

A Note on the Convergence of the Schrödinger Operator along Curve

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. In this paper we set up a convergence property for the fractional Schrödinger operator for $0 < a < 1$. Moreover, we extend the known results to non-tangent convergence and the convergence along Lipschitz curves.

Key Words: Refinement of the Carleson problem, disconvergence set, fractional Schrödinger operator, Hausdorff dimension, Sobolev space.

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1 Introduction

Given a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, we consider the fractional Schrödinger operator defined by

$$S_a(t)f(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{ix\zeta + it|\zeta|^a} \hat{f}(\zeta) d\zeta \quad (1.1)$$

with $a > 0$. It is the solution to the initial data problem of the fractional Schrödinger equation

$$\begin{cases} \partial_t u(x, t) = (-\Delta)^{\frac{a}{2}} u(x, t), & \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x, 0) = f(x). \end{cases} \quad (1.2)$$

From the Plancherel theorem, (1.1) can be easily extend to a bounded operator on L^2 -based Sobolev space $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$. Here we recall the norm of $H^s(\mathbb{R}^n)$ as

$$\|f\|_{H^s(\mathbb{R})} = \left(\int_{\mathbb{R}} (1 + |\zeta|^2)^s |\hat{f}(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} < \infty. \quad (1.3)$$

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When $a = 2$, $S_2(t)$ becomes the classical Schrödinger operator. We take $S(t)$ as its abbreviation. In [3], Carleson posed the following well known problem: To determine the infimum (critical) index s_c such that for any $s > s_c$,

$$\lim_{t \rightarrow 0} S(t)f(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n, \quad \forall f \in H^s(\mathbb{R}^n). \tag{1.4}$$

For one dimensional case, Carleson [3] showed that (1.4) holds for $s \geq \frac{1}{4}$. The corresponding opposite result is obtained by Dahlberg and Kenig [7]. Moreover they showed that (1.4) does not hold for $s < \frac{1}{4}$ in any dimension. Thus we can conclude $s_c = 1/4$ for $n = 1$. After that, there are enumerate literatures devoted to settling the high dimensional problems. Sjölin [16] and Vega [20] proved the convergence if $s > 1/2$ independently. Lee [11] set up (1.4) when $s > 3/8$ and $n = 2$. Bourgain [1] improved these results by showing that the convergence holds for $s > \frac{1}{2} - \frac{1}{4n}$ and the necessary condition is $s \geq \frac{1}{2} - \frac{1}{n}$ for $n \geq 4$. More recently, Bourgain [2] constructed a counter example to show that (1.4) does not hold for $s < \frac{n}{2(n+1)}$. Du, Guth and Li [6] obtained that $s_c = 1/3$ by setting up (1.4) if $s > \frac{1}{3}$ and $n = 2$. Furthermore, Du and Zhang [9] proved the convergence holds if $s > \frac{n}{2(n+1)}$ and $n \geq 3$. Thus the solution to Carleson’s problem is $s_c = \frac{n}{2(n+1)}$ for $n \geq 2$.

It is nature to ask the same question for general $a > 0$. An interesting phenomenon is that when $a > 1$, the results do not depend on the value of a , but when $a < 1$, the results depend on the value of it. For $a > 1$, the convergence were proved to be true if $s > 1/4$, $n = 1$ by Sjölin [16] and Vega [20]. Miao, Yang, and Zheng [14] obtained the convergence when $s > \frac{3}{8}$ and $n = 2$. Cho and Ko [4] proved that the convergence also holds when $s > \frac{n}{2(n+1)}$ and $n \geq 2$. The same result was also obtained by Li, Li and Xiao [12] by setting up the up-bound of Hausdorff dimension of the divergent set.

When $0 < a < 1$, Walther [21, 22] set up the convergence when $s > a/4$ in one dimension and for the radial functions in higher dimensional spaces. Very recently Dimou and Seeger [10] obtained the equivalent condition to time sequence of $\{t_n\}$ such that if $t_n \rightarrow 0$ (1.4) holds. Thus we know that $s_c = \frac{a}{4}$ is the critical index when $n = 1$. For $n \geq 2$, Zhang [24] proved the convergence for $s > \frac{na}{4}$. It is still very open to determine the critical index for the high dimensional case.

An interesting generalization of the point-wise convergence problem is to set up the convergence in a wider approach region instead of vertical lines, for example, the non-tangential limit. It is easy to see that it holds for $s > \frac{n}{2}$ by Sobolev Embedding. Sjölin and Sjögren [15] showed that non-tangential convergence fails for $s \leq \frac{n}{2}$. Cho, Lee and Vargas [5] showed that the non-tangential convergence holds if $s > \frac{\beta(\Theta)+1}{4}$ when $a = 2$ and $n = 2$. $\beta(\Theta)$ denotes the upper Minkowski dimension of the upper cover of the cone which will be given soon. Cho, Lee and Vargas [5] deal with estimating the boundary of the operator along the restricted direction and time localization argument. Shiraki [17] extended result of [5] to $a > 1$. In this paper, we will deal with the case of $0 < a < 1$, $n = 1$.

To state our main results, we need first introduce in some notations. Let $\Theta \subset \mathbb{R}$ be a

fixed compact set of \mathbb{R} , We call

$$\Gamma(x, t) = \{x + s\theta : s \in [-t, t] \text{ and } \theta \in \Theta\}, \quad x \in \mathbb{R} \quad \text{and} \quad t \geq 0, \tag{1.5}$$

as a cone respect to the upper cover Θ . It is clear if $\Theta = [-1, 1]$, it is exactly a classical cone in \mathbb{R}^2 . The upper Minkowski dimension of Θ which can be defined as

$$\beta(\Theta) = \inf \left\{ r > 0 : \limsup_{\delta \rightarrow 0} N(\Theta, \delta) \delta^r = 0 \right\}. \tag{1.6}$$

Here, $N(\Theta, \delta)$ denotes the smallest number of δ -intervals which cover Θ .

