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Completion of $\mathbb{R}^2$ with a Conformal Metric as a Closed Surface

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. In this paper, we obtain some asymptotic behavior results for solutions to the prescribed Gaussian curvature equation. Moreover, we prove that under a conformal metric in $\mathbb{R}^2$, if the total Gaussian curvature is 4π, the conformal area of $\mathbb{R}^2$ is finite and the Gaussian curvature is bounded, then $\mathbb{R}^2$ is a compact $C^{1,\alpha}$ surface after completion at ∞, for any $\alpha \in (0,1)$. If the Gaussian curvature has a Hölder decay at infinity, then the completed surface is C^2. For radial solutions, the same regularity holds if the Gaussian curvature has a limit at infinity.

Key Words: Gaussian curvature, conformal geometry, semilinear equations, entire solutions.

AMS Subject Classifications: 35B08, 35J15, 35J61, 53C18

1 Introduction

In this paper, we consider the prescribed Gaussian curvature equation

$$\Delta u + K(x)e^{2u} = 0 \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where K satisfies

$$\int_{\mathbb{R}^2} K(x)e^{2u(x)}\, dx < \infty. \tag{1.2}$$

(1.1) is equivalent to that K is the Gaussian curvature of $(\mathbb{R}^2, e^{2u}\delta)$, where δ is the Euclidean metric, and hence (1.2) means that the total Gaussian curvature is finite. A natural question is the following:

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Question 1.1. If u is an entire C^2 solution to (1.1), then by assuming what conditions on $K(x)$ can $(\mathbb{R}^2, e^{2u}\delta)$ be a C^2 closed Riemannian surface after completion at ∞?

Note that necessary conditions for this to be true include that

$$\int_{\mathbb{R}^2} K(x)e^{2u(x)}dx = 4\pi \quad \text{(Gauss-Bonnet Theorem)}, \tag{1.3a}$$

$$e^{2u} \in L^1(\mathbb{R}^2) \quad \text{(finite conformal area)}, \tag{1.3b}$$

$$\lim_{|x|\to\infty} K(x) = K_\infty \quad \text{uniformly, for some } K_\infty \in \mathbb{R}. \tag{1.3c}$$

A natural question is the following:

Question 1.2. Are (1.3a)-(1.3c) sufficient to guarantee that $(\mathbb{R}^2, e^{2u}\delta)$ is a C^2 closed Riemannian surface after completion at ∞?

This question is related to a more general question in [6] (Question 8.3) regarding the total area of $\mathbb{R}^2$ equipped with a conformal metric $e^{2u}\delta$ with its Gaussian curvature bigger than 1, i.e., with u being a super solution of (1.1).

Notice that (1.3c) implies that

$$K \in L^\infty(\mathbb{R}^2). \tag{1.4}$$

For the convenience of later discussion, we define

$$\lambda := \frac{1}{2\pi}\int_{\mathbb{R}^2} K(x)e^{2u(x)}dx.$$

In the following, (1.3b) and (1.4) will serve as fundamental assumptions.

Using the stereographic projection, we can identify conformally $\mathbb{R}^2$ with the unit sphere in $\mathbb{R}^3$ without the north pole. To complete the manifold, we need to find a coordinate system near the north pole so the metric is C^2 there. It is natural to use the Kelvin transform $x \mapsto \frac{x}{|x|^2}$ to convert the infinity of $\mathbb{R}^2$ to the origin when $\mathbb{R}^2$ is identified with the complex plane, and hence obtain the local coordinate system near the north pole. From analytic point of view, the completion of $(\mathbb{R}^2, e^{2u}\delta)$ is a closed C^2 Riemannian surface, if and only if

$$h(x) := e^{2u(\frac{x}{|x|^2})}\frac{1}{|x|^4} \tag{1.5}$$

is a C^2 function near $x = 0$, and $\lim_{|x|\to 0} h(x) > 0$, which means the metric at ∞ is nondegenerate. Hence we need to closely study the asymptotic behavior of u at ∞.

We are mostly interested in the case $\lambda = 2$ since it corresponds to (1.3a).

Our first result concerns the asymptotic behavior of u and its partial derivatives when $|x|$ is large.

Theorem 1.1. *Let u be a solution to* (1.1). *We assume that* (1.3b) *and* (1.4) *hold. Then we have, if $\lambda > \frac{3}{2}$, then*

$$u(x) = -\lambda \ln |x| + C_0 + \frac{C_1 x_1 + C_2 x_2}{|x|^2} + \mathcal{O}\Big(\frac{1}{|x|^{1+\alpha}}\Big), \tag{1.6}$$

where C_0 is a constant, $0 < \alpha < 1 \wedge (2\lambda - 3)$, and C_i are constants given by

$$C_i = \int_{\mathbb{R}^2} y_i K(y) e^{2u(y)} dy. \tag{1.7}$$

For general $\lambda > 1$, we have

$$u(x) = -\lambda \ln |x| + C_0 + \mathcal{O}\Big(\frac{1}{|x|^{\alpha}}\Big) \tag{1.8}$$

