partial Q) \subset Q$ .

*Proof.* Assume  $u \in \partial Q$ ,  $v = Au$ . By the definitions

$$\sigma S |u_+|_{L^r(\Omega)}^2 + c_{\mu}^* |u_+|_{L^2(\Omega)}^2 = \nu^2, \tag{2.8a}$$

$$\begin{aligned} & \sigma \int_{\Omega} \nabla v \nabla \varphi \, dx + \langle DH_{\mu}(v), \varphi \rangle + c_{\mu}^* \int_{\Omega} v \varphi \, dx \\ &= \int_{\Omega} f(u) \varphi \, dx + c_{\mu}^* \int_{\Omega} u \varphi \, dx \quad \text{for } \varphi \in H_0^1(\Omega). \end{aligned} \tag{2.8b}$$

Choose  $\varphi = v_+$  as test function in (2.8b). Since

$$\langle DH_{\mu}(v), v_+ \rangle = \langle DH_{\mu}(v_+), v_+ \rangle \geq 0,$$

the left hand side of (2.8b)

$$\begin{aligned} \text{LHS} &\geq \sigma \int_{\Omega} |\nabla v_+|^2 dx + c_{\mu}^* \int_{\Omega} v_+^2 dx \\ &\geq \sigma S |v_+|_{L^r(\Omega)}^2 + c_{\mu}^* |v_+|_{L^2(\Omega)}^2. \end{aligned} \quad (2.9)$$

For any  $\tau > 0$ , by the assumptions  $(f_1)$ ,  $(f_3)$ , there exists  $c_{\tau} > 0$  such that

$$|f(z)| \leq \tau |z| + c_{\tau} |z|^{r-1}.$$

The right hand side of (2.8b),

$$\begin{aligned} \text{RHS} &= \int_{\Omega} f(u) v_+ dx + c_{\mu}^* \int_{\Omega} u v_+ dx \\ &\leq \int_{\Omega} f(u_+) v_+ dx + c_{\mu}^* \int_{\Omega} u_+ v_+ dx \\ &\leq \int_{\Omega} (\tau u_+ + c_{\tau} u_+^{r-1}) v_+ dx + c_{\mu}^* \int_{\Omega} u_+ v_+ dx \\ &\leq \left( \tau |\Omega|^{1-\frac{2}{r}} + c_{\tau} |u_+|_{L^r(\Omega)}^{r-2} \right) |u_+|_{L^r(\Omega)} |v_+|_{L^r(\Omega)} + c_{\mu}^* |u_+|_{L^2(\Omega)} |v_+|_{L^2(\Omega)} \\ &\leq \left( \tau |\Omega|^{1-\frac{2}{r}} + c_{\tau} \left( \frac{\nu^2}{\sigma S} \right)^{\frac{r-2}{2}} \right) |u_+|_{L^r(\Omega)} |v_+|_{L^r(\Omega)} + c_{\mu}^* |u_+|_{L^2(\Omega)} |v_+|_{L^2(\Omega)} \\ &\leq \frac{1}{2} \sigma S |u_+|_{L^r(\Omega)} |v_+|_{L^r(\Omega)} + c_{\mu}^* |u_+|_{L^2(\Omega)} |v_+|_{L^2(\Omega)}, \end{aligned} \quad (2.10)$$

provided we choose  $\tau$  and  $\nu_0$  such that

$$\tau |\Omega|^{1-\frac{2}{r}} \leq \frac{1}{4} \sigma S \quad \text{and} \quad c_{\tau} \left( \frac{\nu_0^2}{\sigma S} \right)^{\frac{r-2}{2}} \leq \frac{1}{4} \sigma S, \quad \nu \leq \nu_0.$$

By (2.9) and (2.10), for  $0 < \nu \leq \nu_0$  we have

$$\sigma S |v_+|_{L^r(\Omega)}^2 + c_{\mu}^* |v_+|_{L^2(\Omega)}^2 \leq \frac{1}{2} \sigma S |u_+|_{L^r(\Omega)}^2 + c_{\mu}^* |u_+|_{L^2(\Omega)}^2 < \nu^2,$$

hence  $v = Au \in Q$  and  $A(\partial Q) \subset Q$ . Similarly  $A(\partial P) \subset P$ .  $\square$

We apply the abstract theorem (Proposition 2.1) to define a sequence of critical values of the functional  $I_{\mu}$ , and estimate the bound of the critical values.