The main results of this paper can be state as follows.

Theorem 1.1. *Let $0 < a < 1$, $\Theta \subset \mathbb{R}$ be a compact set. If $s > \frac{1}{2} - \frac{a}{4}(1 - \beta(\Theta))$, then there exists a constant $C_s > 0$, such that*

$$\left\| \sup_{(t,\theta) \in [-1,1] \times \Theta} |S_a(t)f(\cdot + t\theta)| \right\|_{L^2(-1,1)} \leq C_s \|f\|_{H^s(\mathbb{R})}. \tag{1.7}$$

Corollary 1.1. *Under the condition of Theorem 1.1, we have*

$$\lim_{y \in \Gamma_x, t \rightarrow 0} S_a(t)f(y) = f(x) \quad \text{a.e. } x \in \mathbb{R}, \quad \forall f \in H^s(\mathbb{R}). \tag{1.8}$$

Remark 1.1. When $\Theta = [-1, 1]$, we have $\beta(\Theta) = 1$. By the results of Sjölin and Sjögren [15], our result is sharp in this case. For $\beta(\Theta) < 1$, our results are new. This result is not coincide with the critical index $s_c = \frac{a}{4}$ when $\Theta = \{0\}$. But the latter is only a very special case of $\beta(\Theta) = 0$.

The non-tangential convergence means that the convergence is true along any curve in the cone region. The critic number s_c is $\frac{n}{2}$ when $\beta(\Theta) = 1$. Theorem 1.1 shows that along some curve in $\Gamma(\Theta)$ the convergence can also be true for functions with less regularity. Thus is would be interested to understand convergence for the points along some curves in the cone. Given a continuous curve $\gamma(x, t)$, such that $\lim_{t \rightarrow 0} \gamma(x, 0) = x$, we define the operator along this curve as

$$S_{t,\gamma}f(x) = S_t f(\gamma(x, t)) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\gamma(x,t)\xi + it|\xi|^a} \hat{f}(\xi) d\xi, \quad f \in \mathcal{S}(\mathbb{R}). \tag{1.9}$$

The question now is to determine the lower index $s_{c,\gamma}$, such that for $s > s_{c,\gamma}$,

$$\lim_{t \rightarrow 0} S_{t,\gamma}f(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n, \quad \forall f \in H^s(\mathbb{R}^n). \tag{1.10}$$

For classical Schrödinger operator Lee and Rogers [13], Cho, Lee and Vargas [5] considered the curve $\gamma(x, t)$ satisfies the following conditions:

$$|\gamma(x, t) - \gamma(y, t)| \leq C|t - t'|^\tau, \tag{1.11a}$$

$$c|x - y| \leq |\gamma(x, t) - \gamma(y, t)| \leq C|x - y|. \tag{1.11b}$$

Cho, Lee and Vargas [14] obtained the pointwise convergence holds if

$$s > \max \left\{ \frac{1}{2} - \tau, \frac{1}{4} \right\}.$$

Ding and Niu [8] obtained the convergence along the curve holds if

$$s > \frac{a}{4} \quad \text{for } \frac{1}{2} < \tau < 1$$

or

$$s > \min \left\{ \frac{a}{2}, \frac{a}{4} \left(\frac{1}{\tau} - 1 \right) \right\}, \quad \text{when } a > 1.$$

Furthermore, Ding and Niu [8] show it is sharp when $a \geq 2$ the critical index $s_c = \max \left\{ \frac{1}{2} - \tau, \frac{1}{4} \right\}$. We focus on $0 < a < 1$. For this aim, we need to consider the maximal operator

$$S_{t,\gamma}^* f(x) = \sup_{t \in [0,T]} S_{t,\gamma} f(x) \tag{1.12}$$

with a given constant $T > 0$.

We now state our next result:

Theorem 1.2. *Let $0 < a < 1, 0 < \tau \leq 1$. The curve γ satisfies (1.11a) and (1.11b). We have*

$$\|S_{t,\gamma}^* f\|_{L^2(\mathbb{R})} \leq C \|f\|_{H^s(\mathbb{R})}, \tag{1.13}$$

whenever

$$s > \frac{1}{2} - \frac{a}{4} \quad \text{for } \frac{1}{2} < \tau \leq 1,$$

or

$$s > \min \left\{ \frac{1}{2}, \frac{1}{2} + \frac{a}{4} \left(\frac{1}{\tau} - 3 \right) \right\} \quad \text{for } 0 < \tau \leq \frac{1}{2}.$$

Corollary 1.2. *Under the condition of Theorem 1.2, we have*

$$\lim_{t \rightarrow 0} S_{t,\gamma}(t) f(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}, \quad \forall f \in H^s(\mathbb{R}). \tag{1.14}$$

2 Proof of main results

2.1 Two lemmas

In this section, we collect two lemmas which will be used very frequently in our proof.

Lemma 2.1 (Van der Corput’s lemma, [18, p. 309]). *Suppose $\lambda > 1$ and we have $|\phi^k(x)| \geq 1$ for all (a, b) . If $k = 1$ and ϕ' is monotonic on (a, b) , or simply $k \geq 2$, then there exists a constant C_k such that*

$$\left| \int_b^a e^{i\lambda\phi(x)} \psi(x) dx \right| < C_k \lambda^{-\frac{1}{k}} \left(\int_a^b |\psi'(x)| dx + \|\psi\|_{L^\infty} \right). \tag{2.1}$$

Lemma 2.2 ([19]). Let I denote an open interval in \mathbb{R} . For $g \in C_0^\infty(I)$ and real valued function $F \in C^\infty(I)$ with $F' \neq 0$, if $k \in \mathbb{N}$, then

$$\int_I e^{F(x)} g(x) dx = \int_I e^{F(x)} h_k(x) dx, \quad (2.2)$$

where h_k is a linear combination of functions of the form

$$g^{(s)}(F')^{-k-r} \prod_{q=1}^r F^{(j_q)}$$

with $0 \leq s \leq k$, $0 \leq r \leq k$ and $2 \leq j_q \leq k+1$.