for any $\alpha \in (0, 1 \wedge (2\lambda - 2))$, and

$$D_i u(x) = \begin{cases} -\dfrac{\lambda x_i}{|x|^2} + \mathcal{O}\Big(\dfrac{1}{|x|^{2\lambda-1}}\Big), & \text{if } \lambda \in \Big(1, \dfrac{3}{2}\Big), \\ -\dfrac{\lambda x_i}{|x|^2} + \mathcal{O}\Big(\dfrac{\ln |x|}{|x|^2}\Big), & \text{if } \lambda = \dfrac{3}{2}, \\ -\dfrac{\lambda x_i}{|x|^2} + \mathcal{O}\Big(\dfrac{1}{|x|^2}\Big), & \text{if } \lambda > \dfrac{3}{2}. \end{cases} \tag{1.9}$$

In addition, if we further assume that for $i = 1, 2$,

$$\int_{\mathbb{R}^2} y_i K(y) e^{2u(y)} \, dx = 0, \tag{1.10}$$

then we have better estimates for $\lambda > \frac{3}{2}$ as follows,

$$D_i u(x) = \begin{cases} -\dfrac{\lambda x_i}{|x|^2} + \mathcal{O}\Big(\dfrac{1}{|x|^{2\lambda-1}}\Big), & \text{if } \lambda \in \Big(\dfrac{3}{2}, 2\Big), \\ -\dfrac{\lambda x_i}{|x|^2} + \mathcal{O}\Big(\dfrac{\ln |x|}{|x|^3}\Big), & \text{if } \lambda = 2, \\ -\dfrac{\lambda x_i}{|x|^2} + \mathcal{O}\Big(\dfrac{1}{|x|^3}\Big), & \text{if } \lambda > 2. \end{cases} \tag{1.11}$$

Remark 1.1. (1.8) is already proved in [3] under essentially weaker conditions, and the proof is very technical. The new ingredient in our theorem above is that under more convenient but natural assumptions (1.3b) and (1.4), we have established more precise asymptotic behaviors of u and $D_i u$ for various ranges of λ.

As a consequence of Theorem 1.1, we state an answer to Questions 1.1, 1.2 as follows.

Theorem 1.2. *Assume u and K satisfy* (1.1), (1.3a) *and* (1.3b), *and we also assume that K is a Hölder continuous function and satisfies*

$$K(x) = C + \mathcal{O}(|x|^{-\beta}) \quad \text{for some } \ \beta \in (0,1). \tag{1.12}$$

Then $(\mathbb{R}^2, e^{2u}\delta)$ can be completed as a $C^{2,\alpha}$ closed (compact) surface.

We also study radial solutions to (1.1). First, we show that

Theorem 1.3. *Let u be a radial solution to* (1.1), *and we also assume* (1.3b) *and* (1.4), *then for any $\lambda > 1$, when $r = |x|$ is large, we have that*

$$u_r + \frac{\lambda}{r} = \mathcal{O}\Big(\frac{1}{r^{2\lambda-1}}\Big), \tag{1.13}$$

and that

$$u_{rr} - \frac{\lambda}{r^2} = \mathcal{O}\Big(\frac{1}{r^{2\lambda}}\Big). \tag{1.14}$$

If K further satisfies (1.3c), *then*

$$\lim_{r\to\infty} r^{2\lambda-1}\Big(u_r + \frac{\lambda}{r}\Big) \ \text{ exists and is finite,} \tag{1.15a}$$

$$\lim_{r\to\infty} r^{2\lambda}\Big(u_{rr} - \frac{\lambda}{r^2}\Big) \ \text{ exists and is finite.} \tag{1.15b}$$

As a consequence of Theorem 1.3, it turns out that the answer to Question 1.2 for radial solutions is positive, without (1.12) being assumed.

Corollary 1.1. *If u is a radial solution to* (1.1), *then $(\mathbb{R}^2, e^{2u}\delta)$ can be completed at ∞ such that it becomes a C^2 compact Riemannian surface, if and only if* (1.3a)-(1.3c) *are satisfied.*

We organize the notes as follows. In Section 2, we prove Theorem 1.1 and Theorem 1.2. In Section 3, we prove Theorem 1.3 and Corollary 1.1.

2 Asymptotic behavior of general solutions to (1.1)

In this section, we study asymptotic behavior of solutions to (1.1), and we will prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let

$$w(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} (\ln|x-y| - \ln|y|)K(y)e^{2u(y)}\,dy. \tag{2.1}$$

Then as in [2], $\Delta w(x) = K(x)e^{2u(x)}$, and

$$\frac{w(x)}{\ln|x|} \to \frac{1}{2\pi}\int_{\mathbb{R}^2} K(y)e^{2u(y)}\,dy = \lambda \quad \text{uniformly as } |x| \to \infty.$$

By [1], (1.3b) and (1.4) imply that u is bounded from above. Then by the argument of [2], we have that $u + w \equiv C$ for some constant C.

