

# Compactness of the Commutators of Fractional Hardy Operator with Rough Kernel

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

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**Abstract.** The more explicit decomposition of the operator and the kernel are utilized to investigate a characterization of the central  $BMO(\mathbb{R}^n)$ -closure of  $C_c^\infty(\mathbb{R}^n)$  space via the compactness of the commutators of fractional Hardy operator with rough kernel.

**Key Words:** Fractional Hardy operator, commutator, compactness.

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## 1 Introduction

Problem of commutators draws recently more and more attention of Harmonic analysis, such as its application in the study of elliptic equations [1, 7]. For example, Sun, Wang and Zhang simplify the proof of the famous Wu's theorem on Navier-Stokes equations greatly in [18] and the technique used is some estimates for commutators by Lu and Yan [13]. The commutator formed by an operator  $T$  and a suitable function  $b$  can be recalled as

$$[b, T]f := b(Tf) - T(bf).$$

We call a function  $b \in L_{loc}(\mathbb{R}^n)$  is a central  $BMO(\mathbb{R}^n)$  (the mean oscillation function space) function, denoted by  $CBMO(\mathbb{R}^n)$  which was introduced by Lu and Yang [14], if

$$\|b\|_{CBMO(\mathbb{R}^n)} := \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |b(x) - b_{B_r}| dx < \infty.$$

Here and in what follows,  $B_r := B(0, r)$  is a ball centered at 0 with radius  $r > 0$ .  $CBMO(\mathbb{R}^n)$  can be understood as a local version of  $BMO(\mathbb{R}^n)$  at the origin,  $BMO(\mathbb{R}^n) \subset CBMO(\mathbb{R}^n)$  and they have quite different properties since for  $1 < p < \infty$ ,

$$\|b\|_{BMO(\mathbb{R}^n)} \approx \|b\|_{BMO^p(\mathbb{R}^n)} \quad \text{and} \quad \|b\|_{CBMO(\mathbb{R}^n)} \lesssim \|b\|_{CBMO^p(\mathbb{R}^n)}$$

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with

$$\|b\|_{BMO^p(\mathbb{R}^n)} = \sup_{B \subset \mathbb{R}^n} \left( \frac{1}{|B|} \int_B |b(x) - b_B|^p dx \right)^{\frac{1}{p}},$$

$$\|b\|_{CBMO^p(\mathbb{R}^n)} = \sup_{r>0} \left( \frac{1}{|B_r|} \int_{B_r} |b(x) - b_{B_r}|^p dx \right)^{\frac{1}{p}}.$$

Thus, the John-Nirenberg inequality is not true for  $CBMO(\mathbb{R}^n)$ . We follow the notation used in the existed work:  $VMO(\mathbb{R}^n)$  denotes the  $BMO(\mathbb{R}^n)$ -closure of  $C_c^\infty(\mathbb{R}^n)$  (the space of all functions being infinite-times continuously differential in  $\mathbb{R}^n$  with compact support),  $CVM O(\mathbb{R}^n)$  stands for the  $CBMO(\mathbb{R}^n)$ -closure of  $C_c^\infty(\mathbb{R}^n)$ .

This paper provides a characterization of the  $CVM O(\mathbb{R}^n)$  space by the compactness of  $[b, T]$ , when  $T$  is the following fractional Hardy operator

$$H_{\Omega,\alpha}f(x) = \frac{1}{|x|^{n-\alpha}} \int_{|y|<|x|} \Omega(x-y)f(y)dy,$$

$$H_{\Omega,\alpha}^*f(x) = \int_{|y|\geq|x|} \frac{\Omega(x-y)f(y)}{|y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

Here  $\Omega$  satisfies

$$\Omega(tx) = \Omega(x), \quad \forall t > 0, \quad x \in \mathbb{R}^n, \tag{1.1a}$$

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.1b}$$

$$\Omega \in L^q(S^{n-1}), \quad \forall q \geq 1. \tag{1.1c}$$

The  $L^{q \geq 1}$ -Dini condition of  $\Omega$  can be recalled as

$$\int_0^1 \frac{w_q(\delta)}{\delta} < \infty \quad \text{with} \quad w_q(\delta) = \sup_{\|\tau\| \leq \delta} \left( \int_{S^{n-1}} |\Omega(\tau x') - \Omega(x')|^q d\sigma(x') \right)^{\frac{1}{q}}$$

and  $\tau$  is a rotation on  $S^{n-1}$  with

$$\|\tau\| = \sup_{x' \in S^{n-1}} |\tau x' - x'|.$$

For a suitable function  $h$ ,  $H_{\Omega,\alpha}^*$  is said to be the dual operator of  $H_{\Omega,\alpha}$  in the following sense

$$\int_{\mathbb{R}^n} h(x)H_{\Omega,\alpha}f(x)dx = \int_{\mathbb{R}^n} f(x)H_{\Omega,\alpha}^*h(x)dx.$$

Fu, Lu and Zhao considered the boundedness of  $H_{\Omega,\alpha}$  and  $[b, H_{\Omega,\alpha}]$  on homogeneous Herz spaces and Lebesgue spaces for  $b \in BMO(\mathbb{R}^n)$  in [11]. For  $\Omega = 1$ , see for example [9, 16].

