

Gradient Estimates of Solutions to the Conductivity Problem with Flatter Insulators

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. We study the insulated conductivity problem with inclusions embedded in a bounded domain in \mathbb{R}^n . When the distance of inclusions, denoted by ε , goes to 0, the gradient of solutions may blow up. When two inclusions are strictly convex, it was known that an upper bound of the blow-up rate is of order $\varepsilon^{-1/2}$ for $n = 2$, and is of order $\varepsilon^{-1/2+\beta}$ for some $\beta > 0$ when dimension $n \geq 3$. In this paper, we generalize the above results for insulators with flatter boundaries near touching points.

Key Words: Conductivity problem, harmonic functions, maximum principle, gradient estimates.

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1 Introduction and main results

Let Ω be a bounded domain in \mathbb{R}^n with C^2 boundary, and let D_1^* and D_2^* be two open sets whose closure belongs to Ω , touching only at the origin with the inner normal vector of ∂D_1^* pointing in the positive x_n -direction. Denote $x = (x', x_n)$. Translating D_1^* and D_2^* by $\frac{\varepsilon}{2}$ along x_n -axis, we obtain

$$D_1^\varepsilon := D_1^* + (0', \varepsilon/2) \quad \text{and} \quad D_2^\varepsilon := D_2^* - (0', \varepsilon/2).$$

When there is no confusion, we drop the superscripts ε and denote $D_1 := D_1^\varepsilon$ and $D_2 := D_2^\varepsilon$. Denote $\tilde{\Omega} := \Omega \setminus (\overline{D_1} \cup \overline{D_2})$. A simple model for electric conduction can be formulated as the following elliptic equation:

$$\begin{cases} \operatorname{div}(a_k(x)\nabla u_k) = 0 & \text{in } \Omega, \\ u_k = \varphi(x) & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

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where $\varphi \in C^2(\partial\Omega)$ is given, and

$$a_k(x) = \begin{cases} k \in (0, \infty) & \text{in } D_1 \cup D_2, \\ 1 & \text{in } \tilde{\Omega}, \end{cases}$$

refers to conductivities. The solution u_k and its gradient ∇u_k represent the voltage potential and the electric fields respectively. From an engineering point of view, It is an interesting problem to capture the behavior of ∇u_k . Babuška, et al. [3] numerically analyzed that the gradient of solutions to an analogous elliptic system stays bounded regardless of ε , the distance between the inclusions. Bonnetier and Vogelius [5] proved that for a fixed k , $|\nabla u_k|$ is bounded for touching disks D_1 and D_2 in dimension $n = 2$. A general result was obtained by Li and Vogelius [11] for general second order elliptic equations of divergence form with piecewise Hölder coefficients and general shape of inclusions D_1 and D_2 in any dimension. When k is bounded away from 0 and ∞ , they established a $W^{1,\infty}$ bound of u_k in Ω , and a $C^{1,\alpha}$ bound in each region that do not depend on ε . This result was further extended by Li and Nirenberg [10] to general second order elliptic systems of divergence form. Some higher order estimates with explicit dependence on r_1, r_2, k and ε were obtained by Dong and Li [7] for two circular inclusions of radius r_1 and r_2 respectively in dimension $n = 2$. There are still some related open problems on general elliptic equations and systems. We refer to p. 94 of [11] and p. 894 of [10].

When the inclusions are insulators ($k = 0$), it was shown in [6,9,13] that the gradient of solutions generally becomes unbounded, as $\varepsilon \rightarrow 0$. It was known that (see e.g., Appendix of [4]) when $k \rightarrow 0$, u_k converges to the solution of the following insulated conductivity problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \tilde{\Omega}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, \quad i = 1, 2, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here ν denotes the inward unit normal vectors on $\partial D_i, i = 1, 2$.

The behavior of the gradient in terms of ε has been studied by Ammari et al. in [1] and [2], where they considered the insulated problem on the whole Euclidean space:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{(D_1 \cup D_2)}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, \quad i = 1, 2, \\ u(x) - H(x) = \mathcal{O}(|x|^{n-1}) & \text{as } |x| \rightarrow \infty. \end{cases} \tag{1.3}$$

They established when dimension $n = 2$, D_1^* and D_2^* are disks of radius r_1 and r_2 respectively, and H is a harmonic function in \mathbb{R}^2 , the solution u of (1.3) satisfies

$$\|\nabla u\|_{L^\infty(B_4)} \leq C\varepsilon^{-1/2},$$

for some positive constant C independent of ε . They also showed that the upper bounds are optimal in the sense that for appropriate H ,

$$\|\nabla u\|_{L^\infty(B_4)} \geq \varepsilon^{-1/2}/C.$$

In fact, the equation

$$\begin{cases} \operatorname{div}(a_k(x)\nabla u_k) = 0 & \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}, \\ u(x) - H(x) = \mathcal{O}(|x|^{-1}) & \text{as } |x| \rightarrow \infty, \end{cases}$$

was studied there, and the estimates derived have explicit dependence on r_1, r_2, k and ε .

Yun extended in [14] and [15] these results allowing D_1^* and D_2^* to be any bounded strictly convex smooth domains in \mathbb{R}^2 .

The above upper bound of ∇u was localized and extended to higher dimensions by Bao, Li and Yin in [4], where they considered problem (1.2) and proved

$$\|\nabla u\|_{L^\infty(\tilde{\Omega})} \leq C\varepsilon^{-1/2}\|\varphi\|_{C^2(\partial\Omega)}, \quad \text{when } n \geq 2. \tag{1.4}$$

The upper bound is optimal for $n = 2$ as mentioned earlier. For dimensions $n \geq 3$, the upper bound was recently improved by Li and Yang [12] to

$$\|\nabla u\|_{L^\infty(\tilde{\Omega})} \leq C\varepsilon^{-1/2+\beta}\|\varphi\|_{C^2(\partial\Omega)}, \quad \text{when } n \geq 3, \tag{1.5}$$

for some $\beta > 0$.