Let

$$v(x) := u\Big(\frac{x}{|x|^2}\Big) - \lambda \ln|x|,$$

then $v(x)$ satisfies

$$\Delta v + \tilde{K}(x)e^{2v} = 0 \quad \text{in } B_1(0) \setminus \{0\}, \tag{2.2}$$

where

$$\tilde{K}(x) = K\Big(\frac{x}{|x|^2}\Big)\frac{1}{|x|^{4-2\lambda}}.$$

Since $u + w \equiv C$, and by the asymptotic behavior of w, we have that for any $\epsilon > 0$,

$$e^{2v} = \mathcal{O}\Big(\frac{1}{|x|^{2\epsilon}}\Big)$$

in some ball B_δ, where $\delta = \delta(\epsilon) > 0$. Hence if $\epsilon < \lambda - \frac{3}{2}$, we have

$$\tilde{K}e^{2v} = \mathcal{O}\Big(\frac{1}{|x|^{4-2\lambda+2\epsilon}}\Big) \in L^q(B_\delta),$$

where

$$\begin{cases} q \in \Big(2, \dfrac{1}{2-\lambda}\Big), & \text{if } \dfrac{3}{2} < \lambda < 2, \\ q = \infty, & \text{if } \lambda \geq 2. \end{cases}$$

A standard argument as in the proof of [3, Theorem 1.1] implies that $v \in L^\infty(B_1(0))$, and hence 0 is a removable singularity of v. Hence by the theory of second order elliptic equations, $v \in C^{1,\alpha}$ for any $\alpha \in (0, 1 \wedge (2\lambda - 3))$ if $\lambda > \frac{3}{2}$.

In particular, if $\lambda > \frac{3}{2}$, then near 0 we have

$$v(x) = C_0 + A \cdot x + \mathcal{O}(|x|^{1+\alpha}), \tag{2.3}$$

where C_0 is a constant, A is a constant vector, and $\alpha \in (0, 1 \wedge (2\lambda - 3))$. Hence if $|x|$ is large, we have

$$u(x) = -\lambda \ln|x| + C + \frac{A \cdot x}{|x|^2} + \mathcal{O}\Big(\frac{1}{|x|^{1+\alpha}}\Big). \tag{2.4}$$

Similarly, for general $\lambda > 1$, we have $v \in C^\alpha(B_1(0))$ for any $\alpha \in (0, 1 \wedge (2\lambda - 2))$ if $\lambda \in (1, \frac{3}{2}]$, and thus (1.8) is immediate.

Next, we will prove (1.6). Let

$$\tilde{x} = \frac{x}{|x|^2} = \frac{x}{r^2}$$

and in the following, we will use C to denote various constants, possibly depending on λ and $\|K\|_\infty$.

By the differential property of Newtonian potential, we have

$$D_i w(\tilde{x}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\tilde{x}_i - y_i}{|\tilde{x} - y|^2} K(y) e^{2u(y)}\, dy.$$

Since $u + w$ is a constant, for $i = 1, 2$, by the definition of λ we have

$$D_i u(\tilde{x}) = -D_i w(\tilde{x}) = -\lambda x_i + A_i(\tilde{x}), \tag{2.5}$$

where A_i is given by

$$A_i(\tilde{x}) = \int_{\mathbb{R}^2} \left(\frac{\tilde{x}_i}{|\tilde{x}|^2} - \frac{\tilde{x}_i - y_i}{|\tilde{x} - y|^2} \right) K(y) e^{2u(y)} dy.$$

Now in order to figure out the constant vector A in (2.3), we compute

$$\begin{aligned}
v_{x_1}(x) =& (-\lambda x_1 + A_1(\tilde{x})) \frac{x_2^2 - x_1^2}{|x|^4} + (-\lambda x_2 + A_2(\tilde{x})) \frac{-2x_1 x_2}{|x|^4} - \frac{\lambda x_1}{|x|^2} \\
=& A_1(\tilde{x}) \frac{x_2^2 - x_1^2}{r^4} + A_2(\tilde{x}) \frac{-2x_1 x_2}{r^4} \\
=& \int_{\mathbb{R}^2} \left(\left(r^2 \tilde{x}_1 - \frac{\tilde{x}_1 - y_1}{|\tilde{x} - y|^2} \right) (\tilde{x}_2^2 - \tilde{x}_1^2) + \left(r^2 \tilde{x}_2 - \frac{\tilde{x}_2 - y_2}{|\tilde{x} - y|^2} \right) (-2\tilde{x}_1 \tilde{x}_2) \right) K(y) e^{2u(y)} dy \\
=& \int_{\mathbb{R}^2} \left(-\tilde{x}_1 - \frac{\tilde{x}_1 - y_1}{|\tilde{x} - y|^2} (\tilde{x}_2^2 - \tilde{x}_1^2) + \frac{\tilde{x}_2 - y_2}{|\tilde{x} - y|^2} (2\tilde{x}_1 \tilde{x}_2) \right) K(y) e^{2u(y)} dy \\
=& \int_{\mathbb{R}^2} \frac{y_1 |\tilde{x}|^2 - \tilde{x}_1 |y|^2}{|\tilde{x} - y|^2} K(y) e^{2u(y)} dy.
\end{aligned}$$