2.2 Proof of Theorem 1.1

Let φ be a bump function supported on $[-1, 1]$ and $\psi = \varphi(x/2) - \varphi(2x)$. And we take the notation that $\psi_k(x) = \psi(2^{-k}x)$ for any $k \in \mathbb{N}$. Given $f \in \mathcal{S}(\mathbb{R})$, we denote the projections of the function to the dyadic annulus respectively by

$$\hat{f}_0(\xi) = \hat{f}(\xi)\varphi(\xi) \quad \text{and} \quad \hat{f}_k(\xi) = \hat{f}(\xi)\psi_k(\xi), \quad k \in \mathbb{N}.$$

Then we have the following partition of unit

$$f(x) = f_0(x) + \sum_{k \geq 1} f_k(x).$$

Denote the maximal operator

$$M_\Theta f(x) = \sup\{|S_a(t)f(x+t\theta)| : -1 \leq t \leq 1, \theta \in \Theta\}. \quad (2.3)$$

For fixed k ,

$$M_\Theta f_k(x) = \sup_{(t,\theta) \in B_1 \times \Theta} |S_a(t)f_k(x+t\theta)| \leq \left(\sum_{j=1}^N \sup_{\theta \in \Omega_{k,j}} |S_t f_k(x+t\theta)|^2 \right)^{\frac{1}{2}}, \quad (2.4)$$

where $\Omega_{k,j} = \Omega_j(2^k)$, and $\{\Omega_j(\lambda)\}_{j=1}^N$ is a finite covering of Θ such that

$$\Theta \subset \cup_{j=1}^N \Omega_j(\lambda) \quad \text{and} \quad |\Omega_j(\lambda)| \leq \lambda^{-\frac{n}{2}}. \quad (2.5)$$

By Minkowski's inequality, we have

$$\|M_\Theta f\|_{L^2(I)} \leq \|M_\Theta f_0\|_{L^2(I)} + \sum_{K \geq 1} \|M_\Theta f_K\|_{L^2(I)}. \quad (2.6)$$

For the low frequency part, it is easy to see that

$$\|M_\Theta f_0\|_{L^2(I)} \lesssim \int_{\mathbb{R}} \varphi_0(\xi) |\hat{f}(\xi)| d\xi \lesssim \|f\|_{L^2}. \quad (2.7)$$

We then need to obtain some estimates for $M_{\Theta}f_k$. Moreover,

$$\sum_{k \geq 1} \|M_{\Theta}f_k\|_{L^2(I)}^2 \leq \sum_{K \geq 1} \sum_{j=1}^K \|M_{\Omega_{k,j}}f_k\|_{L^2(I)}^2, \tag{2.8}$$

where

$$M_{\Omega_{k,j}}f_k(x) = \sup\{|S_a(t)f(x+t\theta)| : -1 \leq t \leq 1, \theta \in \Omega_{k,j}\}.$$

Firstly, we claim the following estimate and postpone its proof to the next proposition.

$$\|M_{\Omega}f\|_{L^2(I)} \leq C2^{k(\frac{1}{2}-\frac{a}{4})}\|f\|_{L^2}, \quad \forall \Omega \text{ is an interval with } |\Omega| \leq 2^{k(\frac{a}{2})}. \tag{2.9}$$

And let

$$\widehat{L}_k f = \hat{h}_k \hat{f},$$

where

$$\hat{h} \in C_0^\infty\left(\left(-4, -\frac{1}{4}\right) \cup \left(\frac{1}{4}, 4\right)\right) \quad \text{with } \hat{h} = 1 \quad \text{on } \left(-2, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 2\right).$$

By the definition of the upper Minkowski dimension, there is a constant C_ϵ depending on ϵ to hold the inequality

$$N(\Theta, \lambda^{-\sigma}) \leq C_\epsilon \lambda^{\sigma\beta(\Theta)+\epsilon}$$

for any $\epsilon > 0$. And by (2.8), (2.9), we can obtain that

$$\begin{aligned} \sum_{k \geq 1} \|M_{\Theta}f_k\|_{L^2(I)}^2 &\leq \sum_{K \geq 1} \sum_{j=1}^K \|M_{\Omega_{k,j}}L_k f\|_{L^2(I)}^2 \\ &\leq \sum_{k=1}^{\infty} \sum_{j=1}^K 2^{(1-\frac{a}{2})k} \|L_k f\|_{L^2_2}^2 \\ &\leq \sum_{k=1}^{\infty} 2^{k(1-\frac{a}{2}(1-\beta(\Theta))+\epsilon)} \|L_k f\|_{L^2_2}^2. \end{aligned} \tag{2.10}$$

We conclude that

$$\|M_{\Theta}f\|_{L^2(I)} \lesssim \|f\|_{H^{\frac{1}{2}-\frac{a}{4}(1-\beta(\Theta))+\epsilon}}. \tag{2.11}$$

We now give the proof of (2.9).