Yun [16] considered the problem (1.3) in \mathbb{R}^3 , with unit disks

$$D_1 = B_1(0, 0, 1 + \varepsilon/2), \quad D_2 = B_1(0, 0, -1 - \varepsilon/2),$$

and a harmonic function H . He proved that for some positive constant C independent of ε ,

$$\max_{|x_3| \leq \varepsilon/2} |\nabla u(0, 0, x_3)| \leq C\varepsilon^{\frac{\sqrt{2}-2}{2}}.$$

He also showed that this upper bound of $|\nabla u|$ on the ε -segment connecting D_1 and D_2 is optimal for $H(x) \equiv x_1$.

In this paper, we assume that for some $m \in [2, \infty)$ and a small universal constant R_0 , the portions of ∂D_1^* and ∂D_2^* in $[-R_0, R_0]^n$ are respectively the graphs of two C^2 functions f and g in terms of x' , and

$$f(0') = g(0') = 0, \quad \nabla f(0') = \nabla g(0') = 0, \tag{1.6a}$$

$$\lambda_1|x'|^m \leq (f - g)(x') \leq \lambda_2|x'|^m \quad \text{for } 0 < |x'| < R_0, \tag{1.6b}$$

$$|\nabla(f - g)(x')| \leq \lambda_3|x'|^{m-1} \quad \text{for } 0 < |x'| < R_0, \tag{1.6c}$$

for some $\lambda_1, \lambda_2, \lambda_3 > 0$. Let $a(x) \in C^\alpha(\tilde{\Omega})$, for some $\alpha \in (0, 1)$, be a symmetric, positive definite matrix function satisfying

$$\lambda \leq a(x) \leq \Lambda \quad \text{for } x \in \tilde{\Omega},$$

for some positive constants λ, Λ . Let $\nu = (\nu_1, \dots, \nu_n)$ denote the unit normal vector on ∂D_1 and ∂D_2 , pointing towards the interior of D_1 and D_2 . We consider the following insulated conductivity problem:

$$\begin{cases} -\partial_i(a^{ij}\partial_j u) = 0 & \text{in } \tilde{\Omega}, \\ a^{ij}\partial_j u \nu_i = 0 & \text{on } \partial(D_1 \cup D_2), \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \quad (1.7)$$

where $\varphi \in C^2(\partial\Omega)$ is given. For $0 < r \leq R_0$, we denote

$$\Omega_{x_0, r} := \left\{ (x', x_n) \in \tilde{\Omega} \mid -\frac{\varepsilon}{2} + g(x') < x_n < \frac{\varepsilon}{2} + f(x'), |x' - x'_0| < r \right\}, \quad (1.8a)$$

$$\Gamma_+ := \left\{ x_n = \frac{\varepsilon}{2} + f(x'), |x'| < R_0 \right\}, \quad \Gamma_- := \left\{ x_n = -\frac{\varepsilon}{2} + g(x'), |x'| < R_0 \right\}. \quad (1.8b)$$

Since the blow-up of gradient can only occur in the narrow region between D_1 and D_2 , we will focus on the following problem near the origin:

$$\begin{cases} -\partial_i(a^{ij}\partial_j u) = 0 & \text{in } \Omega_{0, R_0}, \\ a^{ij}\partial_j u \nu_i = 0 & \text{on } \Gamma_+ \cup \Gamma_-, \end{cases} \quad (1.9)$$

where $\nu = (\nu_1, \dots, \nu_n)$ denotes the unit normal vector on Γ_+ and Γ_- , pointing upward and downward respectively.

Theorem 1.1. *Let $m, \Gamma_+, \Gamma_-, a, \alpha$ be as above, and let $u \in H^1(\Omega_{0, R_0})$ be a solution of (1.9). There exist positive constants r_0, β and C depending only on $n, m, \lambda, \Lambda, R_0, \alpha, \lambda_1, \lambda_2, \lambda_3, \|f\|_{C^2(\{|x'| \leq R_0\})}, \|g\|_{C^2(\{|x'| \leq R_0\})}$ and $\|a\|_{C^\alpha(\Omega_{0, R_0})}$, such that*

$$|\nabla u(x_0)| \leq \begin{cases} C \|u\|_{L^\infty(\Omega_{0, R_0})} (\varepsilon + |x'_0|^m)^{-1/m}, & \text{when } n = 2, \\ C \|u\|_{L^\infty(\Omega_{0, R_0})} (\varepsilon + |x'_0|^m)^{-1/m+\beta}, & \text{when } n \geq 3, \end{cases} \quad (1.10)$$

for all $x_0 \in \Omega_{0, r_0}$ and $\varepsilon \in (0, 1)$.

Remark 1.1. For $m = 2$, (1.10) was proved in [4] and [12] for $n = 2$ and $n \geq 3$, respectively.

Let $u \in H^1(\tilde{\Omega})$ be a weak solution of (1.7). By the maximum principle and the gradient estimates of solutions of elliptic equations,

$$\|u\|_{L^\infty(\tilde{\Omega})} \leq \|\varphi\|_{L^\infty(\partial\Omega)}, \quad (1.11a)$$

$$\|\nabla u\|_{L^\infty(\tilde{\Omega} \setminus \Omega_{0, r_0})} \leq C \|\varphi\|_{C^2(\partial\Omega)}. \quad (1.11b)$$

Therefore, a corollary of Theorem 1.1 is as follows.

Corollary 1.1. *Let $u \in H^1(\tilde{\Omega})$ be a weak solution of (1.7). There exist positive constants β and C depending only on $n, m, \lambda, \Lambda, R_0, \alpha, \lambda_1, \lambda_2, \lambda_3, \|\partial D_1\|_{C^2}, \|\partial D_2\|_{C^2}, \|\partial \Omega\|_{C^2}$, and $\|a\|_{C^\alpha(\bar{\Omega})}$, such that*

$$\|\nabla u\|_{L^\infty(\tilde{\Omega})} \leq \begin{cases} C\|\varphi\|_{C^2(\partial\Omega)}\varepsilon^{-\frac{1}{m}}, & \text{when } n = 2, \\ C\|\varphi\|_{C^2(\partial\Omega)}\varepsilon^{-\frac{1}{m}+\beta}, & \text{when } n \geq 3. \end{cases} \tag{1.12}$$

2 Proof of Theorem 1.1

Our proof of Theorem 1.1 is an adaption of the arguments in our earlier paper [12] for $m = 2$, and follows closely the arguments there.