Since

$$\frac{y_1 |\tilde{x}|^2 - \tilde{x}_1 |y|^2}{|\tilde{x} - y|^2} = \frac{\tilde{x}_1 y_1 (\tilde{x}_1 - y_1) + y_1 (\tilde{x}_2 + y_2)(\tilde{x}_2 - y_2) + y_2^2 (y_1 - \tilde{x}_1)}{|\tilde{x} - y|^2},$$

it follows that

$$\frac{y_1 |\tilde{x}|^2 - \tilde{x}_1 |y|^2}{|\tilde{x} - y|^2} \leq C \frac{|\tilde{x}||y| + |y|^2}{|\tilde{x} - y|}. \tag{2.6}$$

By (2.6), and since from (2.4) we have

$$K(y)e^{2u(y)} \le \frac{C}{|y|^{2\lambda}} \le \frac{C}{|\tilde{x}|^{2\lambda}} \quad \text{when} \quad y \in B_{\frac{|\tilde{x}|}{2}}(\tilde{x}),$$

it follows that

$$\int_{B_{\frac{|\tilde{x}|}{2}}(\tilde{x})} \frac{y_1|\tilde{x}|^2 - \tilde{x}_1|y|^2}{|\tilde{x}-y|^2} K(y)e^{2u(y)}dy = \mathcal{O}\Big(\frac{1}{|\tilde{x}|^{2\lambda-3}}\Big) = \mathcal{O}(r^{2\lambda-3}). \tag{2.7}$$

Since

$$\frac{y_1|\tilde{x}|^2 - \tilde{x}_1|y|^2}{|\tilde{x}-y|^2} = \frac{|\tilde{x}|^2(y_1-\tilde{x}_1) + \tilde{x}_1(|\tilde{x}|+|y|)(|\tilde{x}|-|y|)}{|\tilde{x}-y|^2} \le C\frac{|\tilde{x}||y|+|\tilde{x}|^2}{|\tilde{x}-y|},$$

we have

$$\begin{aligned}&\int_{B^c_{\frac{|\tilde{x}|}{2}}(\tilde{x})\cap B^c_{\frac{|\tilde{x}|}{2}}(0)} \frac{y_1|\tilde{x}|^2 - \tilde{x}_1|y|^2}{|\tilde{x}-y|^2} K(y)e^{2u(y)}dy \le C|\tilde{x}|\int_{B^c_{\frac{|\tilde{x}|}{2}}(0)} K(y)e^{2u(y)}dy \\ =&\mathcal{O}\Big(\frac{1}{|\tilde{x}|^{2\lambda-3}}\Big) = \mathcal{O}(r^{2\lambda-3}),\end{aligned} \tag{2.8}$$

where we have used

$$\frac{|\tilde{x}|}{2} < |\tilde{x}-y| \quad \text{and} \quad \frac{|y|}{3} \le |\tilde{x}-y| \quad \text{if} \quad y \in B^c_{\frac{|\tilde{x}|}{2}}(\tilde{x}) \cap B^c_{\frac{|\tilde{x}|}{2}}(0).$$

Also, since as $\lambda > \frac{3}{2}$,

$$\Big|\chi_{B_{\frac{|\tilde{x}|}{2}}(0)} \frac{y_1|\tilde{x}|^2}{|\tilde{x}-y|^2} K(y)e^{2u(y)}\Big| \le 4|y||K(y)|e^{2u(y)} \in L^1(\mathbb{R}^2), \tag{2.9a}$$

$$\Big|\chi_{B_{\frac{|\tilde{x}|}{2}}(0)} \frac{\tilde{x}_1|y|^2}{|\tilde{x}-y|^2} K(y)e^{2u(y)}\Big| \le 2|y||K(y)|e^{2u(y)} \in L^1(\mathbb{R}^2), \tag{2.9b}$$

$$\lim_{|\tilde{x}|\to\infty} \chi_{B_{\frac{|\tilde{x}|}{2}}(0)} \frac{y_1|\tilde{x}|^2 - \tilde{x}_1|y|^2}{|\tilde{x}-y|^2} K(y)e^{2u(y)} = y_1K(y)e^{2u(y)}, \tag{2.9c}$$

by Dominated Convergence Theorem we have

$$\lim_{|x|\to 0} v_{x_1}(x) = \lim_{|\tilde{x}|\to\infty} \int_{B_{\frac{|\tilde{x}|}{2}}(0)} \frac{y_1|\tilde{x}|^2 - \tilde{x}_1|y|^2}{|\tilde{x}-y|^2} K(y)e^{2u(y)}dy = \int_{\mathbb{R}^2} y_1K(y)e^{2u(y)}dy. \tag{2.10}$$

Hence

$$v_{x_1}(0) = \int_{\mathbb{R}^2} y_1K(y)e^{2u(y)}dy. \tag{2.11}$$

Similarly,

$$v_{x_2}(0)=\int_{\mathbb{R}^2} y_2 K(y)e^{2u(y)}dy. \tag{2.12}$$

Therefore, we have explicitly computed the constant vector A in (2.3) and (2.4), namely

$$A=(C_1,C_2):=\Big(\int_{\mathbb{R}^2} y_1 K(y)e^{2u(y)}dy, \int_{\mathbb{R}^2} y_2 K(y)e^{2u(y)}dy\Big). \tag{2.13}$$