The pioneer work on the compactness of operators can be traced to Uchiyama [19], where a characterization of  $VMO(\mathbb{R}^n)$  via the compactness of  $[b, T]$  with  $T$  is the classical Calderón-Zygmund singular integral operator is obtained. To date, much work has been reported in these field. For example, the compactness of  $[b, T]$  on Lebesgue space when  $b$  is in an appropriately  $BMO$  space and  $T$  is the multiplication operator [2]; a characterization of  $VMO(\mathbb{R}^n)$  by the compactness of  $[b, T]$  when  $T$  is the parabolic singular integral [4]; the compactness theory of  $[b, T]$  when  $T$  is the generalized Toeplitz operators by Krantzl and Li [12]; the characterizations of  $VMO(\mathbb{R}^n)$  via the compactness of  $[b, T]$  when  $T$  is the Riesz potential [5] and  $T$  is the singular integral operator [6] on Morrey type space; the compactness of  $[b, T]$  for bilinear operators on Morrey spaces [8]; the characterization of  $CVMO(\mathbb{R}^n)$  by compactness of  $[b, T]$  when  $T$  is the classical Hardy operator and the Hardy operator with homogeneous kernels [10, 15].

The know results for the function characterizations highly depended on the smoothness of  $\Omega$  and there have been many attempts to weak the condition of  $\Omega$  have been undertaken, see e.g., [19] for  $\Omega \in Lip_1(\mathbb{S}^{n-1})$  (Lipschitz functional space), [3, 4] for  $\Omega$  satisfies

$$|\Omega(x') - \Omega(y')| \leq \frac{A}{\left(\log \frac{2}{|x'-y'|}\right)^\gamma} \quad \text{with } A > 0, \quad \gamma > 1 \quad \text{and } x', y' \in \mathbb{S}^{n-1}. \quad (1.2)$$

It is obvious that (1.2) is weaker than the Lipschitz condition  $Lip_{0 < \gamma \leq 1}(\mathbb{S}^{n-1})$  and is stronger than (1.1c). Furthermore, if  $\Omega$  satisfies (1.2), then for  $q \geq 1$ ,

$$\int_0^1 \frac{w_q(\delta)}{\delta} (1 + |\log \delta|) d\delta < \infty. \quad (1.3)$$

The major goal of this paper is to give the following characterization of  $CVMO(\mathbb{R}^n)$  via the compactness of  $[b, H_{\Omega, \alpha}]$  and  $[b, H_{\Omega, \alpha}^*]$ .

**Theorem 1.1.** *Let  $0 < \alpha < n$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ ,  $\Omega$  satisfy (1.1a), (1.1b), (1.2) and  $b \in BMO(\mathbb{R}^n)$ . Then  $b \in CVMO(\mathbb{R}^n) \iff$  Both  $[b, H_{\Omega, \alpha}]$  and  $[b, H_{\Omega, \alpha}^*]$  are compact from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ .*

**Remark 1.1.** The assumption  $b \in BMO(\mathbb{R}^n)$  in Theorem 1.1 can not be weakened in the proof of the necessity part since the John-Nirenberg inequality of  $BMO(\mathbb{R}^n)$  function is used and it is not true for  $CBMO(\mathbb{R}^n)$ . Since  $H_{\Omega, \alpha}$  is centrosymmetric, the method used to consider the Calderón-Zygmund singular integral [4] can not be applied to  $H_{\Omega, \alpha}$  directly.

Section 2 devoted to the basic lemmas for the proof Theorem 1.1; in Section 3, we shall give the proof of Theorem 1.1 by more general case.

In what follows, the symbol  $C$  stands for a positive constant which may vary from line to line.  $A \lesssim B$  means  $A \leq CB$  and  $A \simeq B$  whenever  $A \lesssim B$  and  $B \lesssim A$ .  $\mathbb{Z}$  denotes the set of all integers.  $B_k := B_{2^k}$ ,  $C_k := B_k \setminus B_{k-1}$  and  $\chi_k := \chi_{C_k}$  with  $k \in \mathbb{Z}$ .

## 2 Preparation

Four lemmas will be described in this section which are useful for the analysis of Theorem 1.1. We first recall the John-Nirenberg type inequality of  $BMO(\mathbb{R}^n)$  function and some properties of  $CBMO(\mathbb{R}^n)$  type function from [15, Lemma 2.1].