**Definition 2.3.** Let  $0 < \lambda_1 < \lambda_2 \leq \dots$  be the eigenvalues of the Laplacian operator  $-\Delta$  in  $H_0^1(\Omega) \cap H^2(\Omega)$ ,  $e_1, e_2, \dots$ , be the corresponding eigenfunctions. Denote

$$E_l = \text{span}\{e_1, e_2, \dots, e_l\}, \quad B_l = \{u \mid u \in E_l, \|u\| \leq R_l\},$$

where  $R_l > 0$  satisfies that  $I_{\mu}(u) \leq I(u) < 0$  for  $u \in E_l$ ,  $\|u\| \geq R_l$ . Define

$$c_l = \inf_{\varphi \in \Gamma_l} \sup_{u \in \varphi(B_l) \setminus W} I_{\mu}(u), \quad l = 2, 3, \dots,$$

$$\Gamma_l = \{\varphi \mid \varphi \in C(B_l, H_0^1(\Omega)), \varphi \text{ is odd}, \varphi|_{\partial B_l} = Id\}.$$

**Lemma 2.8.** *There exist  $0 < \alpha_l < \beta_l, l = 2, 3, \dots$ , such that*

$$\alpha_l \leq c_l(\mu) \leq \beta_l, \quad l = 2, 3, \dots, \quad \text{and} \quad \alpha_l \rightarrow +\infty \quad \text{as} \quad l \rightarrow +\infty.$$

*Proof.* Since  $Id \in \Gamma_l$ , we have

$$c_l(\mu) \leq \sup_{B_l \setminus W} I_\mu(u) \leq \sup_{E_l} I(u) := \beta_l.$$

By the condition  $(f_1)$ , we can choose  $\tau > 0$  such that

$$\begin{aligned} I_\mu(u) &\geq \frac{1}{2}\sigma \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} \left( \frac{1}{2}\tau u^2 + \frac{1}{r}c_\tau |u|^r \right) dx \\ &\geq \frac{1}{4}\sigma \int_{\Omega} |\nabla u|^2 dx - \frac{1}{r}c_\tau \int_{\Omega} |u|^r dx := L(u). \end{aligned}$$

Define

$$M = \left\{ u \mid u \in H_0^1(\Omega), \frac{\sigma}{2} \int_{\Omega} |\nabla u|^2 dx - c_\tau \int_{\Omega} |u|^r dx \geq 0 \right\}.$$

By choosing  $R_l$  large enough, we can assume

$$\partial B_l \cap M = \emptyset.$$

By Lemma 4.5 [16] for  $\nu_0$  sufficiently small it holds that

$$(\varphi(B_l) \setminus W) \cap \partial M \cap E_{l-2}^\perp \neq \emptyset \quad \text{for} \quad \varphi \in \Gamma_l, \quad l = 2, 3, \dots \quad (2.11)$$

Hence

$$c_l(\mu) = \inf_{\varphi \in \Gamma_l} \sup_{u \in \varphi(B_l) \setminus W} I_\mu(u) \geq \inf_{u \in \partial M \cap E_{l-2}^\perp} I_\mu(u) \geq \inf_{u \in \partial M \cap E_{l-2}^\perp} L(u).$$

We claim that there exist positive constants  $\tau$  and  $c$ , independent of  $\mu$  and  $l$  such that

$$c_l(\mu) \geq \inf_{u \in \partial M \cap E_{l-2}^\perp} L(u) \geq c\lambda_{l-1}^\tau := \alpha_l, \quad l = 2, 3, \dots \quad (2.12)$$

Choose  $p \in (r, 2^*)$ ,  $t \in (0, 1)$  such that  $\frac{1}{r} = \frac{t}{2} + \frac{1-t}{p}$ . For  $u \in \partial M \cap E_{l-2}^\perp$ , we have

$$\begin{aligned} \frac{\sigma}{2} \int_{\Omega} |\nabla u|^2 dx &= c_\tau \int_{\Omega} |u|^r dx \leq c_\tau \left( \int_{\Omega} u^2 dx \right)^{\frac{t}{2}} \left( \int_{\Omega} |u|^p dx \right)^{\frac{t}{p}(1-t)} \\ &\leq c_\tau \left( \lambda_{l-1}^{-1} \int_{\Omega} |\nabla u|^2 dx \right)^{\frac{t}{2}} \cdot \left( c_p \int_{\Omega} |\nabla u|^2 dx \right)^{\frac{t}{2}(1-t)} \\ &= c\lambda_{l-1}^{-\frac{t}{2}} \left( \int_{\Omega} |\nabla u|^2 dx \right)^{\frac{t}{2}}, \end{aligned}$$

hence

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 dx &\geq c\lambda_{l-1}^{\frac{r}{2}t}, \\ L(u) &= \left( \frac{1}{4} - \frac{1}{2r} \right) \sigma \int_{\Omega} |\nabla u|^2 dx \geq c\lambda_{l-1}^{\frac{r}{2}t}. \end{aligned}$$

Thus, we complete the proof. □

We are in a position to prove Theorem 1.1. To do it, we need the following regularity result for solutions of the quasilinear elliptic equation (2.13), the proof of which will be given in Appendix.