This proves (1.6). Now we consider

$$\begin{aligned}
D_i w(x)-\frac{\lambda x_i}{|x|^2}&=\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\frac{x_i-y_i}{|x-y|^2}-\frac{x_i}{|x|^2}\right)K(y)e^{2u(y)}dy\\
=&\frac{1}{2\pi}\left(\int_{B_{\frac{|x|}{2}}(0)}+\int_{B_{\frac{|x|}{2}}(x)}+\int_{B^c_{\frac{|x|}{2}(0)}\cap B^c_{\frac{|x|}{2}}(x)}\right)\left(\frac{x_i-y_i}{|x-y|^2}-\frac{x_i}{|x|^2}\right)K(y)e^{2u(y)}dy\\
=&:I+II+III.
\end{aligned}$$

Since

$$|K(y)|e^{2u(y)}\le C\Big(1\wedge\frac{1}{|y|^{2\lambda}}\Big),$$

and when $y\in B_{\frac{|x|}{2}}(x)$, $|y|>\frac{|x|}{2}$, we have

$$|II|\le C\int_{B_{\frac{|x|}{2}}(x)}\left(\frac{1}{|x-y||x|^{2\lambda}}+\frac{1}{|x||x|^{2\lambda}}\right)dy\le\frac{C}{|x|^{2\lambda-1}}.$$

Also, since when

$$y\in B^c_{\frac{|x|}{2}(0)}\cap B^c_{\frac{|x|}{2}}(x),\quad |y-x|>\frac{|x|}{2}\quad\text{and}\quad |y|\ge\frac{|x|}{2},$$

we have

$$|III|\le C\int_{|y|\ge\frac{|x|}{2}}\frac{1}{|x|}K(y)e^{2u(y)}dy\le C\int_{|y|\ge\frac{|x|}{2}}\frac{1}{|x||y|^{2\lambda}}dy\le\frac{C}{|x|^{2\lambda-1}}.$$

It remains to estimate

$$\begin{aligned}
I=&\frac{1}{2\pi}\int_{B_{\frac{|x|}{2}}(0)}\left(\frac{2x\cdot y}{|x-y|^2|x|^2}x_i-\frac{|y|^2}{|x-y|^2|x|^2}x_i-\frac{y_i}{|x-y|^2}\right)K(y)e^{2u(y)}dy\\
=&:I_1-I_2-I_3.
\end{aligned}$$

Since

$$|y|^2|K(y)|e^{2u(y)}\le C\Big(1\wedge\frac{1}{|y|^{2\lambda-2}}\Big)$$

and that when $y \in B_{\frac{|x|}{2}}(0)$, $|x-y| > \frac{|x|}{2}$, we have

$$\begin{aligned}|I_2| \le& \frac{C}{|x|^3}\int_{B_{\frac{|x|}{2}}(0)} |y|^2|K(y)|e^{2u(y)}dy\\ \le& \frac{C}{|x|^3}\left(1+\int_{B_{\frac{|x|}{2}}(0)\setminus B_1}\frac{1}{|y|^{2\lambda-2}}dy\right)\\ \le& \begin{cases} C\left(\dfrac{1}{|x|^3}+\dfrac{1}{|x|^{2\lambda-1}}\right), & \text{if } \lambda\neq 2,\\ C\left(\dfrac{1}{|x|^3}+\dfrac{\ln|x|}{|x|^3}\right), & \text{if } \lambda = 2.\end{cases}\end{aligned}$$

Using that $|x\cdot y| \le |x||y|$, we similarly have that

$$\begin{aligned}|I_1| \le& \frac{C}{|x|^2}\int_{B_{\frac{|x|}{2}}(0)} |y||K(y)|e^{2u(y)}dy\\ \le& \frac{1}{|x|^2}\left(1+\int_{B_{\frac{|x|}{2}}(0)\setminus B_1}\frac{1}{|y|^{2\lambda-1}}dy\right)\\ \le& \begin{cases} C\left(\dfrac{1}{|x|^2}+\dfrac{1}{|x|^{2\lambda-1}}\right), & \text{if } \lambda\neq \dfrac{3}{2},\\ C\left(\dfrac{1}{|x|^2}+\dfrac{\ln|x|}{|x|^2}\right), & \text{if } \lambda = \dfrac{3}{2}.\end{cases}\end{aligned}$$

Similarly,

$$|I_3| \le \frac{C}{|x|^2}\int_{B_{\frac{|x|}{2}}(0)} |y||K(y)|e^{2u(y)}dy \le \begin{cases} C\left(\dfrac{1}{|x|^2}+\dfrac{1}{|x|^{2\lambda-1}}\right), & \text{if } \lambda\neq \dfrac{3}{2},\\ C\left(\dfrac{1}{|x|^2}+\dfrac{\ln|x|}{|x|^2}\right), & \text{if } \lambda = \dfrac{3}{2}.\end{cases}$$