We fix a $\gamma \in (0, 1)$, and let $r_0 > 0$ denote a constant depending only on $n, m, \gamma, R_0, \lambda_1, \lambda_2, \|f\|_{C^2}$ and $\|g\|_{C^2}$, whose value will be fixed in the proof. For any $x_0 \in \Omega_{0,r_0}$, we define

$$\delta := (\varepsilon + |x'_0|^m)^{\frac{1}{m}}. \tag{2.1}$$

We will always consider $0 < \varepsilon \leq r_0^m$. First, we require r_0 small so that for $|x'_0| < r_0$,

$$10\delta < \delta^{1-\gamma} < \frac{R_0}{4}.$$

Lemma 2.1. *For $n \geq 3$, there exists a small r_0 , depending only on n, m, γ , and R_0 , such that for any $x_0 \in \Omega_{0,r_0}, 5|x'_0| < r < \delta^{1-\gamma}$, if $u \in H^1(\Omega_{x_0,2r} \setminus \Omega_{x_0,r/4})$ is a positive solution to the equation*

$$\begin{cases} -\partial_i(a^{ij}(x)\partial_j u(x)) = 0 & \text{in } \Omega_{x_0,2r} \setminus \Omega_{x_0,r/4}, \\ a^{ij}(x)\partial_j u(x)v_i(x) = 0 & \text{on } (\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega_{x_0,2r} \setminus \Omega_{x_0,r/4}}, \end{cases}$$

then

$$\sup_{\Omega_{x_0,r} \setminus \Omega_{x_0,r/2}} u \leq C \inf_{\Omega_{x_0,r} \setminus \Omega_{x_0,r/2}} u, \tag{2.2}$$

for some constant $C > 0$ depending only on $n, m, \lambda, \Lambda, R_0, \lambda_1, \lambda_2, \|f\|_{C^2}$ and $\|g\|_{C^2}$ but independent of r and u .

Proof. We only need to prove (2.2) for $|x'_0| > 0$, since the $|x'_0| = 0$ case follows from the result for $|x'_0| > 0$ and then sending $|x'_0|$ to 0. We denote

$$h_r := \varepsilon + f\left(x'_0 - \frac{r}{4} \frac{x'_0}{|x'_0|}\right) - g\left(x'_0 - \frac{r}{4} \frac{x'_0}{|x'_0|}\right),$$

and perform a change of variables by setting

$$\begin{cases} y' = x' - x'_0, \\ y_n = 2h_r \left(\frac{x_n - g(x') + \varepsilon/2}{\varepsilon + f(x') - g(x')} - \frac{1}{2} \right), \end{cases} \quad (x', x_n) \in \Omega_{x_0,2r} \setminus \Omega_{x_0,r/4}. \tag{2.3}$$

This change of variables maps the domain $\Omega_{x_0,2r} \setminus \Omega_{x_0,r/4}$ to an annular cylinder of height h_r , denoted by $Q_{2r,h_r} \setminus Q_{r/4,h_r}$, where

$$Q_{s,t} := \{y = (y', y_n) \in \mathbb{R}^n \mid |y'| < s, |y_n| < t\}, \tag{2.4}$$

for $s, t > 0$. We will show that the Jacobian matrix of the change of variables (2.3), denoted by $\partial_x y$, and its inverse matrix $\partial_y x$ satisfy

$$|(\partial_x y)^{ij}| \leq C, \quad |(\partial_y x)^{ij}| \leq C \quad \text{for } y \in Q_{2r,h_r} \setminus Q_{r/4,h_r}, \tag{2.5}$$

where $C > 0$ depends only on $n, m, R_0, \lambda_1, \lambda_2, \|f\|_{C^2}$ and $\|g\|_{C^2}$.

Let $v(y) = u(x)$, then v satisfies

$$\begin{cases} -\partial_i(b^{ij}(y)\partial_j v(y)) = 0 & \text{in } Q_{2r,h_r} \setminus Q_{r/4,h_r}, \\ b^{nj}(y)\partial_j v(y) = 0 & \text{on } \{y_n = -h_r\} \cup \{y_n = h_r\}, \end{cases} \tag{2.6}$$

where the matrix $(b^{ij}(y))$ is given by

$$(b^{ij}(y)) = \frac{(\partial_x y)(a^{ij})(\partial_x y)^t}{\det(\partial_x y)}, \tag{2.7}$$

$(\partial_x y)^t$ is the transpose of $\partial_x y$.

It is easy to see that (2.5) implies, using $\lambda \leq (a^{ij}) \leq \Lambda$,

$$\frac{\lambda}{C} \leq (b^{ij}(y)) \leq C\Lambda \quad \text{for } y \in Q_{2r,h_r} \setminus Q_{r/4,h_r}, \tag{2.8}$$

for some constant $C > 0$ depending only on $n, m, R_0, \lambda_1, \lambda_2, \|f\|_{C^2}$ and $\|g\|_{C^2}$.

In the following and throughout this section, we will denote $A \sim B$, if there exists a positive universal constant C , which might depend on $n, m, \lambda, \Lambda, R_0, \lambda_1, \lambda_2, \|f\|_{C^2}$ and $\|g\|_{C^2}$, but not depend on ε , such that $C^{-1}B \leq A \leq CB$.