**Lemma 2.1.** (a) Let  $b \in BMO(\mathbb{R}^n)$ . Then for  $C_2 > C_1 > 2$  and  $\forall x_0 \in \mathbb{R}^n$ , there exist positive constants  $C_3, C_4, C_5$  (depending on  $C_1, C_2$  and  $b$ ), such that

$$\begin{aligned} & |\{C_1 r < |x - x_0| < C_2 r : |b(x) - b_{B(x_0, r)}| \rangle^v + C_3\}| \\ & \leq C_4 |B(x_0, r)| e^{-C_5 v} \quad \text{with } 0 < v < \infty. \end{aligned} \quad (2.1)$$

(b) Write

$$\Phi(b, B_r) := \inf_{c \in \mathbb{R}} \frac{1}{|B_r|} \int_{B_r} |b(y) - c| dy$$

and assume that  $b \in CBMO(\mathbb{R}^n)$ , then  $b \in CVMO(\mathbb{R}^n)$  if and only if  $b$  satisfies the following two conditions:

$$\limsup_{r \rightarrow 0} \frac{\Phi(b, B_r)}{r} = 0, \quad (2.2a)$$

$$\limsup_{r \rightarrow \infty} \frac{\Phi(b, B_r)}{r} = 0. \quad (2.2b)$$

(c)  $\|b\|_{CBMO(\mathbb{R}^n)} \simeq \sup_r \Phi(b, B_r)$ .

Some estimates for  $\Omega$  will be concluded in the next lemma, part of which can be deduced from [10, Lemma 2.1] directly.

**Lemma 2.2.** Let  $\Omega$  satisfy (1.1a) and (1.2). Then

(a)  $|\Omega(x - y) - \Omega(x)| \leq \frac{C}{(\log(|x|/|y|))^\gamma}$  with  $|x| \geq 4|y|$  and  $\gamma$  be given in (1.2).

(b) if furthermore  $\Omega$  satisfies the  $L^{q \geq 1}$ -Dini condition, then there is a constant  $C > 0$  such that for  $0 < C < 1/2, r > 0, x \in \mathbb{R}^n$  with  $|x| < Cr$ , one has

$$\begin{cases} \left( \int_{r < |y| < 2r} |\Omega(y - x) - \Omega(y)|^q dy \right)^{1/q} \leq Cr^{\frac{n}{q}} \int_{|x|/2r}^{|x|/r} \frac{w_q(\delta)}{\delta} d\delta, \\ \left( \int_{r < |y| < 2r} \frac{|\Omega(y - x) - \Omega(y)|^q}{|y|^{(n-\alpha)q}} dy \right)^{1/q} \leq Cr^{-\frac{n}{q} + \alpha} \int_{|x|/2r}^{|x|/r} \frac{w_q(\delta)}{\delta} d\delta. \end{cases}$$

*Proof.* We only need to show the second part of (b) since (a) and the first part of (b) is just [10, Lemma 2.1]. This can be done by the fact that  $\Omega$  satisfies the  $L^q$ -Dini condition.

In fact,

$$\begin{aligned} & \left( \int_{r < |y| < 2r} \frac{|\Omega(y-x) - \Omega(y)|^q}{|y|^{(n-\alpha)q}} dy \right)^{1/q} \\ &= Cr^{-\frac{n}{q} + \alpha} \left( \int_r^{2r} \int_{S^{n-1}} |\Omega(y' - t^{-1}x') - \Omega(y')|^q d\delta(y') \frac{dt}{t} \right)^{1/q} \\ &\leq Cr^{-\frac{n}{q} + \alpha} \int_{|x|/2r}^{|x|/r} \frac{w_q(\delta)}{\delta} d\delta. \end{aligned}$$

Thus, we complete the proof.  $\square$

The following known estimates from [17] and [20] will help us to complete the proof of Theorem 1.1.

**Lemma 2.3.** (a) Let  $g(x)$  be a measurable function,

$$\lambda(\mu) = |\{x \in \mathbb{R}^n : |g(x)| > \mu > 0\}|$$

and  $S$  be a measurable set. Define

$$g^*(t) = \inf\{\mu : \lambda(\mu) \leq t\} \quad \text{for } t > 0,$$

then

$$\int_S |g(x)|^p dx \leq \int_0^{|S|} |g^*(t)|^p dt \quad \text{with } 1 \leq p < \infty.$$

(b) Let  $0 < \alpha < n$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$  and  $\Omega$  satisfy (1.1a) and (1.1c). Then both  $H_{\Omega, \alpha}$  and  $H_{\Omega, \alpha}^*$  are bounded operators from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ .

In the end of this section, we give the boundedness for the truncated operators of  $H_{\Omega, \alpha}$  and  $H_{\Omega, \alpha}^*$ , which can be seen as a fractional case of [10, Lemma 2.5].

