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Gradient Estimates of Solutions to the Conductivity Problem with Flatter Insulators

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. We study the insulated conductivity problem with inclusions embedded in a bounded domain in $\mathbb{R}^n$. When the distance of inclusions, denoted by ε, goes to 0, the gradient of solutions may blow up. When two inclusions are strictly convex, it was known that an upper bound of the blow-up rate is of order $\varepsilon^{-1/2}$ for $n = 2$, and is of order $\varepsilon^{-1/2+\beta}$ for some $\beta > 0$ when dimension $n \geq 3$. In this paper, we generalize the above results for insulators with flatter boundaries near touching points.

Key Words: Conductivity problem, harmonic functions, maximum principle, gradient estimates.

AMS Subject Classifications: 35B44, 35J25, 35J57, 74B05, 74G70, 78A48

1 Introduction and main results

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary, and let D_1^* and D_2^* be two open sets whose closure belongs to Ω, touching only at the origin with the inner normal vector of ∂D_1^* pointing in the positive x_n-direction. Denote $x = (x', x_n)$. Translating D_1^* and D_2^* by $\frac{\varepsilon}{2}$ along x_n-axis, we obtain

$$D_1^\varepsilon := D_1^* + \left(0', \varepsilon/2\right) \quad \text{and} \quad D_2^\varepsilon := D_2^* - \left(0', \varepsilon/2\right).$$

When there is no confusion, we drop the superscripts ε and denote $D_1 := D_1^\varepsilon$ and $D_2 := D_2^\varepsilon$. Denote $\widetilde{\Omega} := \Omega \setminus \overline{(D_1 \cup D_2)}$. A simple model for electric conduction can be formulated as the following elliptic equation:

$$\begin{cases} \operatorname{div}\Big(a_k(x)\nabla u_k\Big) = 0 & \text{in } \Omega, \\ u_k = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\varphi \in C^2(\partial\Omega)$ is given, and

$$a_k(x) = \begin{cases} k \in (0,\infty) & \text{in } D_1 \cup D_2, \\ 1 & \text{in } \widetilde{\Omega}, \end{cases}$$

refers to conductivities. The solution u_k and its gradient ∇u_k represent the voltage potential and the electric fields respectively. From an engineering point of view, It is an interesting problem to capture the behavior of ∇u_k. Babuška, et al. [3] numerically analyzed that the gradient of solutions to an analogous elliptic system stays bounded regardless of ε, the distance between the inclusions. Bonnetier and Vogelius [5] proved that for a fixed k, $|\nabla u_k|$ is bounded for touching disks D_1 and D_2 in dimension $n=2$. A general result was obtained by Li and Vogelius [11] for general second order elliptic equations of divergence form with piecewise Hölder coefficients and general shape of inclusions D_1 and D_2 in any dimension. When k is bounded away from 0 and ∞, they established a $W^{1,\infty}$ bound of u_k in Ω, and a $C^{1,\alpha}$ bound in each region that do not depend on ε. This result was further extended by Li and Nirenberg [10] to general second order elliptic systems of divergence form. Some higher order estimates with explicit dependence on r_1, r_2, k and ε were obtained by Dong and Li [7] for two circular inclusions of radius r_1 and r_2 respectively in dimension $n=2$. There are still some related open problems on general elliptic equations and systems. We refer to p. 94 of [11] and p. 894 of [10].

When the inclusions are insulators ($k=0$), it was shown in [6,9,13] that the gradient of solutions generally becomes unbounded, as $\varepsilon \to 0$. It was known that (see e.g., Appendix of [4]) when $k \to 0$, u_k converges to the solution of the following insulated conductivity problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \widetilde{\Omega}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, \quad i = 1,2, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here ν denotes the inward unit normal vectors on ∂D_i, $i=1,2$.

The behavior of the gradient in terms of ε has been studied by Ammari et al. in [1] and [2], where they considered the insulated problem on the whole Euclidean space:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{(D_1 \cup D_2)}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, \quad i = 1,2, \\ u(x) - H(x) = \mathcal{O}(|x|^{n-1}) & \text{as } |x| \to \infty. \end{cases} \tag{1.3}$$

They established when dimension $n=2$, D_1^* and D_2^* are disks of radius r_1 and r_2 respectively, and H is a harmonic function in $\mathbb{R}^2$, the solution u of (1.3) satisfies

$$\|\nabla u\|_{L^\infty(B_4)} \le C\varepsilon^{-1/2},$$

for some positive constant C independent of ε. They also showed that the upper bounds are optimal in the sense that for appropriate H,

$$\|\nabla u\|_{L^\infty(B_4)} \geq \varepsilon^{-1/2}/C.$$

In fact, the equation

$$\begin{cases} \operatorname{div}\Big(a_k(x)\nabla u_k\Big) = 0 & \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}, \\ u(x) - H(x) = \mathcal{O}(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases}$$

was studied there, and the estimates derived have explicit dependence on r_1, r_2, k and ε.

Yun extended in [14] and [15] these results allowing D_1^* and D_2^* to be any bounded strictly convex smooth domains in $\mathbb{R}^2$.

The above upper bound of ∇u was localized and extended to higher dimensions by Bao, Li and Yin in [4], where they considered problem (1.2) and proved

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leq C\varepsilon^{-1/2}\|\varphi\|_{C^2(\partial\Omega)}, \quad \text{when } n \geq 2. \tag{1.4}$$

The upper bound is optimal for $n = 2$ as mentioned earlier. For dimensions $n \geq 3$, the upper bound was recently improved by Li and Yang [12] to

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leq C\varepsilon^{-1/2+\beta}\|\varphi\|_{C^2(\partial\Omega)}, \quad \text{when } n \geq 3, \tag{1.5}$$

for some $\beta > 0$.