From (2.3), one can compute that

$$\begin{aligned} (\partial_x y)^{ii} &= 1 \quad \text{for } 1 \leq i \leq n-1, \\ (\partial_x y)^{nn} &= \frac{2h_r}{\varepsilon + f(x'_0 + y') - g(x'_0 + y')}, \\ (\partial_x y)^{ni} &= -\frac{2h_r \partial_i g(x'_0 + y') + 2y_n [\partial_i f(x'_0 + y') - \partial_i g(x'_0 + y')]}{\varepsilon + f(x'_0 + y') - g(x'_0 + y')} \quad \text{for } 1 \leq i \leq n-1, \\ (\partial_x y)^{ij} &= 0 \quad \text{for } 1 \leq i \leq n-1, \quad j \neq i. \end{aligned}$$

By (1.6b), one can see that

$$h_r \sim \varepsilon + \left| x'_0 - \frac{r}{4} \frac{x'_0}{|x'_0|} \right|^m.$$

Since $|y_n| \leq h_r$, by using (1.6a) and (1.6b), we have that, for $1 \leq i \leq n-1$,

$$\begin{aligned} |(\partial_x y)^{ni}| &\leq C \frac{h_r |\partial_i g(x'_0 + y')| + h_r [|\partial_i f(x'_0 + y')| + |\partial_i g(x'_0 + y')|]}{\varepsilon + f(x'_0 + y') - g(x'_0 + y')} \\ &\leq C \frac{h_r}{\varepsilon + f(x'_0 + y') - g(x'_0 + y')} [|\partial_i f(x'_0 + y')| + |\partial_i g(x'_0 + y')|] \\ &\leq C \frac{\varepsilon + \left| x'_0 - \frac{r}{4} \frac{x'_0}{|x'_0|} \right|^m}{\varepsilon + |x'_0 + y'|^m} |x'_0 + y'|. \end{aligned}$$

Since $r/4 < |y'| < 2r < 2\delta^{1-\gamma}$ and $|x'_0| < \delta$, we can estimate

$$|(\partial_x y)^{ni}| \leq C |x'_0 + y'| \leq C(|x'_0| + |y'|) \leq C\delta^{1-\gamma}.$$

Next, we will show that

$$(\partial_x y)^{nn} \sim 1 \quad \text{for } y \in Q_{2r, h_r} \setminus Q_{r/4, h_r}. \quad (2.9)$$

Indeed, by (1.6b), we have

$$(\partial_x y)^{nn} = \frac{2h_r}{\varepsilon + f(x'_0 + y') - g(x'_0 + y')} \sim \frac{\varepsilon + \left| x'_0 - \frac{r}{4} \frac{x'_0}{|x'_0|} \right|^m}{\varepsilon + |x'_0 + y'|^m}.$$

Since $|y'| > r/4$, it is easy to see

$$(\partial_x y)^{nn} \leq C \frac{\varepsilon + \left| x'_0 - \frac{r}{4} \frac{x'_0}{|x'_0|} \right|^m}{\varepsilon + |x'_0 + y'|^m} \leq C.$$

On the other hand, since $|y'| < 2r$ and $|x'_0| < r/5$, we have

$$\begin{aligned} \varepsilon + \left| x'_0 - \frac{r}{4} \frac{x'_0}{|x'_0|} \right|^m &\geq \varepsilon + \left(\left| \frac{r}{4} \frac{x'_0}{|x'_0|} \right| - |x'_0| \right)^m \geq \varepsilon + \left(\frac{r}{4} - \frac{r}{5} \right)^m \geq \frac{1}{C} (\varepsilon + r^m), \\ \varepsilon + |x'_0 + y'|^m &\leq \varepsilon + m|x'_0|^m + m|y'|^m \leq C(\varepsilon + r^m). \end{aligned}$$

Therefore,

$$(\partial_x y)^{nn} \geq \frac{1}{C} \frac{\varepsilon + \left| x'_0 - \frac{r}{4} \frac{x'_0}{|x'_0|} \right|^m}{\varepsilon + |x'_0 + y'|^m} \geq \frac{1}{C}$$

and (2.9) is verified.

We have shown $(\partial_x y)^{ii} \sim 1$ for all $i = 1, \dots, n$, and $|(\partial_x y)^{ij}| \leq C\delta^{(1-\gamma)}$ for $i \neq j$. We further require r_0 to be small enough so that off-diagonal entries of $\partial_x y$ are small. Therefore (2.5) follows. As mentioned earlier, (2.8) follows from (2.5).

Now we define, for any integer l ,

$$A_l := \left\{ y \in \mathbb{R}^n \mid \frac{r}{4} < |y'| < 2r, (l-1)h_r < z_n < (l+1)h_r \right\}.$$

Note that $A_0 = Q_{2r,h_r} \setminus Q_{r/4,h_r}$. For any $l \in \mathbb{Z}$, we define a new function \tilde{v} by

$$\tilde{v}(y) := v\left(y', (-1)^l (y_n - 2lh_r)\right), \quad \forall y \in A_l.$$

We also define the corresponding coefficients, for $k = 1, 2, \dots, n-1$,

$$\tilde{b}^{nk}(y) = \tilde{b}^{kn}(y) := (-1)^l b^{nk}\left(y', (-1)^l (y_n - 2lh_r)\right), \quad \forall y \in A_l,$$

and for other indices,

$$\tilde{b}^{ij}(y) := b^{ij}\left(y', (-1)^l (y_n - 2lh_r)\right), \quad \forall y \in A_l.$$

Therefore, $\tilde{v}(y)$ and $\tilde{b}^{ij}(y)$ are defined in the infinite cylinder shell $Q_{2r,\infty} \setminus Q_{r/4,\infty}$. By (2.6), $\tilde{v} \in H^1(Q_{2r,\infty} \setminus Q_{r/4,\infty})$ satisfies

$$-\partial_i(\tilde{b}^{ij}(y)\partial_j\tilde{v}(y)) = 0 \quad \text{in } Q_{2r,\infty} \setminus Q_{r/4,\infty}.$$

