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A Note on the Convergence of the Schrödinger Operator along Curve

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. In this paper we set up a convergence property for the fractional Schödinger operator for $0 < a < 1$. Moreover, we extend the known results to non-tangent convergence and the convergence along Lipschitz curves.

Key Words: Refinement of the Carleson problem, disconvergence set, fractional Schrödinger operator, Hausdorff dimension, Sobolev space.

AMS Subject Classifications: 42B25, 35Q20

1 Introduction

Given a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, we consider the fractional Schrödinger operator defined by

$$S_a(t)f(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{ix\xi+it|\xi|^a} \hat{f}(\xi)d\xi \tag{1.1}$$

with $a > 0$. It is the solution to the initial data problem of the fractional Schrödinger equation

$$\begin{cases} \partial_t u(x,t) = (-\Delta)^{\frac{a}{2}} u(x,t), & \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.2}$$

From the Plancherel theorem, (1.1) can be easily extend to a bounded operator on L^2-based Sobolev space $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$. Here we recall the norm of $H^s(\mathbb{R}^n)$ as

$$\|f\|_{H^s(\mathbb{R})} = \left(\int_{\mathbb{R}} \left(1+|\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} < \infty. \tag{1.3}$$

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When $a = 2$, $S_2(t)$ becomes the classical Schrödinger operator. We take $S(t)$ as its abbreviation. In [3], Carleson posed the following well known problem: To determine the infimum (critical) index s_c such that for any $s > s_c$,

$$\lim_{t\to 0} S(t)f(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n, \quad \forall f \in H^s(\mathbb{R}^n). \tag{1.4}$$

For one dimensional case, Carleson [3] showed that (1.4) holds for $s \geq \frac{1}{4}$. The corresponding opposite result is obtained by Dahlberg and Kenig [7]. Moreover they showed that (1.4) does not hold for $s < \frac{1}{4}$ in any dimension. Thus we can conclude $s_c = 1/4$ for $n = 1$. After that, there are enumerate literatures devoted to settling the high dimensional problems. Sjölin [16] and Vega [20] proved the convergence if $s > 1/2$ independently. Lee [11] set up (1.4) when $s > 3/8$ and $n = 2$. Bourgain [1] improved these results by showing that the convergence holds for $s > \frac{1}{2} - \frac{1}{4n}$ and the necessary condition is $s \geq \frac{1}{2} - \frac{1}{n}$ for $n \geq 4$. More recently, Bourgain [2] constructed a counter example to show that (1.4) does not hold for $s < \frac{n}{2(n+1)}$. Du, Guth and Li [6] obtained that $s_c = 1/3$ by setting up (1.4) if $s > \frac{1}{3}$ and $n = 2$. Forthermore, Du and Zhang [9] proved the convergence holds if $s > \frac{n}{2(n+1)}$ and $n \geq 3$. Thus the solution to Carleson's problem is $s_c = \frac{n}{2(n+1)}$ for $n \geq 2$.

It is nature to ask the same question for general $a > 0$. An interesting phenomenon is that when $a > 1$, the results do not depend on the value of a, but when $a < 1$, the results depend on the value of it. For $a > 1$, the convergence were proved to be true if $s > 1/4$, $n = 1$ by Sjölin [16] and Vega [20]. Miao, Yang, and Zheng [14] obtained the convergence when $s > \frac{3}{8}$ and $n = 2$. Cho and Ko [4] proved that the convergence also holds when $s > \frac{n}{2(n+1)}$ and $n \geq 2$. The same result was also obtained by Li, Li and Xiao [12] by setting up the up-bound of Hausdorff dimension of the divergent set.

When $0 < a < 1$, Walther [21, 22] set up the convergence when $s > a/4$ in one dimension and for the radial functions in higher dimensional spaces. Very recently Dimou and Seeger [10] obtained the equivalent condition to time sequence of $\{t_n\}$ such that if $t_n \to 0$ (1.4) holds. Thus we know that $s_c = \frac{a}{4}$ is the critical index when $n = 1$. For $n \geq 2$, Zhang [24] proved the convergence for $s > \frac{na}{4}$. It is still very open to determine the critical index for the high dimensional case.

An interesting generalization of the point-wise convergence problem is to set up the convergence in a wider approach region instead of vertical lines, for example, the non-tangential limit. It is easy to see that it holds for $s > \frac{n}{2}$ by Sobolev Embeding. Sjölin and Sjögren [15] showed that non-tangential convergence fails for $s \leq \frac{n}{2}$. Cho, Lee and Vargas [5] showed that the non-tangential convergence holds if $s > \frac{\beta(\Theta)+1}{4}$ when $a = 2$ and $n = 2$. $\beta(\Theta)$ denotes the upper Minkowski dimension of the upper cover of the cone which will be given soon. Cho, Lee and Vargas [5] deal with estimating the boundary of the operator along the restricted direction and time localization argument. Shiraki [17] extended result of [5] to $a > 1$. In this paper, we will deal with the case of $0 < a < 1$, $n = 1$.

To state our main results, we need first introduce in some notations. Let $\Theta \subset \mathbb{R}$ be a fixed compact set of $\mathbb{R}$, We call

$$\Gamma(x,t)=\{x+s\theta : s\in[-t,t] \text{ and } \theta\in\Theta\}, \quad x\in\mathbb{R} \quad \text{and} \quad t\geq 0, \tag{1.5}$$

as a cone respect to the upper cover Θ. It is clear if $\Theta=[-1,1]$, it is exactly a classical cone in $\mathbb{R}^2$. The upper Minkowski dimension of Θ which can be defined as

$$\beta(\Theta)=\inf\Big\{r>0 : \lim_{\delta\to 0}\sup N(\Theta,\delta)\delta^r=0\Big\}. \tag{1.6}$$

Here, $N(\Theta,\delta)$ denotes the smallest number of δ-intervals which cover Θ.