Proposition 2.1. *Let $k \geq 1$ and Ω be an interval with $|\Omega| \leq 2^{k(\frac{a}{2})}$. Then, there exists a constant $C > 0$ that*

$$\|M_{\Omega}f\|_{L^2(I)} \leq C2^{k(\frac{1}{2}-\frac{a}{4})}\|f\|_{L^2}. \tag{2.12}$$

Proof. Set $\lambda = 2^k$ and denote

$$Tf(x, t, \theta) = \chi(x, t, \theta) \int_{\mathbb{R}} e^{i((x+t\theta)\xi+t|\xi|^a)} \hat{f}(\xi) \psi\left(\frac{\xi}{\lambda}\right) d\xi, \tag{2.13}$$

where $\chi \in C_0^\infty(I \times [-1, 1] \times \Omega)$. The result follows from

$$\|Tf\|_{L_x^2 L_{t,\theta}^\infty} \leq \lambda^{\frac{1}{2}-\frac{a}{4}} \|f\|_{L^2}. \tag{2.14}$$

By duality, it is need to show that

$$\|T^*F\|_{L^2} \leq C\lambda^{\frac{1}{2}-\frac{a}{4}} \|F\|_{L_x^2 L_t^1 L_\theta^1}, \tag{2.15}$$

where

$$T^*F(\xi) = \psi\left(\frac{\xi}{\lambda}\right) \int_{\mathbb{R}} e^{i((y+t\theta')\xi+t|\xi|^a)} F(y, t, \theta') \chi(x, t, \theta') dx dt d\theta'.$$

It is sufficient to show

$$\|TT^*F\|_{L^2 L_{t,\theta}^\infty} \leq C\lambda^{(\frac{1}{2}-\frac{a}{4})} \|F\|_{L^2 L_{t,\theta}^1}. \tag{2.16}$$

We note that

$$TT^*F(x, t, \theta) = \chi(x, t, \theta) \iiint K_\lambda(t, t', x, y, \theta, \theta') \chi(y, t', \theta') F(y, t', \theta') dy dt' d\theta', \tag{2.17a}$$

$$K_\lambda(t, t', x, y, \theta, \theta') = \chi(x, t, \theta) \chi(y, t', \theta') \lambda \int e^{i(\lambda^a(t'-t)|\xi|^a + \lambda(x-y+t\theta-t\theta')\xi)} \psi^2(\xi) d\xi. \tag{2.17b}$$

We have the following estimates for the kernel K_λ .

(i) The case $|x - y| \geq 4|t - t'|$ and $|x - y| \geq 4\lambda^{-\frac{a}{2}}$. We have

$$\begin{cases} \phi'(\xi) = \lambda(x - y + t\theta - t'\theta') + a\lambda^a(t - t')|\xi|^{a-1}, \\ \phi''(\xi) = a(a - 1)\lambda^a(t - t')|\xi|^{a-2}. \end{cases} \tag{2.18}$$

Then,

$$\begin{aligned} |\phi'(\xi)| &\geq \lambda|x - y + t\theta - t'\theta'| - \lambda^a|t - t'| |\xi|^{a-1} \\ &\gtrsim \lambda|x - y| - \lambda^a|t - t'| |\xi|^{a-1} \\ &\gtrsim \lambda|x - y|. \end{aligned} \tag{2.19}$$

Since $\phi''(\xi)$ is single-signed on $(-\infty, -1]$ and $[1, \infty)$, so $\phi'(\xi)$ is monotonic on $|\xi| \geq 1$. By Lemma 2.1, we can obtain that

$$K_\lambda \lesssim \lambda(\lambda|x - y|)^{-1} \lesssim \lambda^{\frac{a}{2}}|x - y|^{-\frac{1}{2}}, \tag{2.20}$$

when $|x - y| \geq 4\lambda^{-\frac{a}{2}}$.

(ii) The case $|x - y| \leq C\lambda^{-\frac{a}{2}}$ and $|x - y| \geq C|t - t'|$. It's obviously that $K_\lambda \lesssim \lambda$.

(iii) The case $|x - y| \leq C|t - t'|$. By Lemma 2.2, we have

$$|K_\lambda| \lesssim \lambda^{1-\frac{\alpha}{2}}(|x - y|)^{-\frac{1}{2}}. \tag{2.21}$$

It follows from Hölder's inequality and Young's inequality that

$$\int (K * |h|)(x)|h(x)|dx \leq \|K\|_{L^1} \|h\|_{L^2}^2. \tag{2.22}$$

By Fubini theorem and previous argument,

$$\begin{cases} \lambda^{\frac{\alpha}{2}} \int_{-1}^1 \int_{-1}^1 \|F(x, \cdot)\|_{L^1_{t,\theta}} \|F(y, \cdot)\|_{L^1_{t,\theta}} |x - y|^{-\frac{1}{2}} dx dy \lesssim \lambda^{\frac{\alpha}{2}} \|F\|_{L^2 L^1_{t,\theta}}^2, \\ \lambda \int_{-1}^1 \int_{-1}^1 \|F(x, \cdot)\|_{L^1_{t,\theta}} \|F(y, \cdot)\|_{L^1_{t,\theta}} |X_{[-C\lambda^{-\frac{\alpha}{2}}, C\lambda^{-\frac{\alpha}{2}}]}(x - y) dx dy \lesssim \lambda^{1-\frac{\alpha}{2}} \|F\|_{L^2 L^1_{t,\theta}}^2, \\ \lambda^{1-\frac{\alpha}{2}} \int_{-1}^1 \int_{-1}^1 \|F(x, \cdot)\|_{L^1_{t,\theta}} \|F(y, \cdot)\|_{L^1_{t,\theta}} |x - y|^{-\frac{1}{2}} dx dy \lesssim \lambda^{1-\frac{\alpha}{2}} \|F\|_{L^2 L^1_{t,\theta}}^2. \end{cases} \tag{2.23}$$

We compare the exponent of λ , the proof of proposition is completed. □

We finish the proof of Theorem 1.1.