Note that for any $l \in \mathbb{Z}$ and $y \in A_l$, $\tilde{b}(y) = (\tilde{b}^{ij}(y))$ is orthogonally conjugated to $b(y', (-1)^l (y_n - 2lh_r))$. Hence, by (2.8), we have

$$\frac{\lambda}{C} \leq \tilde{b}(y) \leq C\Lambda \quad \text{for } y \in Q_{2r,\infty} \setminus Q_{r/4,\infty}.$$

We restrict the domain to be $Q_{2r,r} \setminus Q_{r/4,r}$, and make the change of variables $z = y/r$. Set $\bar{v}(z) = \tilde{v}(y)$, $\bar{b}^{ij}(z) = \tilde{b}^{ij}(y)$, we have

$$\begin{aligned} -\partial_i(\bar{b}^{ij}(z)\partial_j\bar{v}(z)) &= 0 && \text{in } Q_{2,1} \setminus Q_{1/4,1}, \\ \frac{\lambda}{C} \leq \bar{b}(z) &\leq C\Lambda && \text{for } z \in Q_{2,1} \setminus Q_{1/4,1}. \end{aligned}$$

Then by the Harnack inequality for uniformly elliptic equations of divergence form, see e.g., [8, Theorem 8.20], there exists a constant C depending only on $n, m, \lambda, \Lambda, R_0, \lambda_1, \lambda_2, \|f\|_{C^2}$ and $\|g\|_{C^2}$, such that

$$\sup_{Q_{1,1/2} \setminus Q_{1/2,1/2}} \bar{v} \leq C \inf_{Q_{1,1/2} \setminus Q_{1/2,1/2}} \bar{v}.$$

In particular, we have

$$\sup_{Q_{1,h_r/r} \setminus Q_{1/2,h_r/r}} \bar{v} \leq C \inf_{Q_{1,h_r/r} \setminus Q_{1/2,h_r/r}} \bar{v},$$

which is (2.2) after reversing the change of variables. □

Remark 2.1. Lemma 2.1 does not hold for dimension $n = 2$, since $Q_{2,1} \setminus Q_{1/4,1} \subset \mathbb{R}^2$ is the union of two disjoint rectangular domains, and the Harnack inequality cannot be applied on it. Therefore, we will separate the cases $n = 2$ and $n \geq 3$ in our proof of Theorem 1.1.

For any domain $A \subset \tilde{\Omega}$, we denote the oscillation of u in A by $\text{osc}_A u := \sup_A u - \inf_A u$. Using Lemma 2.1, we obtain a decay of $\text{osc}_{\Omega_{x_0,\delta}} u$ in δ as follows.

Lemma 2.2. For $n \geq 3$, let u be a solution of (1.9). For any $x_0 \in \Omega_{0,r_0}$, where r_0 is as in Lemma 2.1, there exist positive constants σ and C , depending only on $n, m, \lambda, \Lambda, R_0, \lambda_1, \lambda_2, \|f\|_{C^2}$ and $\|g\|_{C^2}$ such that

$$\text{osc}_{\Omega_{x_0,\delta}} u \leq C \|u\|_{L^\infty(\Omega_{x_0,\delta^{1-\gamma}})} \delta^{\gamma\sigma}. \quad (2.10)$$

Proof. For simplicity, we drop the x_0 subscript and denote $\Omega_r = \Omega_{x_0,r}$ in this proof. Let $5|x'_0| < r < \delta^{1-\gamma}$ and $u_1 = \sup_{\Omega_{2r}} u - u, u_2 = u - \inf_{\Omega_{2r}} u$. By Lemma 2.1, we have

$$\sup_{\Omega_r \setminus \Omega_{r/2}} u_1 \leq C_1 \inf_{\Omega_r \setminus \Omega_{r/2}} u_1, \quad \sup_{\Omega_r \setminus \Omega_{r/2}} u_2 \leq C_1 \inf_{\Omega_r \setminus \Omega_{r/2}} u_2,$$

where $C_1 > 1$ is a constant independent of r . Since both u_1 and u_2 satisfy Eq. (1.9), by the maximum principle,

$$\sup_{\Omega_r \setminus \Omega_{r/2}} u_i = \sup_{\Omega_r} u_i, \quad \inf_{\Omega_r \setminus \Omega_{r/2}} u_i = \inf_{\Omega_r} u_i,$$

for $i = 1, 2$. Therefore,

$$\sup_{\Omega_r} u_1 \leq C_1 \inf_{\Omega_r} u_1, \quad \sup_{\Omega_r} u_2 \leq C_1 \inf_{\Omega_r} u_2.$$

Adding up the above two inequalities, we have

$$\text{osc}_{\Omega_r} u \leq \left(\frac{C_1 - 1}{C_1 + 1} \right) \text{osc}_{\Omega_{2r}} u.$$

Now we take $\sigma > 0$ such that $2^{-\sigma} = \frac{C_1 - 1}{C_1 + 1}$, then

$$\text{osc}_{\Omega_r} u \leq 2^{-\sigma} \text{osc}_{\Omega_{2r}} u. \quad (2.11)$$

We start with $r = r_0 = \delta^{1-\gamma}/2$, and set $r_{i+1} = r_i/2$. Keep iterating (2.11) $k + 1$ times, where k satisfies $5\delta \leq r_k < 10\delta$, we will have

$$\text{osc}_{\Omega_\delta} u \leq \text{osc}_{\Omega_{r_k}} u \leq 2^{-(k+1)\sigma} \text{osc}_{\Omega_{2r_0}} u \leq 2^{1-(k+1)\sigma} \|u\|_{L^\infty(\Omega_{\delta^{1-\gamma}})}.$$