The main results of this paper can be state as follows.

Theorem 1.1. *Let $0<a<1$, $\Theta\subset\mathbb{R}$ be a compact set. If $s>\frac{1}{2}-\frac{a}{4}(1-\beta(\Theta))$, then there exists a constant $C_s>0$, such that*

$$\Big\|\sup_{(t,\theta)\in[-1,1]\times\Theta}|S_a(t)f(\cdot+t\theta)|\Big\|_{L^2(-1,1)}\leq C_s\|f\|_{H^s(\mathbb{R})}. \tag{1.7}$$

Corollary 1.1. *Under the condition of Theorem 1.1, we have*

$$\lim_{y\in\Gamma_x,t\to 0}S_a(t)f(y)=f(x) \quad a.e.\ x\in\mathbb{R}, \quad \forall f\in H^s(\mathbb{R}). \tag{1.8}$$

Remark 1.1. When $\Theta=[-1,1]$, we have $\beta(\Theta)=1$. By the results of Sjölin and Sjögren [15], our result is sharp in this case. For $\beta(\Theta)<1$, our results are new. This result is not coincide with the critical index $s_c=\frac{a}{4}$ when $\Theta=\{0\}$. But the latter is only a very special case of $\beta(\Theta)=0$.

The non-tangential convergence means that the convergence is true along any curve in the cone region. The critic number s_c is $\frac{n}{2}$ when $\beta(\Theta)=1$. Theorem 1.1 shows that along some curve in $\Gamma(\Theta)$ the convergence can also be true for functions with less regularity. Thus is would be interested to understand convergence for the points along some curves in the cone. Given a continuous curve $\gamma(x,t)$, such that $\lim_{t\to 0}\gamma(x,0)=x$, we define the operator along this curve as

$$S_{t,\gamma}f(x)=S_tf(\gamma(x,t))=\frac{1}{2\pi}\int_{\mathbb{R}}e^{i\gamma(x,t)\xi+it|\xi|^a}\hat{f}(\xi)d\xi, \quad f\in\mathcal{S}(\mathbb{R}). \tag{1.9}$$

The question now is to determine the lower index $s_{c,\gamma}$, such that for $s>s_{c,\gamma}$,

$$\lim_{t\to 0}S_{t,\gamma}f(x)=f(x) \quad \text{a.e. } x\in\mathbb{R}^n, \quad \forall f\in H^s(\mathbb{R}^n). \tag{1.10}$$

For classical Schrödinger operator Lee and Rogers [13], Cho, Lee and Vargas [5] considered the curve $\gamma(x,t)$ satisfies the following conditions:

$$|\gamma(x,t)-\gamma(y,t)|\leq C|t-t'|^\tau, \tag{1.11a}$$

$$c|x-y|\leq|\gamma(x,t)-\gamma(y,t)|\leq C|x-y|. \tag{1.11b}$$

Cho, Lee and Vargas [14] obtained the pointwise convergence holds if

$$s > \max\Big\{\frac{1}{2}-\tau, \frac{1}{4}\Big\}.$$

Ding and Niu [8] obtained the convergence along the curve holds if

$$s > \frac{a}{4} \quad \text{for} \quad \frac{1}{2} < \tau < 1$$

or

$$s > \min\Big\{\frac{a}{2}, \frac{a}{4}\Big(\frac{1}{\tau}-1\Big)\Big\}, \quad \text{when} \quad a > 1.$$

Furthermore, Ding and Niu [8] show it is sharp when $a \geq 2$ the critical index $s_c = \max\{\frac{1}{2}-\tau, \frac{1}{4}\}$. We focus on $0 < a < 1$. For this aim, we need to consider the maximal operator

$$S^*_{t,\gamma} f(x) = \sup_{t\in[0,T]} S_{t,\gamma} f(x) \tag{1.12}$$

with a given constant $T > 0$.

We now state our next result:

Theorem 1.2. *Let $0 < a < 1$, $0 < \tau \leq 1$. The curve γ satisfies* (1.11a) *and* (1.11b)*. We have*

$$\|S^*_{t,\gamma} f\|_{L^2(\mathbb{R})} \leq C\|f\|_{H^s(\mathbb{R})}, \tag{1.13}$$

whenever

$$s > \frac{1}{2}-\frac{a}{4} \quad \textit{for} \quad \frac{1}{2} < \tau \leq 1,$$

or

$$s > \min\Big\{\frac{1}{2}, \frac{1}{2}+\frac{a}{4}\Big(\frac{1}{\tau}-3\Big)\Big\} \quad \textit{for} \quad 0 < \tau \leq \frac{1}{2}.$$

Corollary 1.2. *Under the condition of Theorem* 1.2*, we have*

$$\lim_{t\to 0} S_{t,\gamma}(t) f(x) = f(x) \quad a.e. \ x \in \mathbb{R}, \quad \forall f \in H^s(\mathbb{R}). \tag{1.14}$$

2 Proof of main results

2.1 Two lemmas

In this section, we collect two lemmas which will be used very frequently in our proof.

Lemma 2.1 (Van der Corput's lemma, [18, p. 309])**.** *Suppose $\lambda > 1$ and we have $|\phi^k(x)| \geq 1$ for all (a,b). If $k = 1$ and ϕ' is monotonic on (a,b), or simply $k \geq 2$, then there exists a constant C_k such that*

$$\Big|\int_b^a e^{i\lambda\phi(x)}\psi(x)dx\Big| < C_k\lambda^{-\frac{1}{k}}\Big(\int_a^b |\psi'(x)|dx + \|\psi\|_{L^\infty}\Big). \tag{2.1}$$

Lemma 2.2 ([19]). *Let I denote an open interval in $\mathbb{R}$. For $g \in C_0^\infty(I)$ and real valued function $F \in C^\infty(I)$ with $F' \neq 0$, if $k \in \mathbb{N}$, then*

$$\int_I e^{F(x)} g(x) dx = \int_I e^{F(x)} h_k(x) dx, \tag{2.2}$$

where h_k is a linear combination of functions of the form

$$g^{(s)} (F')^{-k-r} \prod_{q=1}^{r} F^{(j_q)}$$

with $0 \leq s \leq k$, $0 \leq r \leq k$ and $2 \leq j_q \leq k+1$.