Based on the estimates on II, III, I_1, I_2, I_3 above, we therefore have

$$D_i w(x) = \begin{cases} \dfrac{\lambda x_i}{|x|^2} + \mathcal{O}\Big(\dfrac{1}{|x|^{2\lambda-1}}\Big), & \text{if } \lambda \in \Big(1,\dfrac{3}{2}\Big),\\ \dfrac{\lambda x_i}{|x|^2} + \mathcal{O}\Big(\dfrac{\ln|x|}{|x|^2}\Big), & \text{if } \lambda = \dfrac{3}{2},\\ \dfrac{\lambda x_i}{|x|^2} + \mathcal{O}\Big(\dfrac{1}{|x|^2}\Big), & \text{if } \lambda > \dfrac{3}{2}.\end{cases}$$

This proves (1.9).

Next, if K further satisfies (1.10), then II, III and I_2 still bear the same estimates as above, while I_1 and I_3 can enjoy better estimates when $\lambda > \frac{3}{2}$. In fact,

$$\begin{aligned}
|I_3| =& \frac{1}{2\pi}\int_{B_{\frac{|x|}{2}}(0)} \frac{y_i}{|x-y|^2}K(y)e^{2u(y)}dy \\
=& \frac{1}{2\pi}\int_{B_{\frac{|x|}{2}}(0)} \left(\frac{y_i}{|x-y|^2}-\frac{y_i}{|x|^2}\right)K(y)e^{2u(y)}dy - \frac{1}{2\pi}\int_{B^c_{\frac{|x|}{2}}(0)} \frac{y_i}{|x|^2}K(y)e^{2u(y)}dy \\
=& \frac{1}{2\pi}\int_{B_{\frac{|x|}{2}}(0)} \frac{2x\cdot y-|y|^2}{|x|^2|x-y|^2}y_iK(y)e^{2u(y)}dy + \mathcal{O}\Big(\frac{1}{|x|^{2\lambda-1}}\Big).
\end{aligned}$$

Using $|x\cdot y|\le |x||y|$, $|x-y|\ge \frac{|x|}{2}$ and $|y|\le \frac{|x|}{2}$ for $y\in B_{\frac{|x|}{2}}(0)$, we have

$$|I_3| \le \int_{B_{\frac{|x|}{2}}(0)} \frac{C}{|x|^3}|y|^2|K(y)|e^{2u(y)}dy + \mathcal{O}\Big(\frac{1}{|x|^{2\lambda-1}}\Big).$$

Using the exact estimate of $|I_2|$ above, we have

$$|I_3| \le \begin{cases} C\left(\dfrac{1}{|x|^3}+\dfrac{1}{|x|^{2\lambda-1}}\right), & \text{if } \lambda > \dfrac{3}{2} \text{ and } \lambda\neq 2, \\ C\left(\dfrac{1}{|x|^3}+\dfrac{\ln|x|}{|x|^3}\right), & \text{if } \lambda = 2. \end{cases}$$

Also, since

$$I_1 = \sum_{j=1}^{2}\frac{2x_jx_i}{2\pi|x|^2}\int_{B_{\frac{|x|}{2}}(0)} \frac{y_j}{|x-y|^2}K(y)e^{2u(y)}dy,$$

by the same estimate of I_3 as above, we have

$$|I_1| \le \begin{cases} C\left(\dfrac{1}{|x|^3}+\dfrac{1}{|x|^{2\lambda-1}}\right), & \text{if } \lambda > \dfrac{3}{2} \text{ and } \lambda\neq 2, \\ C\left(\dfrac{1}{|x|^3}+\dfrac{\ln|x|}{|x|^3}\right), & \text{if } \lambda = 2. \end{cases}$$

Therefore, combing above, we have

$$\begin{aligned}
&\Big|D_iw(x)-\frac{\lambda x_i}{|x|^2}\Big| \le |I_1|+|I_2|+|I_3|+|II|+|III| \\
\le & \begin{cases} C\left(\dfrac{1}{|x|^3}+\dfrac{1}{|x|^{2\lambda-1}}\right), & \text{if } \lambda > \dfrac{3}{2} \text{ and } \lambda\neq 2, \\ C\left(\dfrac{1}{|x|^3}+\dfrac{\ln|x|}{|x|^3}\right), & \text{if } \lambda = 2. \end{cases}
\end{aligned}$$

$$= \begin{cases} \mathcal{O}\Big(\dfrac{1}{|x|^{2\lambda-1}}\Big), & \text{if } \lambda \in \Big(\dfrac{3}{2},2\Big), \\ \mathcal{O}\Big(\dfrac{\ln|x|}{|x|^3}\Big), & \text{if } \lambda = 2, \\ \mathcal{O}\Big(\dfrac{1}{|x|^3}\Big), & \text{if } \lambda > 2. \end{cases}$$