Yun [16] considered the problem (1.3) in $\mathbb{R}^3$, with unit disks

$$D_1 = B_1\left(0,0,1+\varepsilon/2\right), \quad D_2 = B_1\left(0,0,-1-\varepsilon/2\right),$$

and a harmonic function H. He proved that for some positive constant C independent of ε,

$$\max_{|x_3|\leq\varepsilon/2} |\nabla u(0,0,x_3)| \leq C\varepsilon^{\frac{\sqrt{2}-2}{2}}.$$

He also showed that this upper bound of $|\nabla u|$ on the ε-segment connecting D_1 and D_2 is optimal for $H(x) \equiv x_1$.

In this paper, we assume that for some $m \in [2,\infty)$ and a small universal constant R_0, the portions of ∂D_1^* and ∂D_2^* in $[-R_0, R_0]^n$ are respectively the graphs of two C^2 functions f and g in terms of x', and

$$f(0') = g(0') = 0, \quad \nabla f(0') = \nabla g(0') = 0, \tag{1.6a}$$

$$\lambda_1|x'|^m \leq (f-g)(x') \leq \lambda_2|x'|^m \qquad \text{for } 0 < |x'| < R_0, \tag{1.6b}$$

$$|\nabla(f-g)(x')| \leq \lambda_3|x'|^{m-1} \qquad \text{for } 0 < |x'| < R_0, \tag{1.6c}$$

for some $\lambda_1,\lambda_2,\lambda_3>0$. Let $a(x)\in C^\alpha(\overline{\widetilde{\Omega}})$, for some $\alpha\in(0,1)$, be a symmetric, positive definite matrix function satisfying

$$\lambda\le a(x)\le\Lambda\quad\text{for}\quad x\in\widetilde{\Omega},$$

for some positive constants λ,Λ. Let $\nu=(\nu_1,\cdots,\nu_n)$ denote the unit normal vector on ∂D_1 and ∂D_2, pointing towards the interior of D_1 and D_2. We consider the following insulated conductivity problem:

$$\begin{cases} -\partial_i(a^{ij}\partial_j u)=0 & \text{in}\ \ \widetilde{\Omega},\\ a^{ij}\partial_j u\nu_i=0 & \text{on}\ \ \partial(D_1\cup D_2),\\ u=\varphi & \text{on}\ \ \partial\Omega, \end{cases}\tag{1.7}$$

where $\varphi\in C^2(\partial\Omega)$ is given. For $0<r\le R_0$, we denote

$$\Omega_{x_0,r}:=\left\{(x',x_n)\in\widetilde{\Omega}\ \Big|\ -\frac{\varepsilon}{2}+g(x')<x_n<\frac{\varepsilon}{2}+f(x'),\ |x'-x_0'|<r\right\},\tag{1.8a}$$

$$\Gamma_+:=\left\{x_n=\frac{\varepsilon}{2}+f(x'),\ |x'|<R_0\right\},\quad \Gamma_-:=\left\{x_n=-\frac{\varepsilon}{2}+g(x'),\ |x'|<R_0\right\}.\tag{1.8b}$$

Since the blow-up of gradient can only occur in the narrow region between D_1 and D_2, we will focus on the following problem near the origin:

$$\begin{cases} -\partial_i(a^{ij}\partial_j u)=0 & \text{in}\ \ \Omega_{0,R_0},\\ a^{ij}\partial_j u\nu_i=0 & \text{on}\ \ \Gamma_+\cup\Gamma_-, \end{cases}\tag{1.9}$$

where $\nu=(\nu_1,\cdots,\nu_n)$ denotes the unit normal vector on Γ_+ and Γ_-, pointing upward and downward respectively.

Theorem 1.1. *Let m, Γ_+, Γ_-, a, α be as above, and let $u\in H^1(\Omega_{0,R_0})$ be a solution of* (1.9). *There exist positive constants r_0,β and C depending only on n, m, λ, Λ, R_0, α, λ_1, λ_2, λ_3, $\|f\|_{C^2(\{|x'|\le R_0\})}$, $\|g\|_{C^2(\{|x'|\le R_0\})}$ and $\|a\|_{C^\alpha(\Omega_{0,R_0})}$, such that*

$$|\nabla u(x_0)|\le\begin{cases} C\|u\|_{L^\infty(\Omega_{0,R_0})}\left(\varepsilon+|x_0'|^m\right)^{-1/m}, & \text{when}\ \ n=2,\\ C\|u\|_{L^\infty(\Omega_{0,R_0})}\left(\varepsilon+|x_0'|^m\right)^{-1/m+\beta}, & \text{when}\ \ n\ge3, \end{cases}\tag{1.10}$$

for all $x_0\in\Omega_{0,r_0}$ and $\varepsilon\in(0,1)$.

Remark 1.1. For $m=2$, (1.10) was proved in [4] and [12] for $n=2$ and $n\ge3$, respectively.

Let $u\in H^1(\widetilde{\Omega})$ be a weak solution of (1.7). By the maximum principle and the gradient estimates of solutions of elliptic equations,

$$\|u\|_{L^\infty(\widetilde{\Omega})}\le\|\varphi\|_{L^\infty(\partial\Omega)},\tag{1.11a}$$

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega}\setminus\Omega_{0,r_0})}\le C\|\varphi\|_{C^2(\partial\Omega)}.\tag{1.11b}$$

Therefore, a corollary of Theorem 1.1 is as follows.