2.2 Proof of Theorem 1.1

Let φ be a bump function supported on $[-1,1]$ and $\psi = \varphi(x/2) - \varphi(2x)$. And we take the notation that $\psi_k(x) = \psi(2^{-k}x)$ for any $k \in \mathbb{N}$. Given $f \in \mathcal{S}(\mathbb{R})$, we denote the projections of the function to the dyadic annulus respectively by

$$\hat{f}_0(\xi) = \hat{f}(\xi)\varphi(\xi) \quad \text{and} \quad \hat{f}_k(\xi) = \hat{f}(\xi)\psi_k(\xi), \quad k \in N.$$

Then we have the following partition of unit

$$f(x) = f_0(x) + \sum_{k \geq 1} f_k(x).$$

Denote the maximal operator

$$M_\Theta f(x) = \sup\{|S_a(t) f(x + t\theta)| : \ -1 \leq t \leq 1, \ \theta \in \Theta\}. \tag{2.3}$$

For fixed k,

$$M_\Theta f_k(x) = \sup_{(t,\theta) \in B_1 \times \Theta} |S_a(t) f_k(x + t\theta)| \leq \left(\sum_{j=1}^{N} \sup_{\theta \in \Omega_{k,j}} |S_t f_k(x + t\theta)|^2 \right)^{\frac{1}{2}}, \tag{2.4}$$

where $\Omega_{k,j} = \Omega_j(2^k)$, and $\{\Omega_j(\lambda)\}_{j=1}$ is a finite covering of Θ such that

$$\Theta \subset \cup_{j=1}^{N} \Omega_j(\lambda) \quad \text{and} \quad |\Omega_j(\lambda)| \leq \lambda^{-\frac{a}{2}}. \tag{2.5}$$

By Minkowski's inequality, we have

$$\|M_\Theta f\|_{L^2(I)} \leq \|M_\Theta f_0\|_{L^2(I)} + \sum_{K \geq 1} \|M_\Theta f_k\|_{L^2(I)}. \tag{2.6}$$

For the low frequency part, it is easy to see that

$$\|M_\Theta f_0\|_{L^2(I)} \lesssim \int_{\mathbb{R}} \varphi_0(\xi) |\hat{f}(\xi)| d\xi \lesssim \|f\|_{L^2}. \tag{2.7}$$

We then need to obtain some estimates for $M_\Theta f_k$. Moreover,

$$\sum_{k\geq 1}\|M_\Theta f_k\|^2_{L^2(I)} \leq \sum_{K\geq 1}\sum_{j=1}\|M_{\Omega_{k,j}} f_k\|^2_{L^2(I)}, \tag{2.8}$$

where

$$M_{\Omega_{k,j}} f_k(x) = \sup\{|S_a(t)f(x+t\theta)| : -1\leq t\leq 1,\ \theta\in\Omega_{k,j}\}.$$

Firstly, we claim the following estimate and postpone its proof to the next proposition.

$$\|M_\Omega f\|_{L^2(I)} \leq C2^{k(\frac{1}{2}-\frac{a}{4})}\|f\|_{L^2}, \quad \forall\Omega \text{ is an interval with } |\Omega|\leq 2^{k(\frac{a}{2})}. \tag{2.9}$$

And let

$$\widehat{L_k f} = \hat{h}_k\hat{f},$$

where

$$\hat{h}\in C_0^\infty\Big(\Big(-4,-\frac{1}{4}\Big)\cup\Big(\frac{1}{4},4\Big)\Big) \quad \text{with } \hat{h}=1 \quad \text{on } \Big(-2,-\frac{1}{2}\Big)\cup\Big(\frac{1}{2},2\Big).$$

By the definition of the upper Minkowski dimension, there is a constant C_ϵ depending on ϵ to hold the inequality

$$N(\Theta,\lambda^{-\sigma}) \leq C_\epsilon\lambda^{\sigma\beta(\Theta)+\epsilon}$$

for any $\epsilon>0$. And by (2.8), (2.9), we can obtain that

$$\begin{aligned}\sum_{k\geq 1}\|M_\Theta f_k\|^2_{L^2(I)} &\leq \sum_{K\geq 1}\sum_{j=1}\|M_{\Omega_{k,j}} L_k f\|^2_{L^2(I)}\\ &\leq \sum_{k=1}\sum_{j=1}2^{(1-\frac{a}{2})k}\|L_k f\|^2_{L^2_2}\\ &\leq \sum_{k=1}2^{k(1-\frac{a}{2}(1-\beta(\Theta))+\epsilon)}\|L_k f\|^2_{L^2_2}.\end{aligned} \tag{2.10}$$

We conclude that

$$\|M_\Theta f\|_{L^2(I)} \lesssim \|f\|_{H^{\frac{1}{2}-\frac{a}{4}(1-\beta(\Theta))+\epsilon}}. \tag{2.11}$$

We now give the proof of (2.9).