Since

$$10\delta > r^k = 2^{-k} r_0 = 2^{-(k+1)} \delta^{1-\gamma},$$

we have

$$2^{-(k+1)} < 10\delta^\gamma$$

and hence (2.10) follows immediately. \square

Proof of Theorem 1.1. First we consider the case when $n \geq 3$. Let $u \in H^1(\Omega_{0,R_0})$ be a solution of (1.9). For $x_0 \in \Omega_{0,r_0}$, we have, using Lemma 2.2,

$$\|u - u_0\|_{L^\infty(\Omega_{x_0,\delta})} \leq C \|u\|_{L^\infty(\Omega_{x_0,\delta^{1-\gamma}})} \delta^{\gamma\sigma} \tag{2.12}$$

for some constant u_0 . We denote $v := u - u_0$, and v satisfies the same equation (1.9). We work on the domain $\Omega_{x_0,\delta/4}$, and perform a change of variables by setting

$$\begin{cases} y' = \delta^{-1}(x' - x'_0), \\ y_n = \delta^{-1}x_n. \end{cases} \tag{2.13}$$

The domain $\Omega_{x_0,\delta/4}$ becomes

$$\left\{ y \in \mathbb{R}^n \mid |y'| \leq \frac{1}{4}, \delta^{-1} \left(-\frac{1}{2}\varepsilon + g(x'_0 + \delta y') \right) < y_n < \delta^{-1} \left(\frac{1}{2}\varepsilon + f(x'_0 + \delta y') \right) \right\}.$$

We make a change of variables again by

$$\begin{cases} z' = 4y', \\ z_n = 2\delta^{m-1} \left(\frac{\delta y_n - g(x'_0 + \delta y') + \varepsilon/2}{\varepsilon + f(x'_0 + \delta y') - g(x'_0 + \delta y')} - \frac{1}{2} \right). \end{cases} \tag{2.14}$$

Now the domain in z -variables becomes a thin plate $Q_{1,\delta^{m-1}}$, where $Q_{s,t}$ is defined as in (2.4). Let $w(z) = v(x)$, then w satisfies

$$\begin{cases} -\partial_i(b^{ij}(z)\partial_j w(z)) = 0 & \text{in } Q_{1,\delta^{m-1}}, \\ b^{nj}(z)\partial_j w(z) = 0 & \text{on } \{z_n = -\delta\} \cup \{z_n = \delta\}, \end{cases} \tag{2.15}$$

where the matrix $b(z) = (b^{ij}(z))$ is given by

$$(b^{ij}(z)) = \frac{(\partial_y z)(a^{ij})(\partial_y z)^t}{\det(\partial_y z)}. \tag{2.16}$$

Similar to the proof of Lemma 2.1, we will show that the Jacobian matrix of the change of variables (2.14), denoted by $\partial_y z$, and its inverse matrix $\partial_z y$ satisfy

$$|(\partial_y z)^{ij}| \leq C, \quad |(\partial_z y)^{ij}| \leq C \quad \text{for } z \in Q_{1,\delta^{m-1}}, \tag{2.17}$$

where $C > 0$ depends only on $n, \kappa, R_0, \lambda_1, \lambda_2, \|f\|_{C^2}$ and $\|g\|_{C^2}$. This leads to

$$\frac{\lambda}{C} \leq b(z) \leq C\Lambda \quad \text{for } z \in Q_{1,\delta^{m-1}}. \tag{2.18}$$

From (2.14), one can compute that

$$\begin{aligned} (\partial_y z)^{ii} &= 4 \quad \text{for } 1 \leq i \leq n-1, \\ (\partial_y z)^{nn} &= \frac{2\delta^m}{\varepsilon + f(x'_0 + \delta z'/4) - g(x'_0 + \delta z'/4)}, \\ (\partial_y z)^{ni} &= -\frac{2\delta^m \partial_i g(x'_0 + \delta z'/4) + (z_n + \delta^{m-1})\delta[\partial_i f(x'_0 + \delta z'/4) - \partial_i g(x'_0 + \delta z'/4)]}{\varepsilon + f(x'_0 + \delta z'/4) - g(x'_0 + \delta z'/4)} \\ &\quad \text{for } 1 \leq i \leq n-1, \\ (\partial_y z)^{ij} &= 0 \quad \text{for } 1 \leq i \leq n-1, \quad j \neq i. \end{aligned}$$

First we will show that

$$(\partial_y z)^{nn} \sim 1 \quad \text{for } z \in Q_{1, \delta^{m-1}}. \quad (2.19)$$

Since $|z'| < 1$ and $|x'_0| < \delta$, it is easy to see that

$$(\partial_y z)^{nn} \geq \frac{1}{C} \frac{\delta^m}{\varepsilon + |x'_0 + \delta z'/4|^m} \geq \frac{1}{C} \frac{\delta^m}{\varepsilon + C\delta^m} \geq \frac{1}{C} \quad \text{for } z \in Q_{1, \delta^{m-1}}.$$

On the other hand, when $|x'_0| \leq \varepsilon^{\frac{1}{m}}$, we have $\delta \leq (2\varepsilon)^{\frac{1}{m}}$, and hence

$$(\partial_y z)^{nn} \leq \frac{C\delta^m}{\varepsilon + |x'_0 + \delta z'/4|^m} \leq \frac{C\varepsilon}{\varepsilon + |x'_0 + \delta z'/4|^m} \leq C \quad \text{for } z \in Q_{1, \delta^{m-1}}.$$

When $|x'_0| \geq \varepsilon^{\frac{1}{m}}$, we have $|\delta z'/4| \leq |x'_0|/2$, and hence

$$\begin{aligned} (\partial_y z)^{nn} &\leq \frac{C\delta^m}{\varepsilon + |x'_0 + \delta z'/4|^m} \leq \frac{C\delta^m}{\varepsilon + (|x'_0| - |\delta z'/4|)^m} \\ &\leq \frac{2\delta^m}{\varepsilon + (|x'_0|/2)^m} \leq C \quad \text{for } z \in Q_{1, \delta^{m-1}}. \end{aligned}$$