This implies (1.11). □

Proof of Theorem 1.2. We rewrite (1.5) as

$$h(x) = e^{2u(\frac{x}{|x|^2})-\lambda\ln|x|}|x|^{2\lambda-4} = e^{2v(x)}|x|^{2\lambda-4},$$

where $v(x)$ is defined in the proof of Theorem 1.1. In Theorem 1.1, we have shown that v is $C^{1,\alpha}(B_1(0))$ if $\lambda > \frac{3}{2}$. If we further assume (1.12), then by Hölder estimates, $v \in C^{2,\alpha}(B_1(0))$. Hence

$$h \in C^{2,\alpha}(B_1(0)) \quad \text{when} \quad \lambda = 2.$$

Since $h(0) > 0$, the proof is completed. □

Remark 2.1. From the proof above one can see that h is not C^1 if $\lambda \in (\frac{3}{2},2) \cup (2,\frac{5}{2})$, and h is C^1 but not C^2 when $\lambda \in (\frac{5}{2},3)$, and h is C^2 when $\lambda \geq 3$ or $\lambda = 2$.

Also from Theorem 1.1, we have

Corollary 2.1. *Let u be a solution to* (1.1)*, $\lambda \geq 2$ and* (1.3b) *holds. Moreover, if K is Hölder continuous and satisfies* (1.12)*, then*

$$u(x) = -\lambda\ln|x| + C_0 + \frac{C_1x_1 + C_2x_2}{|x|^2} + \sum_{i=1}^{2}\frac{C_{ij}x_ix_j}{|x|^4} + \mathcal{O}\Big(\frac{1}{|x|^{2+\beta}}\Big), \tag{2.14}$$

where C_i, $i = 1,2$ are constants given by (1.7) *and C_{ij} are also constants.*

Remark 2.2. One can see from above that under the assumptions of Corollary 2.1, and if (1.10) is assumed, then

$$u(x) = -\lambda\ln|x| + C_0 + \mathcal{O}\Big(\frac{1}{|x|^2}\Big), \tag{2.15}$$

which is sharp in the sense that standard bubble solutions do have such asymptotic behavior.

3 Radial solutions to (1.1)

In this section, we study radial solutions to (1.1) and we will prove Theorem 1.3 and Corollary 1.1.

When $K(x)$ is radially symmetric, under certain conditions it can be shown that a solution to (1.1) is also radially symmetric (see, e.g., [3] Theorem 1.7 and [7] Theorem 5.2).

Before giving the proofs, we remark that by [4, Corollary 1.4 and Corollary 1.5], there always exists a C^2 radial solution to (1.1) if $K(x)$ is Hölder continuous, radial, nonpositive near 0, and nonpositive near infinity (this last condition can be dropped if K further satisfies (1.12)). Therefore, one should not worry about the existence of solutions to (1.1) within our assumptions.

Proof of Theorem 1.3. If u is a radial solution to (1.1), then integrating (1.1) over B_r, and by the divergence theorem, we have

$$\int_{B_r} Ke^{2u}dx = \int_{\partial B_r} -u_r ds = -2\pi r u_r.$$

Hence

$$\begin{aligned} u_r + \frac{\lambda}{r} =& \frac{1}{2\pi r}\left(\int_{B_r} -Ke^{2u}dx + \int_{\mathbb{R}^2} Ke^{2u}dx\right) \\ =& \frac{1}{2\pi r}\int_{\mathbb{R}^2\setminus B_r} Ke^{2u}dx \\ \le& \frac{1}{2\pi r}\int_{\mathbb{R}^2\setminus B_r} \frac{C}{|x|^{2\lambda}}dx \le \frac{C}{r^{2\lambda-1}}. \end{aligned}$$

This proves (1.13). Moreover, (1.14) follows from (1.13) and Eq. (1.1), since

$$u_{rr} = -Ke^{2u} - \frac{u_r}{r} = \frac{\lambda}{r^2} + \mathcal{O}\Big(\frac{1}{r^{2\lambda}}\Big).$$

If (1.3c) is further satisfied, then it is easy to see from the above computations that (1.15a) and (1.15b) hold, since

$$\lim_{r\to\infty}\frac{r^{2\lambda-2}}{2\pi}\int_{\mathbb{R}^2\setminus B_r} Ke^{2u}dx = \frac{K_\infty e^{2C_0}}{2\lambda-2},$$

where C_0 is the coefficient in (1.8). □

Proof of Corollary 1.1. We simply write $u(x) = u(|x|)$. The corollary follows from the better regularity theory for radial functions. Here we show the assertion directly. Note that by checking the proof of Theorem 1.1, we know that (1.15a) and (1.15b) are true if K satisfies (1.3a) and (1.3c). Hence as $r \to 0$,