Corollary 1.1. *Let $u \in H^1(\widetilde{\Omega})$ be a weak solution of* (1.7). *There exist positive constants β and C depending only on n, m, λ, Λ, R_0, α, λ_1, λ_2, λ_3, $\|\partial D_1\|_{C^2}$, $\|\partial D_2\|_{C^2}$, $\|\partial\Omega\|_{C^2}$, and $\|a\|_{C^\alpha(\overline{\widetilde{\Omega}})}$, such that*

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le \begin{cases} C\|\varphi\|_{C^2(\partial\Omega)}\varepsilon^{-\frac{1}{m}}, & \text{when } n=2, \\ C\|\varphi\|_{C^2(\partial\Omega)}\varepsilon^{-\frac{1}{m}+\beta}, & \text{when } n\ge 3. \end{cases} \tag{1.12}$$

2 Proof of Theorem 1.1

Our proof of Theorem 1.1 is an adaption of the arguments in our earlier paper [12] for $m=2$, and follows closely the arguments there.

We fix a $\gamma \in (0,1)$, and let $r_0>0$ denote a constant depending only on n, m, γ, R_0, λ_1, λ_2, $\|f\|_{C^2}$ and $\|g\|_{C^2}$, whose value will be fixed in the proof. For any $x_0 \in \Omega_{0,r_0}$, we define

$$\delta := \left(\varepsilon + |x_0'|^m\right)^{\frac{1}{m}}. \tag{2.1}$$

We will always consider $0<\varepsilon \le r_0^m$. First, we require r_0 small so that for $|x_0'|<r_0$,

$$10\delta < \delta^{1-\gamma} < \frac{R_0}{4}.$$

Lemma 2.1. *For $n\ge 3$, there exists a small r_0, depending only on n, m, γ, and R_0, such that for any $x_0\in\Omega_{0,r_0}$, $5|x_0'|<r<\delta^{1-\gamma}$, if $u\in H^1(\Omega_{x_0,2r}\setminus\Omega_{x_0,r/4})$ is a positive solution to the equation*

$$\begin{cases} -\partial_i(a^{ij}(x)\partial_j u(x)) = 0 & \text{in } \ \Omega_{x_0,2r}\setminus\Omega_{x_0,r/4}, \\ a^{ij}(x)\partial_j u(x)\nu_i(x) = 0 & \text{on } \ (\Gamma_+\cup\Gamma_-)\cap\overline{\Omega_{x_0,2r}\setminus\Omega_{x_0,r/4}}, \end{cases}$$

then

$$\sup_{\Omega_{x_0,r}\setminus\Omega_{x_0,r/2}} u \le C \inf_{\Omega_{x_0,r}\setminus\Omega_{x_0,r/2}} u, \tag{2.2}$$

for some constant $C>0$ depending only on n, m, λ, Λ, R_0, λ_1, λ_2, $\|f\|_{C^2}$ and $\|g\|_{C^2}$ but independent of r and u.

Proof. We only need to prove (2.2) for $|x_0'|>0$, since the $|x_0'|=0$ case follows from the result for $|x_0'|>0$ and then sending $|x_0'|$ to 0. We denote

$$h_r := \varepsilon + f\left(x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right) - g\left(x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right),$$

and perform a change of variables by setting

$$\begin{cases} y' = x' - x_0', \\ y_n = 2h_r\left(\dfrac{x_n - g(x') + \varepsilon/2}{\varepsilon + f(x') - g(x')} - \dfrac{1}{2}\right), \end{cases} \qquad (x',x_n)\in\Omega_{x_0,2r}\setminus\Omega_{x_0,r/4}. \tag{2.3}$$

This change of variables maps the domain $\Omega_{x_0,2r} \setminus \Omega_{x_0,r/4}$ to an annular cylinder of height h_r, denoted by $Q_{2r,h_r} \setminus Q_{r/4,h_r}$, where

$$Q_{s,t} := \{y = (y', y_n) \in \mathbb{R}^n \mid |y'| < s,\ |y_n| < t\}, \tag{2.4}$$

for $s, t > 0$. We will show that the Jacobian matrix of the change of variables (2.3), denoted by $\partial_x y$, and its inverse matrix $\partial_y x$ satisfy

$$|(\partial_x y)^{ij}| \le C, \quad |(\partial_y x)^{ij}| \le C \quad \text{for} \quad y \in Q_{2r,h_r} \setminus Q_{r/4,h_r}, \tag{2.5}$$

where $C > 0$ depends only on n, m, R_0, λ_1, λ_2, $\|f\|_{C^2}$ and $\|g\|_{C^2}$.

Let $v(y) = u(x)$, then v satisfies

$$\begin{cases} -\partial_i(b^{ij}(y)\partial_j v(y)) = 0 & \text{in} \quad Q_{2r,h_r} \setminus Q_{r/4,h_r}, \\ b^{nj}(y)\partial_j v(y) = 0 & \text{on} \quad \{y_n = -h_r\} \cup \{y_n = h_r\}, \end{cases} \tag{2.6}$$

where the matrix $(b^{ij}(y))$ is given by

$$(b^{ij}(y)) = \frac{(\partial_x y)(a^{ij})(\partial_x y)^t}{\det(\partial_x y)}, \tag{2.7}$$

$(\partial_x y)^t$ is the transpose of $\partial_x y$.