Proposition 2.1. *Let $k\geq 1$ and Ω be an interval with $|\Omega|\leq 2^{k(\frac{a}{2})}$. Then, there exists a constant $C>0$ that*

$$\|M_\Omega f\|_{L^2(I)} \leq C2^{k(\frac{1}{2}-\frac{a}{4})}\|f\|_{L^2}. \tag{2.12}$$

Proof. Set $\lambda=2^k$ and denote

$$Tf(x,t,\theta) = \chi(x,t,\theta)\int_{\mathbb{R}} e^{i((x+t\theta)\xi+t|\xi|^a)}\hat{f}(\xi)\psi\Big(\frac{\xi}{\lambda}\Big)d\xi, \tag{2.13}$$

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where$\chi \in C_0^\infty(I\times[-1,1]\times\Omega)$. The result follows from

$$\|Tf\|_{L^2_x L^\infty_{t,\theta}} \le \lambda^{\frac{1}{2}-\frac{a}{4}}\|f\|_{L^2}. \tag{2.14}$$

By duality, it is need to show that

$$\|T^*F\|_{L^2} \le C\lambda^{\frac{1}{2}-\frac{a}{4}}\|F\|_{L^2_x L^1_t L^1_\theta}, \tag{2.15}$$

where

$$T^*F(\xi) = \psi\Big(\frac{\xi}{\lambda}\Big)\int_{\mathbb{R}} e^{i((y+t'\theta')\xi+t|\xi|^a)}F(y,t,\theta')\chi(x,t,\theta')dxdtd\theta'.$$

It is sufficient to show

$$\|TT^*F\|_{L^2L^\infty_{t,\theta}} \le C\lambda^{(\frac{1}{2}-\frac{a}{4})}\|F\|_{L^2L^1_{t,\theta}}. \tag{2.16}$$

We note that

$$TT^*F(x,t,\theta)=\chi(x,t,\theta)\iiint K_\lambda(t,t',x,y,\theta,\theta')\chi(y,t',\theta')F(y,t',\theta')dydt'd\theta', \tag{2.17a}$$

$$K_\lambda(t,t',x,y,\theta,\theta')=\chi(x,t,\theta)\chi(y,t',\theta')\lambda\int e^{i(\lambda^a(t'-t)|\xi|^a+\lambda(x-y+t\theta-t\theta')\xi)}\psi^2(\xi)d\xi. \tag{2.17b}$$

We have the following estimates for the kernel K_λ.

(i) The case $|x-y|\ge 4|t-t'|$ and $|x-y|\ge 4\lambda^{-\frac{a}{2}}$. We have

$$\begin{cases} \phi'(\xi)=\lambda(x-y+t\theta-t'\theta')+a\lambda^a(t-t')|\xi|^{a-1}, \\ \phi''(\xi)=a(a-1)\lambda^a(t-t')|\xi|^{a-2}. \end{cases} \tag{2.18}$$

Then,

$$\begin{aligned} |\phi'(\xi)| \ge& \lambda|(x-y+t\theta-t'\theta')|-\lambda^a|(t-t')||\xi|^{a-1} \\ \gtrsim& \lambda|x-y|-\lambda^a|(t-t')||\xi|^{a-1} \\ \gtrsim& \lambda|x-y|. \end{aligned} \tag{2.19}$$

Since $\phi''(\xi)$ is single-signed on $(-\infty,-1]$ and $[1,\infty)$, so $\phi'(\xi)$ is monotonic on $|\xi|\ge 1$. By Lemma 2.1, we can obtain that

$$K_\lambda \lesssim \lambda(\lambda|x-y|)^{-1} \lesssim \lambda^{\frac{a}{2}}|x-y|^{-\frac{1}{2}}, \tag{2.20}$$

when $|x-y|\ge 4\lambda^{-\frac{a}{2}}$.

(ii) The case $|x-y|\le C\lambda^{-\frac{a}{2}}$ and $|x-y|\ge C|t-t'|$. It's obviously that $K_\lambda\lesssim\lambda$.

(iii) The case $|x-y|\le C|t-t'|$. By Lemma 2.2, we have

$$|K_\lambda|\lesssim \lambda^{1-\frac{a}{2}}(|x-y|)^{-\frac{1}{2}}. \tag{2.21}$$

It follows from Hölder's inequality and Young's inequality that

$$\int (K*|h|)(x)|h(x)|dx\le \|K\|_{L^1}\|h\|_{L^2}^2. \tag{2.22}$$

By Fubini theorem and previous argument,

$$\begin{cases} \lambda^{\frac{a}{2}}\int_{-1}^{1}\int_{-1}^{1}\|F(x,\cdot)\|_{L^1_{t,\theta}}\|F(y,\cdot)\|_{L^1_{t,\theta}}||x-y|^{-\frac{1}{2}}dxdy\lesssim \lambda^{\frac{a}{2}}\|F\|^2_{L^2L^1_{t,\theta}}, \\ \lambda\int_{-1}^{1}\int_{-1}^{1}\|F(x,\cdot)\|_{L^1_{t,\theta}}\|F(y,\cdot)\|_{L^1_{t,\theta}}|X_{[-C\lambda^{-\frac{a}{2}},C\lambda^{-\frac{a}{2}}]}(x-y)dxdy\lesssim^{1-\frac{a}{2}}\|F\|^2_{L^2L^1_{t,\theta}}, \\ \lambda^{1-\frac{a}{2}}\int_{-1}^{1}\int_{-1}^{1}\|F(x,\cdot)\|_{L^1_{t,\theta}}\|F(y,\cdot)\|_{L^1_{t,\theta}}|x-y|^{-\frac{1}{2}}dxdy\lesssim \lambda^{1-\frac{a}{2}}\|F\|^2_{L^2L^1_{t,\theta}}. \end{cases} \tag{2.23}$$

We compare the exponent of λ, the proof of proposition is completed. □

We finish the proof of Theorem 1.1.