**Proposition 2.2.** *Let  $u \in H_0^1(\Omega)$ ,  $DI_\mu(u) = 0$  and  $I_\mu(u) \leq L$ . Then there exist  $K > 0$ ,  $\gamma \in (0, 1)$ , depending on  $L$  only, such that*

$$\|u\|_{C^{1,\gamma}(\bar{\Omega})} \leq K. \tag{2.13}$$

*Proof of Theorem 1.1.* Given an integer  $l \geq 2$ , by Proposition 2.1 there exist  $u_l(\mu) \in H_0^1(\Omega)$ ,  $\mu \in (0, 1]$  such that  $u_l(\mu)$  is sign-changing,  $DI_\mu(u_l(\mu)) = 0$ ,  $\alpha_l \leq I_\mu(u_l(\mu)) \leq \beta_l$ . By Proposition 2.2 there exist  $k_l = k_l(\beta_l) > 0$ ,  $\gamma_l = \gamma_l(\beta_l) \in (0, 1)$  such that  $\|u_l(\mu)\|_{C^{1,\gamma_l}(\bar{\Omega})} \leq k_l$ . Choose  $\mu_l$  such that  $b_+k_l^2 \leq \frac{1}{2\mu_l}$ . Denote  $u_l = u_l(\mu_l)$ , then

$$\begin{aligned} \sum_{i,j=1}^N \beta_{ij}(u_l)D_i u_l D_j u_l &\leq b_+k_l^2 \leq \frac{1}{2\mu_l} && \text{for } x \in \bar{\Omega}, \\ h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u_l)D_i u_l D_j u_l \right) &\equiv 1 && \text{for } x \in \bar{\Omega}. \end{aligned}$$

$u_l$  is a sign-changing solution of the original equation (1.1), and  $I(u_l) = I_{\mu_l}(u_l) = c_l(\mu_l) \geq \alpha_l \rightarrow \infty$  as  $l \rightarrow +\infty$ . We obtain infinitely many sign-changing solutions of Eq. (1.1)  $\square$

**Remark 2.1.** Without the assumptions  $(b_4)$ ,  $(f_4)$ , we have the following theorem of three nontrivial solutions, which is reminiscent of the well known result for semilinear elliptic equations such as in [7,9,23,25].

**Theorem 2.1.** *Assume  $(b_1)$ - $(b_3)$ ,  $(f_1)$ - $(f_3)$ . Then Eq. (1.1) has at least three nontrivial solutions, one is positive, one is negative and the third is sign-changing.*

Again we apply the method of invariant sets of the descending flow, see [6,18,19]. We leave the detail of proof to the interested readers.

## Appendix

**Proposition A.1.** *Assume  $u \in H_0^1(\Omega)$ ,  $DI_\mu(u) = 0$  and  $I_\mu(u) \leq L$ . Then there exist  $K > 0$ ,  $\gamma \in (0, 1)$ , depending on  $L$  only, such that*

$$\|u\|_{C^{1,\gamma}(\bar{\Omega})} \leq K. \tag{A.1}$$

First we apply Moser’s iteration to obtain the  $L^\infty$ -bound.

**Lemma A.1.** *Assume  $u \in H_0^1(\Omega)$ ,  $DI_\mu(u) = 0$  and  $I_\mu(u) \leq L$ . Then there exists  $M > 0$ , depending on  $L$  only, such that*

$$|u|_{L^\infty(\Omega)} \leq M.$$



*Proof.* First we note that  $u$  satisfies the equation

$$\begin{aligned} & \sigma \int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\Omega} h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) \sum_{i,j=1}^N \left( \beta_{ij}(u) D_i u D_j \varphi \right. \\ & \quad \left. + \frac{1}{2} D_z \beta_{ij}(u) D_i u D_j u \varphi \right) dx \\ & = \int_{\Omega} f(u) \varphi \, dx \end{aligned} \tag{A.2}$$