It is easy to see that (2.5) implies, using $\lambda \le (a^{ij}) \le \Lambda$,

$$\frac{\lambda}{C} \le (b^{ij}(y)) \le C\Lambda \quad \text{for} \quad y \in Q_{2r,h_r} \setminus Q_{r/4,h_r}, \tag{2.8}$$

for some constant $C > 0$ depending only on n, m, R_0, λ_1, λ_2, $\|f\|_{C^2}$ and $\|g\|_{C^2}$.

In the following and throughout this section, we will denote $A \sim B$, if there exists a positive universal constant C, which might depend on n, m, λ, Λ, R_0, λ_1, λ_2, $\|f\|_{C^2}$ and $\|g\|_{C^2}$, but not depend on ε, such that $C^{-1}B \le A \le CB$.

From (2.3), one can compute that

$$\begin{aligned}
&(\partial_x y)^{ii} = 1 \quad \text{for} \quad 1 \le i \le n-1, \\
&(\partial_x y)^{nn} = \frac{2h_r}{\varepsilon + f(x_0' + y') - g(x_0' + y')}, \\
&(\partial_x y)^{ni} = -\frac{2h_r \partial_i g(x_0' + y') + 2y_n[\partial_i f(x_0' + y') - \partial_i g(x_0' + y')]}{\varepsilon + f(x_0' + y') - g(x_0' + y')} \quad \text{for} \quad 1 \le i \le n-1, \\
&(\partial_x y)^{ij} = 0 \quad \text{for} \quad 1 \le i \le n-1, \quad j \ne i.
\end{aligned}$$

By (1.6b), one can see that

$$h_r \sim \varepsilon + \left| x_0' - \frac{r}{4} \frac{x_0'}{|x_0'|} \right|^m.$$

Since $|y_n| \le h_r$, by using (1.6a) and (1.6b), we have that, for $1 \le i \le n-1$,

$$
\begin{aligned}
\left|(\partial_x y)^{ni}\right| \le & C\frac{h_r|\partial_i g(x_0'+y')| + h_r[|\partial_i f(x_0'+y')| + |\partial_i g(x_0'+y')|]}{\varepsilon + f(x_0'+y') - g(x_0'+y')} \\
\le & C\frac{h_r}{\varepsilon + f(x_0'+y') - g(x_0'+y')}\left[|\partial_i f(x_0'+y')| + |\partial_i g(x_0'+y')|\right] \\
\le & C\frac{\varepsilon + \left|x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right|^m}{\varepsilon + |x_0'+y'|^m}|x_0'+y'|.
\end{aligned}
$$

Since $r/4 < |y'| < 2r < 2\delta^{1-\gamma}$ and $|x_0'| < \delta$, we can estimate

$$
\left|(\partial_x y)^{ni}\right| \le C|x_0'+y'| \le C(|x_0'| + |y'|) \le C\delta^{1-\gamma}.
$$

Next, we will show that

$$
(\partial_x y)^{nn} \sim 1 \quad \text{for} \quad y \in Q_{2r,h_r} \setminus Q_{r/4,h_r}. \tag{2.9}
$$

Indeed, by (1.6b), we have

$$
(\partial_x y)^{nn} = \frac{2h_r}{\varepsilon + f(x_0'+y') - g(x_0'+y')} \sim \frac{\varepsilon + \left|x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right|^m}{\varepsilon + |x_0'+y'|^m}.
$$

Since $|y'| > r/4$, it is easy to see

$$
(\partial_x y)^{nn} \le C\frac{\varepsilon + \left|x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right|^m}{\varepsilon + |x_0'+y'|^m} \le C.
$$

On the other hand, since $|y'| < 2r$ and $|x_0'| < r/5$, we have

$$
\begin{aligned}
&\varepsilon + \left|x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right|^m \ge \varepsilon + \left(\left|\frac{r}{4}\frac{x_0'}{|x_0'|}\right| - |x_0'|\right)^m \ge \varepsilon + \left(\frac{r}{4} - \frac{r}{5}\right)^m \ge \frac{1}{C}\left(\varepsilon + r^m\right), \\
&\varepsilon + |x_0'+y'|^m \le \varepsilon + m|x_0'|^m + m|y'|^m \le C\left(\varepsilon + r^m\right).
\end{aligned}
$$

Therefore,

$$
(\partial_x y)^{nn} \ge \frac{1}{C}\frac{\varepsilon + \left|x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right|^m}{\varepsilon + |x_0'+y'|^m} \ge \frac{1}{C}
$$

and (2.9) is verified.

We have shown $(\partial_x y)^{ii} \sim 1$ for all $i = 1, \cdots, n$, and $|(\partial_x y)^{ij}| \le C\delta^{(1-\gamma)}$ for $i \ne j$. We further require r_0 to be small enough so that off-diagonal entries of $\partial_x y$ are small. Therefore (2.5) follows. As mentioned earlier, (2.8) follows from (2.5).