2.3 Proof of Theorem 1.2

We denote the linearization of the maximal operator as

$$Tf(x)=\int_{\mathbb{R}} e^{i\gamma(x,t(x))\xi+it(x)|\xi|^a}\hat{f}(\xi)d\xi. \tag{2.24}$$

It is sufficient to set up

$$\|Tf(x)\|_{L^2(\mathbb{R})}\lesssim \|f\|_{H^s(\mathbb{R})}. \tag{2.25}$$

We decompose it

$$\begin{aligned} Tf(x)&=\int_{\mathbb{R}} e^{i\gamma(x,t)\xi+it|\xi|^a}\hat{f}_0(\xi)d\xi+\sum_{k=1}^{\infty}\int_{\mathbb{R}} e^{i\gamma(x,t)\xi+it|\xi|^a}\hat{f}_k(\xi)d\xi \\ &=: T_0f(x)+\sum_{k=1}^{\infty}T_kf(x), \end{aligned} \tag{2.26}$$

where f_0 and f_k are the same as in the last subsection. By Minkowski's inequality, we have

$$\|Tf\|_{L_2(\mathbb{R})}\le \|T_0f\|_{L_2(\mathbb{R})}+\sum_{k=1}^{\infty}\|T_kf\|_{L_2(\mathbb{R})}. \tag{2.27}$$

We first estimate the $\|T_0 f\|_{L^2(\mathbb{R})}$. Let

$$L_0 g(x)=\int_{\mathbb{R}} e^{i\gamma(x,t(x))\xi+it(x)|\xi|^a}\varphi_0(\xi)g(\xi)d\xi,\quad g\in\mathcal{S}(\mathbb{R}). \tag{2.28}$$

Taking function $\rho\in C_0^\infty$, $\rho=1$ if $|x|\le 1$, and $\rho=0$ if $|x|\ge 2$, we denote

$$L_{0,m}g(x)=\rho\Big(\frac{x}{m}\Big)\int_{\mathbb{R}} e^{i\gamma(x,t(x))\xi+it(x)|\xi|^a}\varphi_0(\xi)g(\xi)d\xi,\quad g\in S(\mathbb{R}). \tag{2.29}$$

By duality, its adjoint operator

$$L'_{0,m}h(\xi)=\varphi_0(\xi)\int_{\mathbb{R}} e^{-i\gamma(x,t(x))\xi-it(x)|\xi|^a}\rho\Big(\frac{x}{m}\Big)h(x)dx,\quad m\ge 1. \tag{2.30}$$

Thus, we have

$$\begin{aligned}\|L'_{0,m}h\|^2_{L^2(\mathbb{R})}&=\int_{\mathbb{R}}\Big(\varphi_0(\xi)\int_{\mathbb{R}} e^{-i\gamma(x,t(x))\xi-it(x)|\xi|^a}\rho\Big(\frac{x}{m}\Big)h(x)dx\Big)\\&\qquad\times\Big(\varphi_0(\xi)\int_{\mathbb{R}} e^{i\gamma(x,t(y))\xi+it(y)|\xi|^a}\rho\Big(\frac{y}{m}\Big)h(y)dy\Big)d\xi\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}K_0(x,y)h(x)h(y)dxdy,\end{aligned} \tag{2.31}$$

where

$$K_0(x,y)=\rho\Big(\frac{x}{m}\Big)\rho\Big(\frac{y}{m}\Big)\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi+i(t(y)-t(x))|\xi|^a}\varphi_0^2(\xi)d\xi. \tag{2.32}$$

Using the Hölder's inequality and Young's inequality we obtain

$$\|L'_{0,m}h\|^2_{L^2(\mathbb{R})}\le C\|K_0\|_{L^1(\mathbb{R})}\|h\|^2_{L^2(\mathbb{R})}. \tag{2.33}$$

We claim that $\|K_0\|_{L^1(\mathbb{R})}<C$ and it is independent of m, which we will give the proof in Proposition 2.2. Thus, we have

$$\|L_{0,m}g\|^2_{L^2(\mathbb{R})}\le C\|g\|^2_{L^2(\mathbb{R})}. \tag{2.34}$$

By taking $m\to\infty$, we have

$$\|L_0 g\|^2_{L^2(\mathbb{R})}\le C\|g\|^2_{L^2(\mathbb{R})}. \tag{2.35}$$

We now set up the uniform boundedness of $\|k_0\|_{L^1(\mathbb{R})}$. It is sufficient to set the following proposition.

Proposition 2.2. *Suppose γ satisfy the conditions in Theorem* 1.2 *and $K_0(x,y)$ as* (2.30). *Then*

$$\begin{cases}K_0(x,y)\lesssim\dfrac{1}{(1+|x-y|)^{-1-a}}, & |x-y|\ge C(2T)^\tau,\\ K_0(x,y)\lesssim 1, & |x-y|\le C(2T)^\tau.\end{cases} \tag{2.36}$$

Proof. We decompose $K_0(x,y)$ like that

$$\begin{aligned}K_0(x,y)=&\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi}\Big(\sum_{k=0}^{M}\frac{(i(t(y)-t(x))|\xi|^a)^k}{k!}\Big)\varphi_0^2(\xi)d\xi\\&+\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi}\Big(e^{i(t(y)-t(x))|\xi|^a}\\&-\sum_{k=0}^{M}\frac{(i(t(y)-t(x))|\xi|^a)^k}{k!}\Big)\varphi_0^2(\xi)d\xi\\=:&K_{0,1}(x,y)+K_{0,2}(x,y),\end{aligned}\tag{2.37}$$

where $aM<1<a(M+1)$. It's obvious that

$$K_0\lesssim 1 \quad \text{for} \quad |x-y|\le C(2T)^\tau.$$

So we only consider the case $|x-y|\ge C(2T)^\tau$.

The estimate of $K_{0,1}$.