for  $\varphi \in H_0^1(\Omega)$ . Given  $T > 0$ . Let  $u^T$  be the truncated function:  $u^T = u$  if  $|u| \leq T$ ;  $u^T = \pm T$  if  $\pm u \geq T$ . Assume  $s > 1$ , take  $\varphi = |u^T|^{2s-2} u$  as test function in (A.2). Notice that

$$\int_{\Omega} h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u \right) \sum_{i,j=1}^N \left( \beta_{ij}(u) D_i u D_j \varphi + \frac{1}{2} D_z \beta_{ij}(u) D_i u D_j u \varphi \right) dx \geq 0,$$

we have

$$\sigma \int_{\Omega} \nabla u \nabla \varphi \, dx \leq \int_{\Omega} f(u) \varphi \, dx. \tag{A.3}$$

Choose  $p = \frac{2N}{N-2}$  if  $N \geq 3$ ;  $p \in (r, +\infty)$  if  $N = 1, 2$ . By the Sobolev embedding theorem and (2.3) in Lemma 2.3,

$$|u|_{L^r(\Omega)} \leq c \|u\| \leq c(L).$$

Denote  $d = \frac{2+p-r}{2}$ ,  $d > 1$ . Assume

$$\int_{\Omega} |u|^{sp \cdot \frac{1}{d}} \, dx < +\infty.$$

Let  $T \rightarrow \infty$  in (A.3), we obtain

$$(2s - 1) \sigma \int_{\Omega} |\nabla u|^2 |u|^{2s-2} \, dx \leq \int_{\Omega} |f(u)| |u|^{2s-1} \, dx. \tag{A.4}$$

The left hand side of (A.4),

$$\text{LHS} \geq \frac{c}{s} \int_{\Omega} |\nabla |u|^{s-1} u|^2 \, dx \geq \frac{c}{s} \left( \int_{\Omega} |u|^{sp} \, dx \right)^{\frac{2}{p}}. \tag{A.5}$$

The right hand side of (A.4),

$$\begin{aligned} \text{RHS} & \leq \int_{\Omega} c(1 + |u|^{r-1}) |u|^{2s-1} \, dx \leq c \left( 1 + \int_{\Omega} |u|^{r-2} \cdot |u|^{2s} \, dx \right) \\ & \leq c \left( 1 + \left( \int_{\Omega} |u|^p \, dx \right)^{\frac{r-2}{p}} \left( \int_{\Omega} |u|^{sp \cdot \frac{2}{2+p-r}} \, dx \right)^{\frac{2+p-r}{p}} \right) \\ & \leq c \left( 1 + \left( \int_{\Omega} |u|^{sp \cdot \frac{1}{d}} \, dx \right)^{\frac{2d}{p}} \right) \\ & \leq c \max \left\{ 1, \left( \int_{\Omega} |u|^{sp \cdot \frac{1}{d}} \, dx \right)^{\frac{2d}{p}} \right\}. \end{aligned} \tag{A.6}$$

By (A.5) and (A.6), we have

$$\max \left\{ 1, \left( \int_{\Omega} |u|^{sp} dx \right)^{\frac{1}{sp}} \right\} \leq (cs)^{\frac{1}{2s}} \max \left\{ 1, \left( \int_{\Omega} |u|^{sp \cdot \frac{1}{d}} dx \right)^{\frac{d}{sp}} \right\}.$$

Starting from  $s_0 = d > 1$ , by iteration we obtain

$$|u|_{L^\infty(\Omega)} \leq c(1 + |u|_{L^p(\Omega)}) \leq c(1 + \|u\|) \leq C(L).$$

Thus, we complete the proof.  $\square$

*Proof of Proposition A.1.* We write down the quasilinear equation (1.1) in the divergence form

$$\begin{cases} Q(u) = \operatorname{div} A_\mu(u, \nabla u) + B_\mu(u, \nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (Q)$$

where  $A_\mu(z, p) = (A_\mu^i(z, p))_{i=1}^N$ ,  $z \in \mathbb{R}$ ,  $p = (p_i) \in \mathbb{R}^N$  and

$$A_\mu^i(z, p) = \sigma p_i + h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(z) p_i p_j \right) \sum_{i,j=1}^N \beta_{ij}(z) p_j, \quad i = 1, \dots, N,$$

$$B_\mu(z, p) = -h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(z) p_i p_j \right) \sum_{i,j=1}^N D_z \beta_{ij}(z) p_i p_j + f(z).$$