**Lemma 2.4.** Suppose that  $0 < \alpha < n$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$  and set

$$\begin{cases} H_{\Omega, \alpha}^\eta f(x) = \frac{1}{|x|^{n-\alpha}} \int_{S_1} \Omega(x-z) f(z) dz & \text{with } S_1 = \{z : |z| < |x|, |x-z| > \eta\}, \\ H_{\Omega, \alpha}^{*, \eta} f(x) = \int_{S_2} \frac{\Omega(x-z) f(z)}{|z|^{n-\alpha}} dy & \text{with } S_2 = \{z : |z| \geq |x|, |x-z| > \eta\}. \end{cases}$$

If  $\Omega$  satisfies (1.1a) and the  $L^q$ -Dini condition, then  $H_{\Omega, \alpha}^\eta$  and  $H_{\Omega, \alpha}^{*, \eta}$  are bounded operators from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ .

*Proof.* We only give the outline of the proof since the similarity, more details see [10, Lemma 2.5]. It is sufficient to show that for  $f \in L^p(\mathbb{R}^n)$ , there are constants  $C > 0$  satisfying

$$\|H_{\Omega, \alpha}^\eta f\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \quad \text{and} \quad \|H_{\Omega, \alpha}^{*, \eta} f\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$

Let us first prove the boundedness of  $H_{\Omega,\alpha}^\eta$  after the decomposition that

$$|H_{\Omega,\alpha}^\eta f(x)| = |H_{\Omega,\alpha} f(y) - H_{\Omega,\alpha} f_1(y) - H_{\Omega,\alpha} f_2(y) + H_{\Omega,\alpha}^\eta f(x)|,$$

where  $f_1 = f\chi_{4B}$ ,  $f_2 = f - f_1$  and  $B = B(x, \eta/4)$ . Therefore,

$$\begin{aligned} |H_{\Omega,\alpha}^\eta f(x)| &\leq \frac{1}{|B|} \int_B |H_{\Omega,\alpha} f(y)| dy + \frac{1}{|B|} \int_B |H_{\Omega,\alpha} f_1(y)| dy \\ &\quad + \frac{1}{|B|} \int_B |H_{\Omega,\alpha} f_2(y) - H_{\Omega,\alpha}^\eta f(x)| dy \\ &\leq M(H_{\Omega,\alpha} f)(x) + If(x) + IIf(x). \end{aligned}$$

Combining the  $L^p$ -boundedness of the maximal operator  $M$ , the  $(L^p, L^q)$ -boundedness of the fractional maximal operator  $M_\alpha$  and the  $(L^p, L^q)$ -boundedness of  $H_{\Omega,\alpha}$  [20], we get

$$\begin{aligned} \|M(H_{\Omega,\alpha} f)\|_{L^q(\mathbb{R}^n)} &\leq \|H_{\Omega,\alpha} f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \\ \|If\|_{L^q(\mathbb{R}^n)} &\leq C\|M_\alpha(|f|^p)^{\frac{1}{p}}\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \\ \|IIf\|_{L^q(\mathbb{R}^n)} &\leq C\|f\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

as desired.

The task is now to show the boundedness of  $H_{\Omega,\alpha}^{*\eta}$ . Analysis similar to that in the proof of  $H_{\Omega,\alpha}^\eta$  shows that

$$\begin{aligned} |H_{\Omega,\alpha}^{*\eta} f(x)| &\leq M(H_{\Omega,\alpha}^* f)(x) + J(f)(x) + JJ(f)(x), \\ \|M(H_{\Omega,\alpha}^* f)\|_{L^q(\mathbb{R}^n)} &\leq \|H_{\Omega,\alpha}^* f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \\ \|Jf\|_{L^q(\mathbb{R}^n)} &\leq C\|M_\alpha(|f|^p)^{\frac{1}{p}}\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Set  $\tilde{S}_2 = \{z : |z| \geq y, |x - z| > \eta\}$ , we obtain that

$$|JJf(x)| \leq \frac{C}{|B_{\frac{\eta}{4}}|} \int_{B_{\frac{\eta}{4}}} \left| \int_{\tilde{S}_2} \frac{|\Omega(x - y - z) - \Omega(x - z)| |f(z)|}{|z|^{n-\alpha}} dz \right| dy.$$

Accordingly, we conclude from the Minkowski inequality, Lemma 2.2 and the fact  $|x - z| \geq 3|z|/4$  for  $|x| = 2^{k_0-1}\eta$ ,  $|z| > \eta$  and  $|y| < \eta/4$  that

$$\|JJf\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)},$$

whence reaching the required estimation.  $\square$

### 3 Proof of Theorem 1.1

We begin with the proof of the necessity of Theorem 1.1 which is partly inspired by [10, Theorem 4.1]. If  $[b, H_{\Omega, \alpha}]$  and  $[b, H_{\Omega, \alpha}^*]$  are both compact operators from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ , then [20, Theorem 1.1] implies that  $b \in CBMO(\mathbb{R}^n)$ . For simplicity, we assume that  $\|b\|_{CBMO(\mathbb{R}^n)} = 1$ . According to Lemma 2.1, we only need to prove that (2.2a)-(2.2b) holds for  $b$ . This consists of two steps. We follow the notation used in [10, Theorem 4.1]. Step 1-proving that  $b$  satisfies (2.2a). If not, then there exists a  $\tau > 0$  and a sequence of balls  $\{B_i\}_{i=1}^\infty$  with  $\lim_{i \rightarrow \infty} r_i = 0$ , such that for any  $i$ ,  $\Phi(b, B_i) > \tau$ . Upon writing