2.3 Proof of Theorem 1.2

We denote the linearization of the maximal operator as

$$Tf(x) = \int_{\mathbb{R}} e^{i\gamma(x,t)\xi + it|\xi|^a} \hat{f}(\xi) d\xi. \tag{2.24}$$

It is sufficient to set up

$$\|Tf(x)\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}. \tag{2.25}$$

We decompose it

$$\begin{aligned} Tf(x) &= \int_{\mathbb{R}} e^{i\gamma(x,t)\xi + it|\xi|^a} \hat{f}_0(\xi) d\xi + \sum_{k=1}^{\infty} \int_{\mathbb{R}} e^{i\gamma(x,t)\xi + it|\xi|^a} \hat{f}_k(\xi) d\xi \\ &=: T_0f(x) + \sum_{k=1}^{\infty} T_kf(x), \end{aligned} \tag{2.26}$$

where f_0 and f_k are the same as in the last subsection. By Minkowski's inequality, we have

$$\|Tf\|_{L_2(\mathbb{R})} \leq \|T_0f\|_{L_2(\mathbb{R})} + \sum_{k=1}^{\infty} \|T_kf\|_{L_2(\mathbb{R})}. \tag{2.27}$$

We first estimate the $\|T_0 f\|_{L^2(\mathbb{R})}$. Let

$$L_0 g(x) = \int_{\mathbb{R}} e^{i\gamma(x,t(x))\xi + it(x)|\xi|^a} \varphi_0(\xi) g(\xi) d\xi, \quad g \in \mathcal{S}(\mathbb{R}). \quad (2.28)$$

Taking function $\rho \in C_0^\infty$, $\rho = 1$ if $|x| \leq 1$, and $\rho = 0$ if $|x| \geq 2$, we denote

$$L_{0,m} g(x) = \rho\left(\frac{x}{m}\right) \int_{\mathbb{R}} e^{i\gamma(x,t(x))\xi + it(x)|\xi|^a} \varphi_0(\xi) g(\xi) d\xi, \quad g \in \mathcal{S}(\mathbb{R}). \quad (2.29)$$

By duality, its adjoint operator

$$L'_{0,m} h(\xi) = \varphi_0(\xi) \int_{\mathbb{R}} e^{-i\gamma(x,t(x))\xi - it(x)|\xi|^a} \rho\left(\frac{x}{m}\right) h(x) dx, \quad m \geq 1. \quad (2.30)$$

Thus, we have

$$\begin{aligned} \|L'_{0,m} h\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \left(\varphi_0(\xi) \int_{\mathbb{R}} e^{-i\gamma(x,t(x))\xi - it(x)|\xi|^a} \rho\left(\frac{x}{m}\right) h(x) dx \right) \\ &\quad \times \left(\varphi_0(\xi) \int_{\mathbb{R}} e^{i\gamma(x,t(y))\xi + it(y)|\xi|^a} \rho\left(\frac{y}{m}\right) h(y) dy \right) d\xi \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} K_0(x, y) h(x) h(y) dx dy, \end{aligned} \quad (2.31)$$

where

$$K_0(x, y) = \rho\left(\frac{x}{m}\right) \rho\left(\frac{y}{m}\right) \int_{\mathbb{R}} e^{i(\gamma(y,t(y)) - \gamma(x,t(x)))\xi + i(t(y) - t(x))|\xi|^a} \varphi_0^2(\xi) d\xi. \quad (2.32)$$

Using the Hölder's inequality and Young's inequality we obtain

$$\|L'_{0,m} h\|_{L^2(\mathbb{R})}^2 \leq C \|K_0\|_{L^1(\mathbb{R})} \|h\|_{L^2(\mathbb{R})}^2. \quad (2.33)$$

We claim that $\|K_0\|_{L^1(\mathbb{R})} < C$ and it is independent of m , which we will give the proof in Proposition 2.2. Thus, we have

$$\|L_{0,m} g\|_{L^2(\mathbb{R})}^2 \leq C \|g\|_{L^2(\mathbb{R})}^2. \quad (2.34)$$

By taking $m \rightarrow \infty$, we have

$$\|L_0 g\|_{L^2(\mathbb{R})}^2 \leq C \|g\|_{L^2(\mathbb{R})}^2. \quad (2.35)$$

We now set up the uniform boundedness of $\|k_0\|_{L^1(\mathbb{R})}$. It is sufficient to set the following proposition.

Proposition 2.2. *Suppose γ satisfy the conditions in Theorem 1.2 and $K_0(x, y)$ as (2.30). Then*

$$\begin{cases} K_0(x, y) \lesssim \frac{1}{(1 + |x - y|)^{-1-a}}, & |x - y| \geq C(2T)^\tau, \\ K_0(x, y) \lesssim 1, & |x - y| \leq C(2T)^\tau. \end{cases} \quad (2.36)$$

Proof. We decompose $K_0(x, y)$ like that

$$\begin{aligned}
 K_0(x, y) &= \int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi} \left(\sum_{k=0}^M \frac{(i(t(y) - t(x))|\xi|^a)^k}{k!} \right) \varphi_0^2(\xi) d\xi \\
 &\quad + \int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi} \left(e^{i(t(y)-t(x))|\xi|^a} \right. \\
 &\quad \left. - \sum_{k=0}^M \frac{(i(t(y) - t(x))|\xi|^a)^k}{k!} \right) \varphi_0^2(\xi) d\xi \\
 &=: K_{0,1}(x, y) + K_{0,2}(x, y),
 \end{aligned} \tag{2.37}$$

where $aM < 1 < a(M + 1)$. It's obvious that

$$K_0 \lesssim 1 \quad \text{for } |x - y| \leq C(2T)^\tau.$$

So we only consider the case $|x - y| \geq C(2T)^\tau$.

The estimate of $K_{0,1}$.