Denote

$$\begin{aligned} a_\mu^{ij} &= \frac{\partial A_\mu^i}{\partial p_j} = \sigma \delta_{ij} + h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(z) p_i p_j \right) \beta_{ij}(z) \\ &\quad + 2h''_\mu \left( \sum_{i,j=1}^N \beta_{ij}(z) p_i p_j \right) \sum_{i=1}^N \beta_{ij}(z) p_i \sum_{j=1}^N \beta_{ij}(z) p_j, \end{aligned}$$

for  $i, j = 1, \dots, N$ . We verify the structure conditions satisfied by the quasilinear equation (Q) by use of the assumptions (B), (F) and the property of the truncated function  $h_\mu$ .

$$(a_1) \quad |A_\mu(z, p)| \leq \sigma |p| + \left| \left( \sum_{j=1}^N \beta_{ij}(z) p_j \right)_{i=1}^N \right| \leq \sigma |p| + (b_+ - \sigma) |p| = b_+ |p|.$$

$$(a_2) \quad p \cdot A_\mu(z, p) = \sigma |p|^2 + h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(z) p_i p_j \right) \sum_{i,j=1}^N \beta_{ij}(z) p_i p_j \geq \sigma |p|^2.$$

$$(a_3) \quad \sum_{i,j=1}^N a_\mu^{ij}(z, p) \xi_i \xi_j$$

$$\begin{aligned}
 &= \sigma|\xi|^2 + h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(z)p_i p_j \right) \sum_{i,j=1}^N \beta_{ij}(z)\xi_i \xi_j + 2h''_\mu \left( \sum_{i,j=1}^N \beta_{ij}(z)p_i p_j \right) \left( \sum_{i,j=1}^N \beta_{ij}(z)p_i \xi_j \right)^2 \\
 &\geq \sigma|\xi|^2 - 2 \left| h''_\mu \left( \sum_{i,j=1}^N \beta_{ij}(z)p_i p_j \right) \right| \cdot \sum_{i,j=1}^N \beta_{ij}(z)p_i p_j \cdot \sum_{i,j=1}^N \beta_{ij}(z)\xi_i \xi_j \\
 &\geq \sigma|\xi|^2 - 2\epsilon b_+ |\xi|^2 \geq \frac{1}{2}\sigma|\xi|^2 \quad \text{for } \xi = (\xi_i) \in \mathbb{R}^N.
 \end{aligned}$$

(a<sub>4</sub>) For  $z, w \in \mathbb{R}$ , denote  $y_t = tz + (1 - t)w, t \in (0, 1]$ . Then

$$\begin{aligned}
 &\left| A_\mu^i(z, p) - A_\mu^i(w, p) \right| \\
 &= \left| \int_0^1 dt \left\{ h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(y_t)p_i p_j \right) \sum_{i,j=1}^N \beta_{ij}(y_t)p_j \right\} \right| \\
 &\leq \left| \int_0^1 h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(z)p_i p_j \right) \sum_{i,j=1}^N D_z \beta_{ij}(y_t)p_j(z - w) dt \right| \\
 &\quad + \left| \int_0^1 h''_\mu \left( \sum_{i,j=1}^N \beta_{ij}p_i p_j \right) \sum_{i,j=1}^N D_z \beta_{ij}(y_t)p_i p_j(z - w) \cdot \sum_{j=1}^N \beta_{ij}(y_t)p_j dt \right| \\
 &\leq c|p| |z - w|.
 \end{aligned}$$

(b) On the other hand, we have

$$\begin{aligned}
 |B_\mu(z, p)| &\leq \left| h'_\mu \left( \sum_{i,j=1}^N \beta_{ij}(z)p_i p_j \right) \sum_{i,j=1}^N D_z \beta_{ij}(z)p_i p_j \right| + |f(z)| \\
 &\leq c|p|^2 + \Lambda(M) \leq \Lambda(M)(1 + |p|^2) \quad \text{for } z \in \mathbb{R}, \quad |z| \leq M,
 \end{aligned}$$

where  $\Lambda(M)$  is an increasing function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$ .

Assume  $u \in H_0^1(\Omega), DI_\mu(u) = 0, I_\mu(u) \leq L$ . By Lemma A.1, there exists  $M = M(L)$  such that  $|u|_{L^\infty(\Omega)} \leq M$ . The quasilinear equation (Q) satisfies the natural structure conditions for elliptic equation. All the assumptions of Corollary 1.5, Theorem 1.7 of [14] (see also [12]) are fulfilled. Therefore there exist  $K > 0, \gamma \in (0, 1)$  depending only on  $M, \Lambda$  such that

$$\|u\|_{C^{1,\gamma}(\bar{\Omega})} \leq K.$$

We complete the proof. □

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