$$f_i(y) = \frac{1}{|B_i|^{\frac{1}{p}}} [\text{sgn}(b(y) - b_{B_i}) - a_0] \chi_{B_i}(y), \quad i = 1, 2, \dots,$$

$$\text{with } a_0 = \frac{1}{|B_i|} \int_{B_i} \text{sgn}(b(y) - b_{B_i}) dy,$$

we find

$$\begin{cases} \text{supp } f_i \subset B_i, & f_i(y)(b(y) - b_{B_i}) > 0, \\ |f_i(y)| \leq 2|B_i|^{-\frac{1}{p}} & \text{with } y \in B_i, \\ \|f_i\|_{L^p(\mathbb{R}^n)} \leq C, & \int_{\mathbb{R}^n} f_i(y) dy = 0. \end{cases} \quad (3.1)$$

The argument is completed by showing that  $\{[b, H_{\Omega, \alpha}]f_i\}_{i=1}^\infty$  is not a compact set from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ . From now on,  $C_k$ , ( $k \in \mathbb{Z}$ ) stands for a positive constant depending only on  $\Omega$ ,  $p$ ,  $\alpha$ ,  $\tau$  with  $C_i$ , ( $1 \leq i < k$ ). We continue to choose

$$\begin{cases} D = \left\{ x' \in \mathbb{S}^{n-1} : \Omega(x') \geq \frac{2A}{(\log(2/C_1))^\gamma} \right\} & \text{with } A, \gamma \text{ be the same as that of in (1.2),} \\ E = \{x \in \mathbb{R}^n : |x| > C_2 r, x' \in D\} & \text{with } C_2 = 3C_1^{-1} + 1 > 4. \end{cases}$$

Using (1.1b) and (1.2), we obtain that there exists a  $0 < C_1 < 1$  such that

$$\sigma(D) > 0, \quad |x| > C_2|y| \quad \text{for } y \in B_i, \quad x \in E.$$

In view of the fact that

$$\Omega(x') \geq \frac{2A}{(\log(2/C_1))^\gamma}$$

and (1.2), we are interested in finding that for  $x' \in D$  and  $y' \in \mathbb{S}^{n-1}$  with  $|x' - y'| \leq C_1$ ,

$$\Omega(y') = \Omega(x') - (\Omega(x') - \Omega(y')) \geq |\Omega(x')| - |\Omega(x') - \Omega(y')| \geq \frac{A}{(\log(2/C_1))^\gamma}.$$

This in turn implies that

$$\Omega((x - y)') \geq \frac{A}{(\log(2/C_1))^\gamma}.$$

And hence, we get from (3.1) that for  $x \in E$ ,

$$H_{\Omega,\alpha}((b - b_{B_i})f_i)(x) \geq \frac{C}{|B_i|^{\frac{1}{p}}|x|^{n-\alpha}} \int_{B_i} (|b(y) - b_{B_i}| - a_0(b(y) - b_{B_i})) dy.$$

Consequently,

$$H_{\Omega,\alpha}((b - b_{B_i})f_i)(x) \geq \frac{C|B_i|^{1/p'}}{|x|^{n-\alpha}} \Phi(b, B_i) \geq \frac{C\tau|B_i|^{1/p'}}{|x|^{n-\alpha}}.$$

On the other hand, (3.1) and Hölder's inequality allow us to obtain

$$\begin{aligned} & |H_{\Omega,\alpha}((b - b_{B_i})f_i)(x)| \\ & \leq \frac{1}{|x|^{n-\alpha}} \int_{B_i} |\Omega((x - y)')(b(y) - b_{B_i})f_i(y)| dy \\ & \leq \frac{C|B_i|^{1/p'}}{|x|^{n-\alpha}} \left( \frac{1}{|B_i|} \int_{B_i} |b(y) - b_{B_i}|^{p'} dy \right)^{1/p'} \left( \int_{B_i} |f_i(y)|^p dy \right)^{1/p}, \end{aligned}$$

namely,

$$|H_{\Omega,\alpha}((b - b_{B_i})f_i)(x)| \leq \frac{C|B_i|^{1/p'}}{|x|^{n-\alpha}}. \quad (3.2)$$