Now we define, for any integer l,

$$A_l := \left\{ y \in \mathbb{R}^n \;\Big|\; \frac{r}{4} < |y'| < 2r,\ (l-1)h_r < z_n < (l+1)h_r \right\}.$$

Note that $A_0 = Q_{2r,h_r} \setminus Q_{r/4,h_r}$. For any $l \in \mathbb{Z}$, we define a new function $\tilde{v}$ by

$$\tilde{v}(y) := v\left(y', (-1)^l \left(y_n - 2lh_r\right)\right), \quad \forall y \in A_l.$$

We also define the corresponding coefficients, for $k = 1, 2, \cdots, n-1$,

$$\tilde{b}^{nk}(y) = \tilde{b}^{kn}(y) := (-1)^l b^{nk}\left(y', (-1)^l \left(y_n - 2lh_r\right)\right), \quad \forall y \in A_l,$$

and for other indices,

$$\tilde{b}^{ij}(y) := b^{ij}\left(y', (-1)^l \left(y_n - 2lh_r\right)\right), \quad \forall y \in A_l.$$

Therefore, $\tilde{v}(y)$ and $\tilde{b}^{ij}(y)$ are defined in the infinite cylinder shell $Q_{2r,\infty} \setminus Q_{r/4,\infty}$. By (2.6), $\tilde{v} \in H^1(Q_{2r,\infty} \setminus Q_{r/4,\infty})$ satisfies

$$-\partial_i(\tilde{b}^{ij}(y)\partial_j \tilde{v}(y)) = 0 \quad \text{in} \quad Q_{2r,\infty} \setminus Q_{r/4,\infty}.$$

Note that for any $l \in \mathbb{Z}$ and $y \in A_l$, $\tilde{b}(y) = (\tilde{b}^{ij}(y))$ is orthogonally conjugated to $b\left(y', (-1)^l \left(y_n - 2lh_r\right)\right)$. Hence, by (2.8), we have

$$\frac{\lambda}{C} \leq \tilde{b}(y) \leq C\Lambda \quad \text{for} \quad y \in Q_{2r,\infty} \setminus Q_{r/4,\infty}.$$

We restrict the domain to be $Q_{2r,r} \setminus Q_{r/4,r}$, and make the change of variables $z = y/r$. Set $\bar{v}(z) = \tilde{v}(y)$, $\bar{b}^{ij}(z) = \tilde{b}^{ij}(y)$, we have

$$\begin{aligned} &-\partial_i(\bar{b}^{ij}(z)\partial_j \bar{v}(z)) = 0 && \text{in} \quad Q_{2,1} \setminus Q_{1/4,1}, \\ &\frac{\lambda}{C} \leq \bar{b}(z) \leq C\Lambda && \text{for} \quad z \in Q_{2,1} \setminus Q_{1/4,1}. \end{aligned}$$

Then by the Harnack inequality for uniformly elliptic equations of divergence form, see e.g., [8, Theorem 8.20], there exists a constant C depending only on n, m, λ, Λ, R_0, λ_1, λ_2, $\|f\|_{C^2}$ and $\|g\|_{C^2}$, such that

$$\sup_{Q_{1,1/2} \setminus Q_{1/2,1/2}} \bar{v} \leq C \inf_{Q_{1,1/2} \setminus Q_{1/2,1/2}} \bar{v}.$$

In particular, we have

$$\sup_{Q_{1,h_r/r} \setminus Q_{1/2,h_r/r}} \bar{v} \leq C \inf_{Q_{1,h_r/r} \setminus Q_{1/2,h_r/r}} \bar{v},$$

which is (2.2) after reversing the change of variables. □

Remark 2.1. Lemma 2.1 does not hold for dimension $n=2$, since $Q_{2,1}\setminus Q_{1/4,1}\subset\mathbb{R}^2$ is the union of two disjoint rectangular domains, and the Harnack inequality cannot be applied on it. Therefore, we will separate the cases $n=2$ and $n\geq 3$ in our proof of Theorem 1.1.

For any domain $A\subset\widetilde{\Omega}$, we denote the oscillation of u in A by $\mathrm{osc}_A u:=\sup_A u-\inf_A u$. Using Lemma 2.1, we obtain a decay of $\mathrm{osc}_{\Omega_{x_0,\delta}}u$ in δ as follows.

Lemma 2.2. *For $n\geq 3$, let u be a solution of* (1.9). *For any $x_0\in\Omega_{0,r_0}$, where r_0 is as in Lemma 2.1, there exist positive constants σ and C, depending only on n, m, λ, Λ, R_0, λ_1, λ_2, $\|f\|_{C^2}$ and $\|g\|_{C^2}$ such that*

$$osc_{\Omega_{x_0,\delta}}u\leq C\|u\|_{L^\infty(\Omega_{x_0,\delta^{1-\gamma}})}\delta^{\gamma\sigma}. \tag{2.10}$$

Proof. For simplicity, we drop the x_0 subscript and denote $\Omega_r=\Omega_{x_0,r}$ in this proof. Let $5|x_0'|<r<\delta^{1-\gamma}$ and $u_1=\sup_{\Omega_{2r}}u-u$, $u_2=u-\inf_{\Omega_{2r}}u$. By Lemma 2.1, we have

$$\sup_{\Omega_r\setminus\Omega_{r/2}}u_1\leq C_1\inf_{\Omega_r\setminus\Omega_{r/2}}u_1,\qquad \sup_{\Omega_r\setminus\Omega_{r/2}}u_2\leq C_1\inf_{\Omega_r\setminus\Omega_{r/2}}u_2,$$

where $C_1>1$ is a constant independent of r. Since both u_1 and u_2 satisfy Eq. (1.9), by the maximum principle,

$$\sup_{\Omega_r\setminus\Omega_{r/2}}u_i=\sup_{\Omega_r}u_i,\qquad \inf_{\Omega_r\setminus\Omega_{r/2}}u_i=\inf_{\Omega_r}u_i,$$

for $i=1,2$. Therefore,

$$\sup_{\Omega_r}u_1\leq C_1\inf_{\Omega_r}u_1,\qquad \sup_{\Omega_r}u_2\leq C_1\inf_{\Omega_r}u_2.$$

Adding up the above two inequalities, we have

$$\mathrm{osc}_{\Omega_r}u\leq\left(\frac{C_1-1}{C_1+1}\right)\mathrm{osc}_{\Omega_{2r}}u.$$