In the view of (2.37), it is need to show

$$\int e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi}|\xi|^a\varphi_0^2(\xi)d\xi\le C|x-y|^{-1-a},\tag{2.38}$$

where the constant C is independent of x, and $x\ge 1$. Let $\psi=1-\varphi$ and $\psi_m(\xi)=\psi(m\xi)$. Integrating by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi}|\xi|^a\varphi_0^2(\xi)d\xi\\=&\lim_{m\to\infty}\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi}|\xi|^a\psi_m(\xi)\varphi_0(\xi)d\xi\\=&\frac{-1}{i(\gamma(y,t(y))-\gamma(x,t(x)))}\bigg(\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi}a\mathrm{sgn}(\xi)|\xi|^{a-1}\varphi_0(\xi)d\xi\\&+\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi}|\xi|^a\varphi_0'(\xi)d\xi\\&+\lim_{m\to\infty}\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi}|\xi|^a\psi_m'(\xi)\varphi_0(\xi)d\xi\bigg)\\=&\frac{-1}{i(\gamma(y,t(y))-\gamma(x,t(x)))}(I_1(x-y)+I_2(x-y)+\lim_{m\to\infty}I_{3,m}(x-y)).\end{aligned}\tag{2.39}$$

We denote $h:\xi\to sgn(\xi)|\xi|^{a-1}$. Since h is odd and homogeneous of degree $a-1$ its inverse Fourier transform is odd and homogeneous of degree $-a$. Thus the convolution $\check{h}*\check{\varphi}_0=I_1/C$ is bounded and continuous and that it veryfies the estimate. I_2 decays rapidly at infinity.

$$|I_{3,m}(x-y)|\le\lim_{m\to\infty}2\int_{\frac{1}{m}}^{\frac{2}{m}}|\xi|^a\psi_m'(\xi)d\xi\le Cm^{-a}.\tag{2.40}$$

Thus, we have

$$\Big|\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi}|\xi|^a\varphi_0^2(\xi)d\xi\Big| \le |x-y|^{-1-a}. \tag{2.41}$$

The estimate of $K_{0,2}$.

Set

$$\begin{aligned} K_{0,2,m}(x,y) = \int e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi}\Big(&e^{i(t(y)-t(x))|\xi|^a} \\ &-\sum_{k=0}^{M}\frac{(i(t(y)-t(x))|\xi|^a)^k}{k!}\Big)\varphi_0^2(\xi)\psi_m(\xi)d\xi \\ =:\int e^{iP(\xi)}Q(\xi)d\xi,& \end{aligned} \tag{2.42}$$

where

$$P(\xi) = (\gamma(y,t(y))-\gamma(x,t(x)))\xi, \tag{2.43a}$$

$$Q(\xi) = \Big(e^{i(t(y)-t(x))|\xi|^a} - \sum_{k=0}^{M}\frac{(i(t(y)-t(x))|\xi|^a)^k}{k!}\Big)\varphi_0^2(\xi)\psi_m(\xi). \tag{2.43b}$$

By integrating by parts twice, we have

$$K_{0,2}(x,y) = -\frac{1}{(\gamma(y,t(y))-\gamma(x,t(x)))^2}\int_{\mathbb{R}} e^{iP(\xi)}Q''(\xi)d\xi, \tag{2.44}$$

where

$$\begin{aligned} &Q''(\xi) \\ =&\sum_{\mu+\beta+\eta=2}\Big(e^{i(t(y)-t(x))|\xi|^a} - \sum_{k=0}^{M}\frac{(i(t(y)-t(x))|\xi|^a)^k}{k!}\Big)^{(\mu)}(\varphi_0^2(\xi))^{(\beta)}(\psi_m(\xi))^{(\eta)}, \end{aligned} \tag{2.45}$$

when $|x-y| \ge C(2T)^\tau$. We have the following that

$$\begin{aligned} K_{0,2,m}(x,y) =& \frac{1}{(|\gamma(y,t(y))-\gamma(x,t(x))|)^2}\int_{\mathbb{R}}|e^{iP(\xi)}||Q''(\xi)|d\xi \\ \lesssim& \frac{1}{(1+|x-y|)^2}\sum_{\mu+\beta+\eta=2} I_{\mu,\beta,\eta}, \end{aligned} \tag{2.46}$$

where

$$I_{\mu,\beta,\eta} = \int\Big|\Big(e^{i(t(y)-t(x))|\xi|^a} - \sum_{k=0}^{M}\frac{(i(t(y)-t(x))|\xi|^a)^k}{k!}\Big)^{(\mu)}\Big|\Big|\big(\varphi_0^2(\xi)\big)^{(\beta)}\Big|\Big|\big(\psi_m(\xi)\big)^{(\eta)}d\xi\Big|.$$

Thus, for $0 < |\xi| < 1$, the following estimate holds

$$\begin{aligned}&\Big|\Big(e^{i(t(y)-t(x))|\xi|^a}-\sum_{k=0}^{M}\frac{(i(t(y)-t(x))|\xi|^a)^k}{k!}\Big)^{(\mu)}\Big|\\=&\Big|\Big(\sum_{M+1}^{\infty}\frac{(i(t(y)-t(x))|\xi|^a)^k}{k!}\Big)^{(\mu)}\Big|\\\le& C|\xi|^{a(M+1)-\mu}.\end{aligned}\tag{2.47}$$

We can obtain the estimate for $1 \le |\xi| \le 2$ in a similar way. For $\mu = 0,1,2$, by the convergence of Taylor series.

$$\Big|\Big(\sum_{M+1}^{\infty}\frac{(i(t(y)-t(x))|\xi|^a)^k}{k!}\Big)^{(\mu)}\Big| \le C, \quad \mu=0,1,2.\tag{2.48}$$

And by the definition of ψ and $1 \le |\xi| \le 2$, we have

$$|(\psi_m(\xi))^{(\eta)}| \le C|\xi|^{-\eta}, \quad \eta = 1,2.\tag{2.49}$$

Thus, if $\eta = 0$,

$$\begin{aligned}I_{\mu,\beta,\eta} \le& C\int_{\frac{1}{m}<|\xi|<1}|\xi|^{a(M+1)-\mu}d\xi+\int_{1<|\xi|<2}d\xi\\\le& C\int_{|\xi|<1}|\xi|^{a(M+1)-2}d\xi + C \le C.\end{aligned}\tag{2.50}$$