At the same time, Lemma 2.2(a) and (3.1) shows

$$\begin{aligned} & |(b(x) - b_{B_i})H_{\Omega,\alpha}(f_i)(x)| \\ & \leq \frac{C|b(x) - b_{B_i}|}{|x|^{n-\alpha}} \int_{B_i} \frac{|f_i(y)|}{(\log(|x|/r_i))^\gamma} dy \leq \frac{C|b(x) - b_{B_i}||B_i|^{1/p'}}{|x|^{n-\alpha}(\log(|x|/r_i))^\gamma}. \end{aligned}$$

This in turn implies that for  $a > C_2$ ,

$$\left( \int_{\{|x|>ar_i\}} |(b(x) - b_{B_i})H_{\Omega,\alpha}(f_i)(x)|^q dx \right)^{1/q} \leq C(\log a)^{1-\gamma} a^{-\frac{n}{p'}},$$

where we used the fact that for  $O = \{x : 2^m r_i < |x| < 2^{m+1} r_i\}$ ,

$$\int_O |b(x) - b_{B_i}|^q dx \leq \int_O |b(x) - b_{2^m B_i}|^q dx + \int_O |b_{2^m B_i} - b_{B_i}|^q dx \leq Cm^q |2^m B_i|.$$

Upon setting  $W = \{x : ar_i < |x| < br_i\}$ , we find according to the above analysis that for  $b > a > C_2$

$$\begin{aligned} & \left( \int_W |[b, H_{\Omega,\alpha}]f_i(x)|^q dx \right)^{1/q} \\ & \geq C\tau|B_i|^{\frac{1}{p'}} \left( \int_{W \cap \{x:x' \in D\}} \frac{dx}{|x|^{(n-\alpha)q}} \right)^{1/q} - C(\log a)^{1-\gamma} a^{-\frac{n}{p'}} \\ & \geq C\tau \left( a^{-\frac{nq}{p'}} - b^{-\frac{nq}{p'}} \right)^{1/q} - C(\log a)^{1-\gamma} a^{-\frac{n}{p'}}. \end{aligned}$$



At the same time, (3.2) shows that

$$\begin{aligned} & \left( \int_{\{|x|>br_i\}} |[b, H_{\Omega,\alpha}]f_i(x)|^q dx \right)^{1/q} \\ & \leq \left( \int_{\{|x|>br_i\}} \frac{|B_i|^{\frac{q}{p'}}}{|x|^{(n-\alpha)q}} dx \right)^{1/q} + C (\log b)^{1-\gamma} b^{-\frac{n}{p'}} \\ & \leq Cb^{-\frac{n}{p'}} + C (\log b)^{1-\gamma} b^{-\frac{n}{p'}}. \end{aligned}$$

Accordingly, there are constants  $C_3 > C_2, C_5$  and  $C := C(\Omega, p, n, \alpha, \tau) > 1$  with  $C_4 = CC_3$  such that

$$\left( \int_{\{C_3r_i < |x| < C_4r_i\}} |[b, H_{\Omega,\alpha}]f_i(x)|^q dx \right)^{1/q} \geq C_5, \tag{3.3a}$$

$$\left( \int_{\{|x|>C_4r_i\}} |[b, H_{\Omega,\alpha}]f_i(x)|^q dx \right)^{1/q} \leq \frac{C_5}{4}. \tag{3.3b}$$

Set  $S \subset \{x : C_3r_i < |x| < C_4r_i\}$  be an arbitrary measurable set. An application of the Minkowski inequality shows that

$$\left( \int_S |[b, H_{\Omega,\alpha}]f_i(x)|^q dx \right)^{1/q} \leq C \left( \frac{|S|}{|B_i|} \right)^{1/q} + C \left( \frac{1}{|B_i|} \int_S |b(x) - b_{B_i}|^q dx \right)^{1/q}. \tag{3.4}$$

Setting

$$g_i(x) = b(x) - b_{B_i} \quad \text{and} \quad \lambda_{g_i}(t) = |\{C_5r_i < |x| < C_6r_i : |g_i(x)| > t\}|, \quad 0 < t < \infty,$$

we obtain from Lemma 2.1 that there are constants  $C_6, C_7$  and  $C_8$  such that

$$\lambda_{g_i}(t + C_6) \leq C_7|B_i|e^{-C_8t} \Rightarrow \lambda_{g_i}(t) \leq C_7|B_i|e^{-C_8(t-C_6)}.$$

Upon choosing  $g_i^*(\mu) = \inf\{t : \lambda_{g_i}(t) \leq \mu\}$ , it is easy to check that for  $0 < \mu < C_7|B_i|$ ,

$$g_i^*(\mu) \leq \frac{1}{C_8} \ln \frac{C_7|B_i|}{\mu} + C_6.$$

Using Lemma 2.3, we get that for  $|S| \ll C_7|B_i|$ ,

$$\begin{aligned} \frac{1}{|B_i|} \int_S |b(x) - b_{B_i}|^q dx & \leq \frac{1}{|B_i|} \int_0^{|S|} |g_i^*(\mu)|^q d\mu \\ & \leq \frac{C|S|}{|B_i|} |1 + \ln(C_7|B_i|/|S|)|^{[q]+1}. \end{aligned} \tag{3.5}$$