Now we take $\sigma>0$ such that $2^{-\sigma}=\frac{C_1-1}{C_1+1}$, then

$$\mathrm{osc}_{\Omega_r}u\leq 2^{-\sigma}\mathrm{osc}_{\Omega_{2r}}u. \tag{2.11}$$

We start with $r=r_0=\delta^{1-\gamma}/2$, and set $r_{i+1}=r_i/2$. Keep iterating (2.11) $k+1$ times, where k satisfies $5\delta\leq r_k<10\delta$, we will have

$$\mathrm{osc}_{\Omega_\delta}u\leq\mathrm{osc}_{\Omega_{r_k}}u\leq 2^{-(k+1)\sigma}\mathrm{osc}_{\Omega_{2r_0}}u\leq 2^{1-(k+1)\sigma}\|u\|_{L^\infty(\Omega_{\delta^{1-\gamma}})}.$$

Since

$$10\delta>r^k=2^{-k}r_0=2^{-(k+1)}\delta^{1-\gamma},$$

we have

$$2^{-(k+1)}<10\delta^\gamma$$

and hence (2.10) follows immediately. □

Proof of Theorem 1.1. First we consider the case when $n \geq 3$. Let $u \in H^1(\Omega_{0,R_0})$ be a solution of (1.9). For $x_0 \in \Omega_{0,r_0}$, we have, using Lemma 2.2,

$$\|u-u_0\|_{L^\infty(\Omega_{x_0,\delta})} \leq C\|u\|_{L^\infty(\Omega_{x_0,\delta^{1-\gamma}})}\delta^{\gamma\sigma} \tag{2.12}$$

for some constant u_0. We denote $v := u-u_0$, and v satisfies the same equation (1.9). We work on the domain $\Omega_{x_0,\delta/4}$, and perform a change of variables by setting

$$\begin{cases} y' = \delta^{-1}(x'-x_0'), \\ y_n = \delta^{-1}x_n. \end{cases} \tag{2.13}$$

The domain $\Omega_{x_0,\delta/4}$ becomes

$$\left\{ y \in \mathbb{R}^n \;\middle|\; |y'| \leq \frac{1}{4},\ \delta^{-1}\left(-\frac{1}{2}\varepsilon + g(x_0'+\delta y')\right) < y_n < \delta^{-1}\left(\frac{1}{2}\varepsilon + f(x_0'+\delta y')\right)\right\}.$$

We make a change of variables again by

$$\begin{cases} z' = 4y', \\ z_n = 2\delta^{m-1}\left(\dfrac{\delta y_n - g(x_0'+\delta y') + \varepsilon/2}{\varepsilon + f(x_0'+\delta y') - g(x_0'+\delta y')} - \dfrac{1}{2}\right). \end{cases} \tag{2.14}$$

Now the domain in z-variables becomes a thin plate $Q_{1,\delta^{m-1}}$, where $Q_{s,t}$ is defined as in (2.4). Let $w(z) = v(x)$, then w satisfies

$$\begin{cases} -\partial_i(b^{ij}(z)\partial_j w(z)) = 0 & \text{in } \ Q_{1,\delta^{m-1}}, \\ b^{nj}(z)\partial_j w(z) = 0 & \text{on } \ \{z_n = -\delta\} \cup \{z_n = \delta\}, \end{cases} \tag{2.15}$$

where the matrix $b(z) = (b^{ij}(z))$ is given by

$$(b^{ij}(z)) = \frac{(\partial_y z)(a^{ij})(\partial_y z)^t}{\det(\partial_y z)}. \tag{2.16}$$

Similar to the proof of Lemma 2.1, we will show that the Jacobian matrix of the change of variables (2.14), denoted by $\partial_y z$, and its inverse matrix $\partial_z y$ satisfy

$$|(\partial_y z)^{ij}| \leq C, \quad |(\partial_z y)^{ij}| \leq C \quad \text{for } z \in Q_{1,\delta^{m-1}}, \tag{2.17}$$

where $C > 0$ depends only on n, κ, R_0, λ_1, λ_2, $\|f\|_{C^2}$ and $\|g\|_{C^2}$. This leads to

$$\frac{\lambda}{C} \leq b(z) \leq C\Lambda \quad \text{for } z \in Q_{1,\delta^{m-1}}. \tag{2.18}$$

From (2.14), one can compute that

$$
\begin{aligned}
&(\partial_y z)^{ii} = 4 \quad \text{for} \quad 1 \le i \le n-1, \\
&(\partial_y z)^{nn} = \frac{2\delta^m}{\varepsilon + f(x_0' + \delta z'/4) - g(x_0' + \delta z'/4)}, \\
&(\partial_y z)^{ni} = -\frac{2\delta^m \partial_i g(x_0' + \delta z'/4) + (z_n + \delta^{m-1})\delta[\partial_i f(x_0' + \delta z'/4) - \partial_i g(x_0' + \delta z'/4)]}{\varepsilon + f(x_0' + \delta z'/4) - g(x_0' + \delta z'/4)} \\
&\qquad\qquad \text{for} \quad 1 \le i \le n-1, \\
&(\partial_y z)^{ij} = 0 \quad \text{for} \quad 1 \le i \le n-1, \quad j \ne i.
\end{aligned}
$$

First we will show that

$$
(\partial_y z)^{nn} \sim 1 \quad \text{for} \quad z \in Q_{1,\delta^{m-1}}. \tag{2.19}
$$