If $\eta = 1$ or $\eta = 2$. We consider $m^{-1} \le |\xi| \le 2m^{-1}$ for m sufficient large.

$$I_{\mu,\beta,\eta} \le C\int_{\frac{1}{m}<|\xi|<\frac{2}{m}}|\xi|^{a(M+1)-\mu-\eta}d\xi \le Cm^{-1}m^{-a(M+1)+\mu+\eta} \le C.\tag{2.51}$$

Thus let $m \to \infty$, so we complete the proof. □

Next, we estimate $\|T_k f\|_{L^2(\mathbb{R})}$. Defining the operator R_λ as

$$R_\lambda g(x) = \lambda^{-s}\int_{\mathbb{R}} e^{i\gamma(x,t(x))\xi}e^{it(x)|\xi|^a}\psi\Big(\frac{\xi}{\lambda}\Big)g(\xi)d\xi, \quad g\in S(\mathbb{R}), \quad \lambda \ge 2.\tag{2.52}$$

Taking ρ as above

$$R_{\lambda,m}g(x) = \lambda^{-s}\rho\Big(\frac{x}{m}\Big)\int_{\mathbb{R}} e^{i\gamma(x,t(x))\xi}e^{it(x)|\xi|^a}\psi\Big(\frac{\xi}{\lambda}\Big)g(\xi)d\xi, \quad g\in S(\mathbb{R}), \quad \lambda \ge 2.\tag{2.53}$$

Noticing that N is a dyadic number, we consider the adjoint operator of it

$$R'_{\lambda,m}h(\xi) = \lambda^{-s}\psi\Big(\frac{\xi}{N}\Big)\int_{\mathbb{R}} e^{i\gamma(x,t(x))\xi}e^{it(x)|\xi|^a}\rho\Big(\frac{x}{m}\Big)h(x)dx, \quad m>1, \quad \lambda \ge 2.\tag{2.54}$$

We have

$$\|R'_{\lambda,m}h(\xi)\|_{L^2_{(\mathbb{R})}} =: \int_{\mathbb{R}}\int_{\mathbb{R}} K_0(x,y)h(x)h(y)dxdy, \tag{2.55}$$

where

$$K_\lambda(x,y)=\rho\Big(\frac{x}{m}\Big)\rho\Big(\frac{y}{m}\Big)\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi+i(t(y)-t(x))|\xi|^a}\psi^2\Big(\frac{\xi}{\lambda}\Big)d\xi. \tag{2.56}$$

Let

$$I_\lambda(x,y)=\lambda^{-2s}\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi+i(t(y)-t(x))|\xi|^a}\psi^2\Big(\frac{\xi}{\lambda}\Big)d\xi. \tag{2.57}$$

Denote $G(\xi)=\psi^2(\xi)$, and by changing the variables, we obtain that

$$I_\lambda(x,y)=\lambda^{1-2s}\int_{\mathbb{R}} e^{i\lambda(\gamma(y,t(y))-\gamma(x,t(x)))\xi+i\lambda^a(t(y)-t(x))|\xi|^a}G(\xi)d\xi. \tag{2.58}$$

Proposition 2.3. *Suppose that γ and $I_\lambda(x-y)$ as above. For $\frac{1}{4}\le\tau\le 1$, we have*

$$\begin{cases} I_\lambda(x,y)\lesssim\lambda^{1-2s}, & 0<|x-y|\le C\lambda^{\epsilon-a},\\ I_\lambda(x,y)\lesssim\lambda^{-\frac{a}{2}+\frac{\epsilon}{2\tau}}(|x-y|)^{-\frac{1}{2\tau}}\lambda^{1-2s}, & \lambda^{\epsilon-a}<|x-y|\le C\lambda^{\epsilon},\\ I_\lambda(x,y)\lesssim(\lambda|x-y|)^{-2}\lambda^{1-2s}, & |x-y|\ge C\lambda^{\epsilon}. \end{cases} \tag{2.59}$$

The constants C are independent of λ.

Proof. For the case $|x-y|\ge C\lambda^{\epsilon}|t(y)-t(x)|^{\tau}$. Let

$$F(\xi)=\lambda(\gamma(y,t(y))-\gamma(x,t(x)))\xi+\lambda^a(t(y)-t(x))|\xi|^a.$$

It's obviously that

$$I_\lambda(x,y)=\lambda^{1-2s}\int e^{iF(\xi)}G(\xi)d\xi, \tag{2.60}$$

and

$$\begin{cases} F'(\xi)=\lambda(\gamma(y,t(y))-\gamma(x,t(x)))+a\lambda^a sgn(\xi)(t(y)-t(x))|\xi|^{a-1},\\ F''(\xi)=a(a-1)\lambda^a(t(y)-t(x))|\xi|^{a-2},\\ F^{(3)}(\xi)=a(a-1)(a-2)\lambda^a(t(y)-t(x))sgn(\xi)|\xi|^{a-3}. \end{cases} \tag{2.61}$$

From γ satisfying the condition (1.11a), (1.11b) and $|F'(\xi)|\ge C\lambda|x-y|$. Noticing that $\frac{1}{2}\le|\xi|\le 2$, $|F^{(j)}(\xi)|\le C\lambda^a$ for $j=2,3$ and by Lemma 2.2, we can obtain

$$\begin{aligned}\int e^{if(\xi)}G(\xi)d\xi\lesssim&\int_{\frac{1}{2}\le|\xi|\le 2}\frac{1}{|F'(\xi)|^2}\left(1+\frac{|F''(\xi)|}{|F'(\xi)|}+\Big(\frac{|F''(\xi)|}{|F'(\xi)|}\Big)^2+\frac{|F^{(3)}(\xi)|}{|F'(\xi)|}\right)d\xi\\ \lesssim&(\lambda|x-y|)^{-2}\sum\Big(\frac{\lambda^a}{\lambda|x-y|}\Big)^r\\ \lesssim&(\lambda|x-y|)^{-2}. \end{aligned} \tag{2.62}$$