Eqs. (3.4) and (3.5) imply that there is a  $C_9 < \min\{C_7^{\frac{1}{n}}, C_4\}$  such that for  $|S|/|B_i| < C_9^n$ ,

$$\begin{aligned} & \left( \int_S |[b, H_{\Omega, \alpha}]f_i(x)|^q dx \right)^{1/q} \\ & \leq C \left( \frac{|S|}{|B_i|} \right)^{1/q} + C \left( \frac{|S|}{|B_i|} \left( 1 + \ln \frac{C_7|B_i|}{|S|} \right)^{[q]+1} \right)^{1/q} \leq \frac{C_5}{4}. \end{aligned}$$

Picking a subsequence  $\{B_{i(m)}\}_m$  from  $\{B_i\}$  with  $r_{i(m+1)}/r_{i(m)} < C_9/C_4$ , we concluded that for  $k > 0$ ,

$$\begin{aligned} & \| [b, H_{\Omega, \alpha}]f_{i(m)} - [b, H_{\Omega, \alpha}]f_{i(m+k)} \|_{L^q(\mathbb{R}^n)} \\ & \geq \left( \int_{G_1} |[b, H_{\Omega, \alpha}]f_{i(m)}(x)|^q dx \right)^{1/q} - \left( \int_{G_2} |[b, H_{\Omega, \alpha}]f_{i(m+k)}(x)|^q dx \right)^{1/q}, \end{aligned}$$

where

$$\begin{cases} G_1 = \{x : C_5r_{i(m)} < |x| < C_6r_{i(m)}\} \setminus \{x : |x| \leq C_6r_{i(m+k)}\} = G - (G_2^c \cap G), \\ G_2 = \{x : |x| > C_6r_{i(m+k)}\}, \\ G = \{x : C_5r_{i(m)} < |x| < C_6r_{i(m)}\}. \end{cases}$$

From (3.3) and what already been proved, we conclude that

$$\| [b, H_{\Omega, \alpha}]f_{i(m)} - [b, H_{\Omega, \alpha}]f_{i(m+k)} \|_{L^q(\mathbb{R}^n)} \geq \left( C_5^p - \left( \frac{C_5}{4} \right)^q \right)^{1/q} - \frac{C_5}{4} \geq \frac{C_5}{4},$$

which clearly shows that  $\{[b, H_{\Omega, \alpha}]f_{i(m)}\}_{m=1}^\infty$  does not have any convergence subsequence in  $L^q(\mathbb{R}^n)$ . This in turn implies that  $[b, H_{\Omega, \alpha}]$  is not a compact operator from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ . Therefore,  $b$  satisfies (2.2a) by the contradiction.

Step 2-showing that  $b$  satisfies (2.2b). This step can be handled in much the same way as the argument for (2.2a), the only difference being in choosing a sequence  $\{B_i\}_i$  such that

$$\Phi(b, B_i) > \tau \quad \text{with} \quad \lim_{i \rightarrow \infty} r_i = +\infty.$$

We proceed to show the sufficiency of Theorem 1.1, which can be deduced by the following more general form.

**Theorem 3.1.** *Suppose that*

$$\begin{cases} 0 < \alpha < n, & \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \\ \Omega \text{ satisfies (1.1a) and (1.3),} \\ b \in \text{CVMO}(\mathbb{R}^n), \end{cases}$$

*then both  $[b, H_{\Omega, \alpha}]$  and  $[b, H_{\Omega, \alpha}^*]$  are compact operators from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ .*

To prove Theorem 3.1, the following two lemmas are needed. We first recall the well known Frechet-Kolmogorov theorem as

**Lemma 3.1.** *Let a set  $S \subset L^p(\mathbb{R}^n)$  and  $G_\alpha = \{x \in \mathbb{R}^n : |x| > \beta\}$ . Then  $S$  is strongly pre-compact, if and only if,*

$$\sup_{f \in S} \|f\|_{L^p(\mathbb{R}^n)} < \infty, \quad (3.6a)$$

$$\lim_{|y| \rightarrow 0} \|f(\cdot + y) - f(\cdot)\|_{L^p(\mathbb{R}^n)} = 0 \quad \text{uniformly in } f \in S, \quad (3.6b)$$

$$\lim_{\beta \rightarrow \infty} \|f\chi_{G_\beta}\|_{L^p(\mathbb{R}^n)} = 0 \quad \text{uniformly in } f \in S. \quad (3.6c)$$

Next, we give the second lemma which can simplify the proof of Theorem 3.1 by considering  $b \in C_c^\infty(\mathbb{R}^n)$  ([10, Lemma 4.4]).