Since $|z'| < 1$ and $|x_0'| < \delta$, it is easy to see that

$$
(\partial_y z)^{nn} \ge \frac{1}{C} \frac{\delta^m}{\varepsilon + |x_0' + \delta z'/4|^m} \ge \frac{1}{C} \frac{\delta^m}{\varepsilon + C\delta^m} \ge \frac{1}{C} \quad \text{for} \quad z \in Q_{1,\delta^{m-1}}.
$$

On the other hand, when $|x_0'| \le \varepsilon^{\frac{1}{m}}$, we have $\delta \le (2\varepsilon)^{\frac{1}{m}}$, and hence

$$
(\partial_y z)^{nn} \le \frac{C\delta^m}{\varepsilon + |x_0' + \delta z'/4|^m} \le \frac{C\varepsilon}{\varepsilon + |x_0' + \delta z'/4|^m} \le C \quad \text{for} \quad z \in Q_{1,\delta^{m-1}}.
$$

When $|x_0'| \ge \varepsilon^{\frac{1}{m}}$, we have $|\delta z'/4| \le |x_0'|/2$, and hence

$$
\begin{aligned}
(\partial_y z)^{nn} \le& \frac{C\delta^m}{\varepsilon + |x_0' + \delta z'/4|^m} \le \frac{C\delta^m}{\varepsilon + (|x_0'| - |\delta z'/4|)^m} \\
\le& \frac{2\delta^m}{\varepsilon + (|x_0'|/2)^m} \le C \quad \text{for} \quad z \in Q_{1,\delta^{m-1}}.
\end{aligned}
$$

Therefore, (2.19) is verified. Since $|z_n| < \delta^{m-1}$, $|z'| < 1$ and $|x_0'| < \delta$, by (1.6a) and (1.6b), for $1 \le i \le n-1$,

$$
\begin{aligned}
|(\partial_y z)^{ni}| \le& \frac{2\delta^m |\partial_i g(x_0' + \delta z'/4)| + 2\delta^m [|\partial_i f(x_0' + \delta z'/4)| + |\partial_i g(x_0' + \delta z'/4)|]}{\varepsilon + f(x_0' + \delta z'/4) - g(x_0' + \delta z'/4)} \\
\le& \frac{C\delta^m}{\varepsilon + f(x_0' + \delta z'/4) - g(x_0' + \delta z'/4)} [|\partial_i f(x_0' + \delta z'/4)| + |\partial_i g(x_0' + \delta z'/4)|] \\
\le& C \frac{\delta^m}{\varepsilon + |x_0' + \delta z'/4|^m} |x_0' + \delta z'/4| \\
\le& C(|x_0'| + \delta|z'|) \le C\delta,
\end{aligned}
$$

where in the last line, we have used the same arguments in showing $(\partial_y z)^{nn} \le C$ earlier.

We have shown $(\partial_y z)^{ii} \sim 1$ for all $i = 1, \cdots, n$, and $|(\partial_y z)^{ij}| \le C\delta$ for $i \ne j$. We further require r_0 to be small enough so that off-diagonal entries are small. Therefore (2.17) follows. As mentioned earlier, (2.18) follows from (2.17).

Next, we will show

$$\|b\|_{C^\alpha(\overline{Q}_{1,\delta^{m-1}})} \le C \tag{2.20}$$

for some $C > 0$ depending only on n, m, R_0, λ_1, λ_2, λ_3, $\|f\|_{C^2}$, $\|g\|_{C^2}$ and $\|a\|_{C^\alpha}$, by showing

$$|\nabla_z(\partial_y z)^{ij}(z)| \le C, \qquad \left|\nabla_z \frac{1}{\det(\partial_y z)}\right| \le C \quad \text{for } z \in Q_{1,\delta^{m-1}}. \tag{2.21}$$

Then (2.20) follows from (2.21), (2.16), and $\|a\|_{C^\alpha} \le C$.

By a straightforward computation, we have, for any $i = 1, \cdots, n-1$,

$$\begin{aligned}
\left|\partial_{z_i} \frac{1}{\det(\partial_y z)}\right| &= \left|\partial_{z_i}\left(\frac{\varepsilon + f(x_0' + \delta z'/4) - g(x_0' + \delta z'/4)}{2 \cdot 4^{n-1}\delta^m}\right)\right| \\
&= \left|\frac{\delta[\partial_i f(x_0' + \delta z'/4) - \partial_i g(x_0' + \delta z'/4)]}{2 \cdot 4^{n-1}\delta^m}\right| \\
&\le \frac{C}{\delta^{m-1}}|x_0' + \delta z'/4|^{m-1} \le C \quad \text{for} \quad z \in Q_{1,\delta},
\end{aligned}$$

where in the last line, (1.6b) and (1.6c) have been used. For any $i = 1, \cdots, n-1$, by (1.6b) and (1.6c),

$$\begin{aligned}
|\partial_{z_i}(\partial_y z)^{nn}| &= \left|\frac{2\delta^{m+1}[\partial_i f(x_0' + \delta z'/4) - \partial_i g(x_0' + \delta z'/4)]}{(\varepsilon + f(x_0' + \delta z'/4) - g(x_0' + \delta z'/4))^2}\right| \\
&\le \frac{C\delta^{m+1}}{(\varepsilon + |x_0' + \delta z'/4|^m)^2}|x_0' + \delta z'/4|^{m-1} \\
&\le \frac{C\delta^{m+1}|x_0' + \delta z'/4|^{m-1}}{\delta^{2m}} \le C \quad \text{for} \quad z \in Q_{1,\delta},
\end{aligned}$$

where in the last line, we have used the same arguments in showing $(\partial_y z)^{nn} \le C$ earlier. Similar computations apply to $\partial_{z_i}(\partial_y z)^{ni}$ for $i = 1, \cdots, n-1$, and we have

$$|\partial_{z_i}(\partial_y z)^{ni}| \le C \quad \text{for} \quad z \in Q_{1,\delta^{m-1}}.$$