In the view of (2.37), it is need to show

$$\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi} |\xi|^a \varphi_0^2(\xi) d\xi \leq C|x - y|^{-1-a}, \tag{2.38}$$

where the constant C is independent of x , and $x \geq 1$. Let $\psi = 1 - \varphi$ and $\psi_m(\xi) = \psi(m\xi)$. Integrating by parts, we have

$$\begin{aligned}
 &\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi} |\xi|^a \varphi_0^2(\xi) d\xi \\
 &= \lim_{m \rightarrow \infty} \int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi} |\xi|^a \psi_m(\xi) \varphi_0(\xi) d\xi \\
 &= \frac{-1}{i(\gamma(y, t(y)) - \gamma(x, t(x)))} \left(\int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi} a \operatorname{sgn}(\xi) |\xi|^{a-1} \varphi_0(\xi) d\xi \right. \\
 &\quad \left. + \int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi} |\xi|^a \varphi_0'(\xi) d\xi \right. \\
 &\quad \left. + \lim_{m \rightarrow \infty} \int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi} |\xi|^a \psi_m'(\xi) \varphi_0(\xi) d\xi \right) \\
 &= \frac{-1}{i(\gamma(y, t(y)) - \gamma(x, t(x)))} (I_1(x - y) + I_2(x - y) + \lim_{m \rightarrow \infty} I_{3,m}(x - y)).
 \end{aligned} \tag{2.39}$$

We denote $h : \xi \rightarrow \operatorname{sgn}(\xi) |\xi|^{a-1}$. Since h is odd and homogeneous of degree $a - 1$ its inverse Fourier transform is odd and homogeneous of degree $-a$. Thus the convolution $\check{h} * \check{\varphi}_0 = I_1/C$ is bounded and continuous and that it verifies the estimate. I_2 decays rapidly at infinity.

$$|I_{3,m}(x - y)| \leq \lim_{m \rightarrow \infty} 2 \int_{\frac{1}{m}}^{\frac{2}{m}} |\xi|^a \psi_m'(\xi) d\xi \leq Cm^{-a}. \tag{2.40}$$

Thus, we have

$$\left| \int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi} |\xi|^a \varphi_0^2(\xi) d\xi \right| \leq |x - y|^{-1-a}. \tag{2.41}$$

The estimate of $K_{0,2}$.

Set

$$\begin{aligned} K_{0,2,m}(x, y) &= \int e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi} \left(e^{i(t(y)-t(x))|\xi|^a} \right. \\ &\quad \left. - \sum_{k=0}^M \frac{(i(t(y) - t(x))|\xi|^a)^k}{k!} \right) \varphi_0^2(\xi) \psi_m(\xi) d\xi \\ &=: \int e^{iP(\xi)} Q(\xi) d\xi, \end{aligned} \tag{2.42}$$

where

$$P(\xi) = (\gamma(y, t(y)) - \gamma(x, t(x)))\xi, \tag{2.43a}$$

$$Q(\xi) = \left(e^{i(t(y)-t(x))|\xi|^a} - \sum_{k=0}^M \frac{(i(t(y) - t(x))|\xi|^a)^k}{k!} \right) \varphi_0^2(\xi) \psi_m(\xi). \tag{2.43b}$$

By integrating by parts twice, we have

$$K_{0,2}(x, y) = -\frac{1}{(\gamma(y, t(y)) - \gamma(x, t(x)))^2} \int_{\mathbb{R}} e^{iP(\xi)} Q''(\xi) d\xi, \tag{2.44}$$

where

$$\begin{aligned} &Q''(\xi) \\ &= \sum_{\mu+\beta+\eta=2} \left(e^{i(t(y)-t(x))|\xi|^a} - \sum_{k=0}^M \frac{(i(t(y) - t(x))|\xi|^a)^k}{k!} \right)^{(\mu)} (\varphi_0^2(\xi))^{(\beta)} (\psi_m(\xi))^{(\eta)}, \end{aligned} \tag{2.45}$$

when $|x - y| \geq C(2T)^\tau$. We have the following that

$$\begin{aligned} K_{0,2,m}(x, y) &= \frac{1}{(|\gamma(y, t(y)) - \gamma(x, t(x))|^2} \int_{\mathbb{R}} |e^{iP(\xi)}| |Q''(\xi)| d\xi \\ &\lesssim \frac{1}{(1 + |x - y|)^2} \sum_{\mu+\beta+\eta=2} I_{\mu,\beta,\eta}, \end{aligned} \tag{2.46}$$

where

$$I_{\mu,\beta,\eta} = \int \left| \left(e^{i(t(y)-t(x))|\xi|^a} - \sum_{k=0}^M \frac{(i(t(y) - t(x))|\xi|^a)^k}{k!} \right)^{(\mu)} \right| \left| (\varphi_0^2(\xi))^{(\beta)} \right| \left| (\psi_m(\xi))^{(\eta)} \right| d\xi.$$

Thus, for $0 < |\xi| < 1$, the following estimate holds

$$\begin{aligned} & \left| \left(e^{i(t(y)-t(x))|\xi|^a} - \sum_{k=0}^M \frac{(i(t(y)-t(x))|\xi|^a)^k}{k!} \right)^{(\mu)} \right| \\ &= \left| \left(\sum_{M+1}^{\infty} \frac{(i(t(y)-t(x))|\xi|^a)^k}{k!} \right)^{(\mu)} \right| \\ &\leq C|\xi|^{a(M+1)-\mu}. \end{aligned} \tag{2.47}$$

We can obtain the estimate for $1 \leq |\xi| \leq 2$ in a similar way. For $\mu = 0, 1, 2$, by the convergence of Taylor series.

$$\left| \left(\sum_{M+1}^{\infty} \frac{(i(t(y)-t(x))|\xi|^a)^k}{k!} \right)^{(\mu)} \right| \leq C, \quad \mu = 0, 1, 2. \tag{2.48}$$

And by the definition of ψ and $1 \leq |\xi| \leq 2$, we have

$$|(\psi_m(\xi))^{(\eta)}| \leq C|\xi|^{-\eta}, \quad \eta = 1, 2. \tag{2.49}$$

Thus, if $\eta = 0$,

$$\begin{aligned} I_{\mu,\beta,\eta} &\leq C \int_{\frac{1}{m} < |\xi| < 1} |\xi|^{a(M+1)-\mu} d\xi + \int_{1 < |\xi| < 2} d\xi \\ &\leq C \int_{|\xi| < 1} |\xi|^{a(M+1)-2} d\xi + C \leq C. \end{aligned} \tag{2.50}$$