$$e^{2u(\frac{1}{r})} = A_0 r^4 + o(r^4), \tag{3.1a}$$

$$u'\Big(\frac{1}{r}\Big) = -2r + A_1 r^3 + o(r^3), \tag{3.1b}$$

$$u''\Big(\frac{1}{r}\Big) = 2r^2 + A_2 r^4 + o(r^4), \tag{3.1c}$$

$$\Big[u'\Big(\frac{1}{r}\Big)\Big]^2 = 4r^2 + A_3 r^4 + o(r^4), \tag{3.1d}$$

where A_i, $i = 0,1,2,3$ are constants. Since u is radial,

$$h(z) = h(r) := e^{2u(\frac{1}{r})}\frac{1}{r^4}.$$

Hence

$$h(0) := \lim_{r\to 0} h(r) = A_0 > 0.$$

By direct computation,

$$\begin{aligned} h'(r) =& e^{2u(\frac{1}{r})}\left(2u'\Big(\frac{1}{r}\Big)\Big(-\frac{1}{r^2}\Big)\frac{1}{r^4} - \frac{4}{r^5}\right) \\ =& e^{2u(\frac{1}{r})}\left(\frac{4r - 2A_1 r^3 + o(r^3)}{r^6} - \frac{4}{r^5}\right) \\ =& \frac{-2(A_1 r^3 + o(r^3))e^{2u(\frac{1}{r})}}{r^6} = \mathcal{O}(r). \end{aligned}$$

Hence $\lim_{r\to 0} h'(r) = 0$, and thus L-Hospital's Rule implies

$$h'(0) := \lim_{r\to 0}\frac{h(r) - h(0)}{r} = \lim_{r\to 0} h'(r) = 0.$$

Hence $h(r)$ is C^1 at $r = 0$. Also, applying (3.1a)-(3.1d), we have

$$\begin{aligned} h''(r) =& e^{2u(\frac{1}{r})}\left(4\Big[u'\Big(\frac{1}{r}\Big)\Big]^2\frac{1}{r^8} + 8u'\Big(\frac{1}{r}\Big)\frac{1}{r^7} + 2u''\Big(\frac{1}{r}\Big)\frac{1}{r^8} + 12u'\Big(\frac{1}{r}\Big)\frac{1}{r^7} + \frac{20}{r^6}\right) \\ =& \Big(A_0 r^4 + o(r^4)\Big)\left(\frac{4A_3 r^4 + 8rA_1 r^3 + 2A_2 r^4 + 12rA_1 r^3 + o(r^4)}{r^8}\right) \\ =& A_0(4A_3 + 20A_1 + 2A_2) + o(1). \end{aligned}$$

Hence

$$\lim_{r\to 0} h''(r) = A_0(4A_3 + 20A_1 + 2A_2).$$

Again, L-Hospital's rule implies that

$$h''(0) = \lim_{r\to 0} \frac{h'(r) - h'(0)}{r} = \lim_{r\to 0} h''(0) = A_0(4A_3 + 20A_1 + 2A_2).$$

Hence $h(r)$ is C^2 at $r = 0$. □

Example 3.1. Let $u = \ln \frac{2}{1+r^2}$ be the standard bubble solution to (1.1) with $K \equiv 1$, then by direct computation,

$$\begin{aligned} &e^{2u} = \frac{4}{(1+r^2)^2} = e^{2u(\frac{1}{r})}\frac{1}{r^4} = h(r), \\ &h'(0) = 0, \quad h''(0) = -16. \end{aligned}$$

Also, one can check that $A_0 = 4$, $A_1 = 2$, $A_2 = -6$, $A_3 = -8$, and hence $A_0(4A_3 + 20A_1 + 2A_2) = -16 = h''(0)$.

Example 3.2. Let $u = \ln \frac{\lambda}{1+r^\lambda}$, $\lambda > 1$, $\lambda \neq 2$. Then u is a solution to (1.1) with $K = r^{\lambda-2}$. In this case,

$$h(r) = e^{2u(\frac{1}{r})}\frac{1}{r^4} = \frac{\lambda^2 r^{2\lambda-4}}{(1+r^\lambda)^2}$$

has a singularity at $r = 0$. Indeed, $h(r) \to \infty$ as $r \to 0^+$ when $\lambda < 2$, while $h(0) = 0$ is degenerate when $\lambda > 2$. Furthermore, $h(r)$ is not C^1 if $2 < \lambda < 5/2$ and is not C^2 if $2 < \lambda < 3$. The metrics correspond to the so-called conformal metrics with conical singularities on S^2.

The reader is referred to [8, 9] for detailed discussions about surfaces with conical singularities.

Remark 3.1. Recently Dong Ye informed the authors that the answer to Question 1.2 should be negative, and a counter example can be constructed using the classic example of the nonexistence of a C^2 solution of the Poisson equation in the unit disc for a continuous but not C^α data (see, e.g., [5, Exercise 4.9]). Indeed, one can choose a solution v to $\Delta v = f$ in $C^2(B_1 \setminus \{0\}) \cap C^{1,\alpha}(B_1)$, $0 < \alpha < 1$ but $v \notin C^2(B_1)$, and extend v to $C^2(\mathbb{R}^2 \setminus \{0\})$ so that $v(x) = -2\ln|x|$ for $|x|$ sufficiently large. Define

$$u(x) = v(x/|x|^2) - 2\ln|x|,$$

then $u \in C^2(\mathbb{R}^2)$. Let

$$K(x) = -e^{-2u}\Delta u,$$

it can be verified that (1.3a)-(1.3c) hold. However, the completion of the surface is only $C^{1,\alpha}$ but not C^2.

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