Finally, we compute, for $i = 1, \cdots, n-1$,

$$\begin{aligned}
|\partial_{z_n}(\partial_y z)^{ni}| &= \left|\frac{2\delta[\partial_i f(x_0' + \delta z'/4) - \partial_i g(x_0' + \delta z'/4)]}{\varepsilon + f(x_0' + \delta z'/4) - g(x_0' + \delta z'/4)}\right| \\
&\le \frac{C\delta|x_0' + \delta z'/4|^{m-1}}{\varepsilon + |x_0' + \delta z'/4|^m} \le C \quad \text{for} \quad z \in Q_{1,\delta}.
\end{aligned}$$

Therefore, (2.21) is verified, and hence (2.20) follows as mentioned above.

Now we define

$$S_l := \left\{ z \in \mathbb{R}^n \;\middle|\; |z'| < 1,\ (l-1)\delta^{m-1} < z_n < (l+1)\delta^{m-1} \right\}$$

for any integer l, and

$$S := \left\{ z \in \mathbb{R}^n \;\middle|\; |z'| < 1,\ |z_n| < 1 \right\}.$$

Note that $Q_{1,\delta^{m-1}} = S_0$. As in the proof of Lemma 2.1, we define, for any $l \in \mathbb{Z}$, a new function $\tilde{w}$ by setting

$$\tilde{w}(z) := w\left(z', (-1)^l \left(z_n - 2l\delta^{m-1}\right)\right), \quad \forall z \in S_l.$$

We also define the corresponding coefficients, for $k = 1, 2, \cdots, n-1$,

$$\tilde{b}^{nk}(z) = \tilde{b}^{kn}(z) := (-1)^l b^{nk}\left(z', (-1)^l \left(z_n - 2l\delta^{m-1}\right)\right), \quad \forall z \in S_l,$$

and for other indices,

$$\tilde{b}^{ij}(z) := b^{ij}\left(z', (-1)^l \left(z_n - 2l\delta^{m-1}\right)\right), \quad \forall y \in S_l.$$

Then $\tilde{w}$ and $\tilde{b}^{ij}$ are defined in the infinite cylinder $Q_{1,\infty}$. By (2.15), $\tilde{w}$ satisfies the equation

$$-\partial_i(\tilde{b}^{ij}\partial_j \tilde{w}) = 0 \quad \text{in} \quad Q_{1,\infty}.$$

Note that for any $l \in \mathbb{Z}$, $\tilde{b}(z)$ is orthogonally conjugated to $b\left(z', (-1)^l \left(z_n - 2l\delta^{m-1}\right)\right)$, for $z \in S_l$. Hence, by (2.18), we have

$$\frac{\lambda}{C} \le \tilde{b}(z) \le C\Lambda \quad \text{for} \quad z \in Q_{1,\infty},$$

and, by (2.20),

$$\|\tilde{b}\|_{C^\alpha(\overline{S}_l)} \le C, \quad \forall l \in \mathbb{Z}.$$

Apply Lemma 2.1 in [12] on S with $N = 1$, we have

$$\|\nabla \tilde{w}\|_{L^\infty(\frac{1}{2}S)} \le C\|\tilde{w}\|_{L^2(S)}.$$

It follows that

$$\|\nabla w\|_{L^\infty(Q_{1/2,\delta^{m-1}})} \le \frac{C}{\delta^{(m-1)/2}}\|w\|_{L^2(Q_{1,\delta^{m-1}})} \le C\|w\|_{L^\infty(Q_{1,\delta^{m-1}})}$$

for some positive constant C, depending only on $n, \alpha, R_0, m, \lambda, \Lambda, \lambda_1, \lambda_2, \lambda_3, \|f\|_{C^2}, \|g\|_{C^2}$ and $\|a\|_{C^\alpha}$.

By (2.17), we have $\|(\partial_z y)\|_{L^\infty(Q_{1,\delta^{m-1}})} \le C$, where C is independent of ε and δ. Reversing the change of variables (2.14) and (2.13), we have, by (2.12)

$$\delta\|\nabla v\|_{L^\infty(\Omega_{x_0,\delta/8})} \le C\|v\|_{L^\infty(\Omega_{x_0,\delta/4})} \le C\|u\|_{L^\infty(\Omega_{x_0,\delta^{1-\gamma}})}\delta^{\gamma\sigma}. \tag{2.22}$$

In particular, this implies

$$|\nabla u(x_0)| \le C\|u\|_{L^\infty(\Omega_{x_0,\delta^{1-\gamma}})}\delta^{-1+\gamma\sigma},$$

and it concludes the proof of Theorem 1.1 for the case $n \ge 3$ after taking $\beta = \gamma\sigma/2$.

For the case $n = 2$, we work with u instead of v, and repeat the argument in deriving the first inequality in (2.22), we have

$$\delta\|\nabla u\|_{L^\infty(\Omega_{x_0,\delta/8})} \le C\|u\|_{L^\infty(\Omega_{x_0,\delta/4})}.$$

In particular,

$$|\nabla u(x_0)| \le C\|u\|_{L^\infty(\Omega_{x_0,\delta/4})}\delta^{-1}.$$

This concludes the proof of Theorem 1.1 for the case $n = 2$. □

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