If $\eta = 1$ or $\eta = 2$. We consider $m^{-1} \leq |\xi| \leq 2m^{-1}$ for m sufficient large.

$$I_{\mu,\beta,\eta} \leq C \int_{\frac{1}{m} < |\xi| < \frac{2}{m}} |\xi|^{a(M+1)-\mu-\eta} d\xi \leq Cm^{-1}m^{-a(M+1)+\mu+\eta} \leq C. \tag{2.51}$$

Thus let $m \rightarrow \infty$, so we complete the proof. □

Next, we estimate $\|T_k f\|_{L^2(\mathbb{R})}$. Defining the operator R_λ as

$$R_\lambda g(x) = \lambda^{-s} \int_{\mathbb{R}} e^{i\gamma(x,t(x))\xi} e^{it(x)|\xi|^a} \psi\left(\frac{\xi}{\lambda}\right) g(\xi) d\xi, \quad g \in S(\mathbb{R}), \quad \lambda \geq 2. \tag{2.52}$$

Taking ρ as above

$$R_{\lambda,m} g(x) = \lambda^{-s} \rho\left(\frac{x}{m}\right) \int_{\mathbb{R}} e^{i\gamma(x,t(x))\xi} e^{it(x)|\xi|^a} \psi\left(\frac{\xi}{\lambda}\right) g(\xi) d\xi, \quad g \in S(\mathbb{R}), \quad \lambda \geq 2. \tag{2.53}$$

Noticing that N is a dyadic number, we consider the adjoint operator of it

$$R'_{\lambda,m} h(\xi) = \lambda^{-s} \psi\left(\frac{\xi}{N}\right) \int_{\mathbb{R}} e^{i\gamma(x,t(x))\xi} e^{it(x)|\xi|^a} \rho\left(\frac{x}{m}\right) h(x) dx, \quad m > 1, \quad \lambda \geq 2. \tag{2.54}$$

We have

$$\|R'_{\lambda,m}h(\xi)\|_{L^2_{(\mathbb{R})}} =: \int_{\mathbb{R}} \int_{\mathbb{R}} K_0(x,y)h(x)h(y)dx dy, \tag{2.55}$$

where

$$K_{\lambda}(x,y) = \rho\left(\frac{x}{m}\right)\rho\left(\frac{y}{m}\right) \int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi+i(t(y)-t(x))|\xi|^a} \psi^2\left(\frac{\xi}{\lambda}\right) d\xi. \tag{2.56}$$

Let

$$I_{\lambda}(x,y) = \lambda^{-2s} \int_{\mathbb{R}} e^{i(\gamma(y,t(y))-\gamma(x,t(x)))\xi+i(t(y)-t(x))|\xi|^a} \psi^2\left(\frac{\xi}{\lambda}\right) d\xi. \tag{2.57}$$

Denote $G(\xi) = \psi^2(\xi)$, and by changing the variables, we obtain that

$$I_{\lambda}(x,y) = \lambda^{1-2s} \int_{\mathbb{R}} e^{i\lambda(\gamma(y,t(y))-\gamma(x,t(x)))\xi+i\lambda^a(t(y)-t(x))|\xi|^a} G(\xi) d\xi. \tag{2.58}$$

Proposition 2.3. *Suppose that γ and $I_{\lambda}(x-y)$ as above. For $\frac{1}{4} \leq \tau \leq 1$, we have*

$$\begin{cases} I_{\lambda}(x,y) \lesssim \lambda^{1-2s}, & 0 < |x-y| \leq C\lambda^{\epsilon-a}, \\ I_{\lambda}(x,y) \lesssim \lambda^{-\frac{a}{2}+\frac{\epsilon}{2\tau}}(|x-y|)^{-\frac{1}{2\tau}}\lambda^{1-2s}, & \lambda^{\epsilon-a} < |x-y| \leq C\lambda^{\epsilon}, \\ I_{\lambda}(x,y) \lesssim (\lambda|x-y|)^{-2}\lambda^{1-2s}, & |x-y| \geq C\lambda^{\epsilon}. \end{cases} \tag{2.59}$$

The constants C are independent of λ .

Proof. For the case $|x-y| \geq C\lambda^{\epsilon}|t(y)-t(x)|^{\tau}$. Let

$$F(\xi) = \lambda(\gamma(y,t(y))-\gamma(x,t(x)))\xi + \lambda^a(t(y)-t(x))|\xi|^a.$$

It's obviously that

$$I_{\lambda}(x,y) = \lambda^{1-2s} \int e^{iF(\xi)} G(\xi) d\xi, \tag{2.60}$$

and

$$\begin{cases} F'(\xi) = \lambda(\gamma(y,t(y))-\gamma(x,t(x))) + a\lambda^a \operatorname{sgn}(\xi)(t(y)-t(x))|\xi|^{a-1}, \\ F''(\xi) = a(a-1)\lambda^a(t(y)-t(x))|\xi|^{a-2}, \\ F^{(3)}(\xi) = a(a-1)(a-2)\lambda^a(t(y)-t(x))\operatorname{sgn}(\xi)|\xi|^{a-3}. \end{cases} \tag{2.61}$$

From γ satisfying the condition (1.11a), (1.11b) and $|F'(\xi)| \geq C\lambda|x-y|$. Noticing that $\frac{1}{2} \leq |\xi| \leq 2$, $|F^{(j)}(\xi)| \leq C\lambda^a$ for $j = 2, 3$ and by Lemma 2.2, we can obtain

$$\begin{aligned} \int e^{iF(\xi)} G(\xi) d\xi &\lesssim \int_{\frac{1}{2} \leq |\xi| \leq 2} \frac{1}{|F'(\xi)|^2} \left(1 + \frac{|F''(\xi)|}{|F'(\xi)|} + \left(\frac{|F''(\xi)|}{|F'(\xi)|} \right)^2 + \frac{|F^{(3)}(\xi)|}{|F'(\xi)|} \right) d\xi \\ &\lesssim (\lambda|x-y|)^{-2} \sum \left(\frac{\lambda^a}{\lambda|x-y|} \right)^r \\ &\lesssim (\lambda|x-y|)^{-2}. \end{aligned} \tag{2.62}$$