Therefore, (2.19) is verified. Since $|z_n| < \delta^{m-1}$, $|z'| < 1$ and $|x'_0| < \delta$, by (1.6a) and (1.6b), for $1 \leq i \leq n-1$,

$$\begin{aligned} |(\partial_y z)^{ni}| &\leq \frac{2\delta^m |\partial_i g(x'_0 + \delta z'/4)| + 2\delta^m [|\partial_i f(x'_0 + \delta z'/4)| + |\partial_i g(x'_0 + \delta z'/4)|]}{\varepsilon + f(x'_0 + \delta z'/4) - g(x'_0 + \delta z'/4)} \\ &\leq \frac{C\delta^m}{\varepsilon + f(x'_0 + \delta z'/4) - g(x'_0 + \delta z'/4)} [|\partial_i f(x'_0 + \delta z'/4)| + |\partial_i g(x'_0 + \delta z'/4)|] \\ &\leq C \frac{\delta^m}{\varepsilon + |x'_0 + \delta z'/4|^m} |x'_0 + \delta z'/4| \\ &\leq C(|x'_0| + \delta|z'|) \leq C\delta, \end{aligned}$$

where in the last line, we have used the same arguments in showing $(\partial_y z)^{nn} \leq C$ earlier.

We have shown $(\partial_{yz})^{ii} \sim 1$ for all $i = 1, \dots, n$, and $|(\partial_{yz})^{ij}| \leq C\delta$ for $i \neq j$. We further require r_0 to be small enough so that off-diagonal entries are small. Therefore (2.17) follows. As mentioned earlier, (2.18) follows from (2.17).

Next, we will show

$$\|b\|_{C^\alpha(\bar{Q}_{1,\delta^{m-1}})} \leq C \tag{2.20}$$

for some $C > 0$ depending only on $n, m, R_0, \lambda_1, \lambda_2, \lambda_3, \|f\|_{C^2}, \|g\|_{C^2}$ and $\|a\|_{C^\alpha}$, by showing

$$|\nabla_z(\partial_{yz})^{ij}(z)| \leq C, \quad \left| \nabla_z \frac{1}{\det(\partial_{yz})} \right| \leq C \quad \text{for } z \in Q_{1,\delta^{m-1}}. \tag{2.21}$$

Then (2.20) follows from (2.21), (2.16), and $\|a\|_{C^\alpha} \leq C$.

By a straightforward computation, we have, for any $i = 1, \dots, n - 1$,

$$\begin{aligned} \left| \partial_{z_i} \frac{1}{\det(\partial_{yz})} \right| &= \left| \partial_{z_i} \left(\frac{\varepsilon + f(x'_0 + \delta z'/4) - g(x'_0 + \delta z'/4)}{2 \cdot 4^{n-1} \delta^m} \right) \right| \\ &= \left| \frac{\delta[\partial_i f(x'_0 + \delta z'/4) - \partial_i g(x'_0 + \delta z'/4)]}{2 \cdot 4^{n-1} \delta^m} \right| \\ &\leq \frac{C}{\delta^{m-1}} |x'_0 + \delta z'/4|^{m-1} \leq C \quad \text{for } z \in Q_{1,\delta}, \end{aligned}$$

where in the last line, (1.6b) and (1.6c) have been used. For any $i = 1, \dots, n - 1$, by (1.6b) and (1.6c),

$$\begin{aligned} |\partial_{z_i}(\partial_{yz})^{nn}| &= \left| \frac{2\delta^{m+1}[\partial_i f(x'_0 + \delta z'/4) - \partial_i g(x'_0 + \delta z'/4)]}{(\varepsilon + f(x'_0 + \delta z'/4) - g(x'_0 + \delta z'/4))^2} \right| \\ &\leq \frac{C\delta^{m+1}}{(\varepsilon + |x'_0 + \delta z'/4|^m)^2} |x'_0 + \delta z'/4|^{m-1} \\ &\leq \frac{C\delta^{m+1}|x'_0 + \delta z'/4|^{m-1}}{\delta^{2m}} \leq C \quad \text{for } z \in Q_{1,\delta}, \end{aligned}$$

where in the last line, we have used the same arguments in showing $(\partial_{yz})^{nn} \leq C$ earlier. Similar computations apply to $\partial_{z_i}(\partial_{yz})^{ni}$ for $i = 1, \dots, n - 1$, and we have

$$|\partial_{z_i}(\partial_{yz})^{ni}| \leq C \quad \text{for } z \in Q_{1,\delta^{m-1}}.$$

Finally, we compute, for $i = 1, \dots, n - 1$,

$$\begin{aligned} |\partial_{z_n}(\partial_{yz})^{ni}| &= \left| \frac{2\delta[\partial_i f(x'_0 + \delta z'/4) - \partial_i g(x'_0 + \delta z'/4)]}{\varepsilon + f(x'_0 + \delta z'/4) - g(x'_0 + \delta z'/4)} \right| \\ &\leq \frac{C\delta|x'_0 + \delta z'/4|^{m-1}}{\varepsilon + |x'_0 + \delta z'/4|^m} \leq C \quad \text{for } z \in Q_{1,\delta}. \end{aligned}$$

Therefore, (2.21) is verified, and hence (2.20) follows as mentioned above.

Now we define

$$S_l := \left\{ z \in \mathbb{R}^n \mid |z'| < 1, (l-1)\delta^{m-1} < z_n < (l+1)\delta^{m-1} \right\}$$

for any integer l , and

$$S := \{z \in \mathbb{R}^n \mid |z'| < 1, |z_n| < 1\}.$$

Note that $Q_{1,\delta^{m-1}} = S_0$. As in the proof of Lemma 2.1, we define, for any $l \in \mathbb{Z}$, a new function \tilde{w} by setting

$$\tilde{w}(z) := w\left(z', (-1)^l (z_n - 2l\delta^{m-1})\right), \quad \forall z \in S_l.$$

We also define the corresponding coefficients, for $k = 1, 2, \dots, n-1$,

$$\tilde{b}^{nk}(z) = \tilde{b}^{kn}(z) := (-1)^l b^{nk}\left(z', (-1)^l (z_n - 2l\delta^{m-1})\right), \quad \forall z \in S_l,$$

and for other indices,

$$\tilde{b}^{ij}(z) := b^{ij}\left(z', (-1)^l (z_n - 2l\delta^{m-1})\right), \quad \forall y \in S_l.$$