For the case $|x-y| \leq C\lambda^{\epsilon}|t(y)-t(x)|^{\tau}$.

$$|F''(\xi)| \geq C\lambda^{a}|t(x)-t(y)| \geq \lambda^{a-\frac{\epsilon}{\tau}}(|x-y|)^{\frac{1}{\tau}}. \tag{2.63}$$

Noticing that $\|G\|_{L^\infty} \leq C$ and $\|G'\|_{L^1} \leq C$, by Lemma 2.2, then

$$|I_\lambda(x,y)| \leq C\lambda^{-\frac{a}{2}+\frac{\epsilon}{2\tau}}(|x-y|)^{-\frac{1}{2\tau}}\lambda^{1-2s}. \tag{2.64}$$

Thus, we complete the proof. □

Let I_λ be as above. By Hölder's inequality and Young's inequality, we have

$$\|R'_{\lambda,m}h\|^2_{L^2(\mathbb{R})} \leq \int_{\mathbb{R}}(I_\lambda * |h(x)|)|h(x)|dx \leq C\|I_\lambda\|\|h\|^2_{L^2(\mathbb{R})}. \tag{2.65}$$

From Proposition 2.3 it follows

$$\|I_\lambda\|_{L^1_{(\mathbb{R})}} \leq \lambda^{-2\delta}, \tag{2.66}$$

where $\delta > 0$. So we have

$$\|R_{\lambda,m}g\|_{L^2(\mathbb{R})} \leq \lambda^{-2\delta}\|g\|_{L^2(\mathbb{R})}, \tag{2.67}$$

the constant C is independent of m and λ. By taking $m \to \infty$ we have

$$\|R_\lambda g\|_{L^2(\mathbb{R})} \leq \lambda^{-2\delta}\|g\|_{L^2(\mathbb{R})}.$$

For $0 < \tau \leq 1$, we have the estimate

$$\begin{cases} I_\lambda(x,y) \lesssim \lambda^{1-2s}, & |x-y| \leq C\lambda^{\epsilon}, \\ I_\lambda(x,y) \lesssim (\lambda|x-y|)^{-2}N^{1-2s}, & |x-y| \geq C\lambda^{\epsilon}. \end{cases} \tag{2.68}$$

We prove that for all $0 < \tau \leq 1$

$$\begin{aligned} \|R'_{n,m}h\|^2_{L^2(\mathbb{R})} \leq & \int_{\mathbb{R}}\int_{\mathbb{R}}|I_N(x,y)||h(x)||h(y)|dxdy \\ \leq & C\int_{|x-y|\leq CN^{\epsilon}} N^{1-2s}|h(x)||h(y)|dxdy \\ & + C\int_{|x-y|>CN^{\epsilon}} N^{1-2s}(N|x-y|^{-2})|h(x)||h(y)|dxdy \\ \leq & CN^{1-2s+\epsilon}\|h\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{2.69}$$

We need to restriction the exponent of λ to negative,

$$1-2s+\epsilon < 0, \tag{2.70}$$

for any $\epsilon > 0$, thus the convergence holds if $s > \frac{1}{2}$.

We consider the case for $\frac{1}{2} < \tau \leq 1$

$$\begin{aligned}\|R'_{n,m}h\|^2_{L^2} \leq & C\int_{|x-y|\leq C\lambda^{\epsilon}} \lambda^{-\frac{a}{2}+\frac{\epsilon}{2\tau}}(|x-y|)^{-\frac{1}{2\tau}}\lambda^{1-2s}|h(x)||h(y)|dxdy \\ & + C\int_{|x-y|>C\lambda^{\epsilon}} \lambda^{1-2s}(\lambda|x-y|)^{-2}|h(x)||h(y)|dxdy \\ \leq & C\lambda^{-\frac{a}{2}+1-2s+\epsilon}\|h\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{2.71}$$

Thus, we have

$$-\frac{a}{2}+1-2s+\epsilon < 0. \tag{2.72}$$

Then,

$$s > \frac{1}{2}-\frac{a}{4}. \tag{2.73}$$

The case for $\frac{1}{4} \leq \tau < \frac{1}{2}$, it is obviously that $-\frac{1}{2\tau} \geq -2$. We obtain that

$$\int_{|x-y|<\lambda^{\epsilon-a}} \lambda^{1-2s}h(x)h(y)dxdy \leq C\lambda^{1-2s+\epsilon-a}\|h\|^2_{L^2(\mathbb{R})}, \tag{2.74a}$$

$$\begin{aligned}&\int_{\lambda^{\epsilon-a}<|x-y|\leq C\lambda^{\epsilon}} \lambda^{-\frac{a}{2}+\frac{\epsilon}{2\tau}}|x-y|^{-\frac{1}{2\tau}}\lambda^{1-2s}|h(x)||h(y)|dxdy \\ &\leq C\lambda^{1-2s+\frac{a}{2}(\frac{1}{\tau}-3)+\epsilon}\|h\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{2.74b}$$

Then,

$$s > \frac{1}{2}+\frac{a}{4}\Big(\frac{1}{\tau}-3\Big). \tag{2.75}$$

We consider the case for $\tau = \frac{1}{2}$. Denote $\tau = \frac{1}{2}-\theta$, $0<\theta<\frac{1}{6}$, as above, we have

$$s > \frac{1}{2}+\frac{a}{4}\Big(\frac{1}{\frac{1}{2}-\theta}-3\Big).$$

So that the convergence holds if $s > \frac{1}{2}-\frac{a}{4}$, when $\tau = \frac{1}{2}$.

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