For the case $|x - y| \leq C\lambda^\epsilon |t(y) - t(x)|^\tau$.

$$|F''(\zeta)| \geq C\lambda^a |t(x) - t(y)| \geq \lambda^{a-\frac{\epsilon}{\tau}} (|x - y|)^{\frac{1}{\tau}}. \tag{2.63}$$

Noticing that $\|G\|_{L^\infty} \leq C$ and $\|G'\|_{L^1} \leq C$, by Lemma 2.2, then

$$|I_\lambda(x, y)| \leq C\lambda^{-\frac{a}{2} + \frac{\epsilon}{2\tau}} (|x - y|)^{-\frac{1}{2\tau}} \lambda^{1-2s}. \tag{2.64}$$

Thus, we complete the proof. □

Let I_λ be as above. By Hölder's inequality and Young's inequality, we have

$$\|R'_{\lambda,m}h\|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}} (I_\lambda * |h(x)|) |h(x)| dx \leq C \|I_\lambda\| \|h\|_{L^2(\mathbb{R})}^2. \tag{2.65}$$

From Proposition 2.3 it follows

$$\|I_\lambda\|_{L^1(\mathbb{R})} \leq \lambda^{-2\delta}, \tag{2.66}$$

where $\delta > 0$. So we have

$$\|R_{\lambda,m}g\|_{L^2(\mathbb{R})} \leq \lambda^{-2\delta} \|g\|_{L^2(\mathbb{R})}, \tag{2.67}$$

the constant C is independent of m and λ . By taking $m \rightarrow \infty$ we have

$$\|R_\lambda g\|_{L^2(\mathbb{R})} \leq \lambda^{-2\delta} \|g\|_{L^2(\mathbb{R})}.$$

For $0 < \tau \leq 1$, we have the estimate

$$\begin{cases} I_\lambda(x, y) \lesssim \lambda^{1-2s}, & |x - y| \leq C\lambda^\epsilon, \\ I_\lambda(x, y) \lesssim (\lambda|x - y|)^{-2} N^{1-2s}, & |x - y| \geq C\lambda^\epsilon. \end{cases} \tag{2.68}$$

We prove that for all $0 < \tau \leq 1$

$$\begin{aligned} \|R'_{n,m}h\|_{L^2(\mathbb{R})}^2 &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |I_N(x, y)| |h(x)| |h(y)| dx dy \\ &\leq C \int_{|x-y| \leq CN^\epsilon} N^{1-2s} |h(x)| |h(y)| dx dy \\ &\quad + C \int_{|x-y| > CN^\epsilon} N^{1-2s} (N|x - y|^{-2}) |h(x)| |h(y)| dx dy \\ &\leq CN^{1-2s+\epsilon} \|h\|_{L^2(\mathbb{R})}^2. \end{aligned} \tag{2.69}$$

We need to restriction the exponent of λ to negative,

$$1 - 2s + \epsilon < 0, \tag{2.70}$$

for any $\epsilon > 0$, thus the convergence holds if $s > \frac{1}{2}$.

We consider the case for $\frac{1}{2} < \tau \leq 1$

$$\begin{aligned} \|R'_{n,m}h\|_{L^2}^2 &\leq C \int_{|x-y|\leq C\lambda^\epsilon} \lambda^{-\frac{a}{2}+\frac{\epsilon}{2\tau}} (|x-y|)^{-\frac{1}{2\tau}} \lambda^{1-2s} |h(x)||h(y)| dx dy \\ &\quad + C \int_{|x-y|>C\lambda^\epsilon} \lambda^{1-2s} (\lambda|x-y|)^{-2} |h(x)||h(y)| dx dy \\ &\leq C \lambda^{-\frac{a}{2}+1-2s+\epsilon} \|h\|_{L^2(\mathbb{R})}^2. \end{aligned} \quad (2.71)$$

Thus, we have

$$-\frac{a}{2} + 1 - 2s + \epsilon < 0. \quad (2.72)$$

Then,

$$s > \frac{1}{2} - \frac{a}{4}. \quad (2.73)$$

The case for $\frac{1}{4} \leq \tau < \frac{1}{2}$, it is obviously that $-\frac{1}{2\tau} \geq -2$. We obtain that

$$\int_{|x-y|<\lambda^{\epsilon-a}} \lambda^{1-2s} h(x)h(y) dx dy \leq C \lambda^{1-2s+\epsilon-a} \|h\|_{L^2(\mathbb{R})}^2, \quad (2.74a)$$

$$\begin{aligned} \int_{\lambda^{\epsilon-a}<|x-y|\leq C\lambda^\epsilon} \lambda^{-\frac{a}{2}+\frac{\epsilon}{2\tau}} |x-y|^{-\frac{1}{2\tau}} \lambda^{1-2s} |h(x)||h(y)| dx dy \\ \leq C \lambda^{1-2s+\frac{a}{2}(\frac{1}{\tau}-3)+\epsilon} \|h\|_{L^2(\mathbb{R})}^2. \end{aligned} \quad (2.74b)$$

Then,

$$s > \frac{1}{2} + \frac{a}{4} \left(\frac{1}{\tau} - 3 \right). \quad (2.75)$$

We consider the case for $\tau = \frac{1}{2}$. Denote $\tau = \frac{1}{2} - \theta$, $0 < \theta < \frac{1}{6}$, as above, we have

$$s > \frac{1}{2} + \frac{a}{4} \left(\frac{1}{\frac{1}{2} - \theta} - 3 \right).$$

So that the convergence holds if $s > \frac{1}{2} - \frac{a}{4}$, when $\tau = \frac{1}{2}$.

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