Then \tilde{w} and \tilde{b}^{ij} are defined in the infinite cylinder $Q_{1,\infty}$. By (2.15), \tilde{w} satisfies the equation

$$-\partial_i(\tilde{b}^{ij}\partial_j\tilde{w}) = 0 \quad \text{in } Q_{1,\infty}.$$

Note that for any $l \in \mathbb{Z}$, $\tilde{b}(z)$ is orthogonally conjugated to $b(z', (-1)^l (z_n - 2l\delta^{m-1}))$, for $z \in S_l$. Hence, by (2.18), we have

$$\frac{\lambda}{C} \leq \tilde{b}(z) \leq C\Lambda \quad \text{for } z \in Q_{1,\infty},$$

and, by (2.20),

$$\|\tilde{b}\|_{C^\alpha(\bar{S}_l)} \leq C, \quad \forall l \in \mathbb{Z}.$$

Apply Lemma 2.1 in [12] on S with $N = 1$, we have

$$\|\nabla\tilde{w}\|_{L^\infty(\frac{1}{2}S)} \leq C\|\tilde{w}\|_{L^2(S)}.$$

It follows that

$$\|\nabla w\|_{L^\infty(Q_{1/2,\delta^{m-1}})} \leq \frac{C}{\delta^{(m-1)/2}}\|w\|_{L^2(Q_{1,\delta^{m-1}})} \leq C\|w\|_{L^\infty(Q_{1,\delta^{m-1}})}$$

for some positive constant C , depending only on $n, \alpha, R_0, m, \lambda, \Lambda, \lambda_1, \lambda_2, \lambda_3, \|f\|_{C^2}, \|g\|_{C^2}$ and $\|a\|_{C^\alpha}$.

By (2.17), we have $\|(\partial_z y)\|_{L^\infty(Q_{1,\delta^{m-1}})} \leq C$, where C is independent of ε and δ . Reversing the change of variables (2.14) and (2.13), we have, by (2.12)

$$\delta\|\nabla v\|_{L^\infty(\Omega_{x_0,\delta/8})} \leq C\|v\|_{L^\infty(\Omega_{x_0,\delta/4})} \leq C\|u\|_{L^\infty(\Omega_{x_0,\delta^{1-\gamma}})}\delta^{\gamma\sigma}. \quad (2.22)$$

In particular, this implies

$$|\nabla u(x_0)| \leq C \|u\|_{L^\infty(\Omega_{x_0, \delta^{1-\gamma}})} \delta^{-1+\gamma\sigma},$$

and it concludes the proof of Theorem 1.1 for the case $n \geq 3$ after taking $\beta = \gamma\sigma/2$.

For the case $n = 2$, we work with u instead of v , and repeat the argument in deriving the first inequality in (2.22), we have

$$\delta \|\nabla u\|_{L^\infty(\Omega_{x_0, \delta/8})} \leq C \|u\|_{L^\infty(\Omega_{x_0, \delta/4})}.$$

In particular,

$$|\nabla u(x_0)| \leq C \|u\|_{L^\infty(\Omega_{x_0, \delta/4})} \delta^{-1}.$$

This concludes the proof of Theorem 1.1 for the case $n = 2$. □

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References

- [1] H. Ammari, H. Kang, H. Lee, J. Lee, and M. Lim, Optimal estimates for the electric field in two dimensions, *J. Math. Pures Appl.*, 88(4) (2007), 307–324.
- [2] H. Ammari, H. Kang, and M. Lim, Gradient estimates for solutions to the conductivity problem, *Math. Ann.*, 332(2) (2005), 277–286.
- [3] I. Babuška, B. Andersson, P. J. Smith, and K. Levin, Damage analysis of fiber composites. I. Statistical analysis on fiber scale, *Comput. Methods Appl. Mech. Eng.*, 172(1-4) (1999), 27–77.
- [4] E. Bao, Y. Y. Li, and B. Yin, Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions, *Commun. Partial Differential Equations*, 35(11) (2010), 1982–2006.
- [5] E. Bonnetier and M. Vogelius, An elliptic regularity result for a composite medium with “touching” fibers of circular cross-section, *SIAM J. Math. Anal.*, 31(3) (2000), 651–677.
- [6] B. Budiansky and G. F. Carrier, High Shear Stresses in Stiff-Fiber Composites, *J. Appl. Mech.*, 51(4) (1984), 733–735.
- [7] H. Dong and H. G. Li, Optimal estimates for the conductivity problem by Green’s function method, *Arch. Ration. Mech. Anal.*, 231(3) (2019), 1427–1453.
- [8] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [9] J. B. Keller, Stresses in narrow regions, *J. Appl. Mech.*, 60(4) (1993), 1054–1056.
- [10] Y. Y. Li and L. Nirenberg, Estimates for elliptic systems from composite material, *Commun. Pure Appl. Math.*, 56(7) (2003), 892–925.
- [11] Y. Y. Li and M. Vogelius, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients, *Arch. Ration. Mech. Anal.*, 153(2) (2000), 91–151.

- [12] Y. Y. Li and Z. Yang, Gradient estimates of solutions to the insulated conductivity problem in dimension greater than two, arXiv:2012.14056.
- [13] X. Markenscoff, Stress amplification in vanishingly small geometries, *Comput. Mech.*, 19(1) (1996), 77–83.
- [14] K. Yun, Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape, *SIAM J. Appl. Math.*, 67(3) (2007), 714–730.
- [15] K. Yun, Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections, *J. Math. Anal. Appl.*, 350(1) (2009), 306–312.
- [16] K. Yun, An optimal estimate for electric fields on the shortest line segment between two spherical insulators in three dimensions, *J. Differential Equations*, 261(1) (2016), 148–188.