**Lemma 3.2.** *Assume that  $[b, T]$  is a compact operator from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$  for  $b \in C_c^\infty(\mathbb{R}^n)$ , then  $[b, T]$  is also a compact operator from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$  for  $b \in \text{CVMO}(\mathbb{R}^n)$ .*

We are now in a position to complete the proof of Theorem 3.1. For  $b \in C_c^\infty(\mathbb{R}^n)$ , we are about to show (3.6a)-(3.6c) for

$$S_1 = \{[b, H_{\Omega, \alpha}]f : f \in Q\} \quad \text{and} \quad S_2 = \{[b, H_{\Omega, \alpha}^*]f : f \in Q\},$$

with  $Q = \{f : f \in L^p(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L^p(\mathbb{R}^n)} \leq C\}$ .

The fact  $b \in C_c^\infty(\mathbb{R}^n)$  allows us to have

$$\sup_{f \in Q} \|[b, H_{\Omega, \alpha}]f\|_{L^q(\mathbb{R}^n)} \leq C \|b\|_{\text{CBMO}(\mathbb{R}^n)} \sup_{f \in Q} \|f\|_{L^p(\mathbb{R}^n)} < \infty$$

and to obtain (3.6a).

Next, to show (3.6b), we only need to prove that for any  $\varepsilon > 0$  and  $|z|$  small enough,

$$\|[b, H_{\Omega, \alpha}]f(\cdot + z) - [b, H_{\Omega, \alpha}]f(\cdot)\|_{L^q(\mathbb{R}^n)} \leq C\varepsilon, \quad \forall f \in Q. \quad (3.7)$$

For  $0 < \varepsilon < 1/2$ , setting

$$\begin{cases} E_1 = \{y : |y| < |x + z|, |x - y| > e^{\frac{1}{\varepsilon}}|z|\}, & E_2 = \{y : |y| < |x + z|, |x - y| \leq e^{\frac{1}{\varepsilon}}|z|\}, \\ E_3 = \{y : |y| < |x|, |x - y| > e^{\frac{1}{\varepsilon}}|z|\}, & E_4 = \{y : |y| < |x|, |x - y| \leq e^{\frac{1}{\varepsilon}}|z|\}, \end{cases}$$

we achieve that for  $z \in \mathbb{R}^n$ ,

$$|[b, H_{\Omega, \alpha}]f(x + z) - [b, H_{\Omega, \alpha}]f(x)| = K_1^b f + K_2^b f + K_3^b f - K_4^b f,$$

where

$$\left\{ \begin{aligned} K_1^b f &= \frac{1}{|x|^{n-\alpha}} \int_{E_3} [\Omega(x-y)(b(x+z) - b(x))] f(y) dy, \\ K_2^b f &= \frac{1}{|x|^{n-\alpha}} \int_{E_3} [\Omega(x-y)(b(y) - b(x+z))] f(y) dy \\ &\quad - \frac{1}{|x+z|^{n-\alpha}} \int_{E_1} [\Omega(x+z-y)(b(y) - b(x+z))] f(y) dy, \\ K_3^b f &= \frac{1}{|x|^{n-\alpha}} \int_{E_4} [\Omega(x-y)(b(y) - b(x))] f(y) dy, \\ K_4^b f &= \frac{1}{|x+z|^{n-\alpha}} \int_{E_2} [\Omega(x+z-y)(b(y) - b(x+z))] f(y) dy. \end{aligned} \right.$$

Combining  $b \in C_c^\infty(\mathbb{R}^n)$ ,  $|b(x+z) - b(x)| \leq C|z|$  and Lemma 2.4, we obtain that for  $f \in Q$ ,

$$\|K_1^b f\|_{L^q(\mathbb{R}^n)} \leq C|z| \left\| H_{\Omega, \alpha}^{e^{\frac{1}{\varepsilon}}|z|} f \right\|_{L^q(\mathbb{R}^n)} \leq C|z| \|f\|_{L^p(\mathbb{R}^n)} \leq C|z|.$$

By Lemma 2.2 and Minkoski’s inequality, one has

$$\begin{aligned} \|K_2^b f\|_{L^q(\mathbb{R}^n)} &\leq \left( \int_{x:|x-y|>e^{\frac{1}{\varepsilon}}|z|} \left| \frac{1}{|x|^{n-\alpha}} \int_{\widetilde{U}_1} [\Omega(y) - \Omega(y+z)] f(x-y) dy \right|^q dx \right)^{1/q} \\ &\leq C \sum_{k=0}^\infty \frac{1}{1+k+\frac{1}{\varepsilon}} \int_{\frac{1}{2^{k+1}e^{\frac{1}{\varepsilon}}}}^{\frac{1}{2^k e^{\frac{1}{\varepsilon}}}} \frac{w_q(\delta)}{\delta} (1+|\log \delta|) d\delta \\ &\leq C\varepsilon, \end{aligned}$$

where

$$\widetilde{E}_1 := \left\{ y : 2^{k+1}e^{\frac{1}{\varepsilon}}|z| < |y| < 2^k e^{\frac{1}{\varepsilon}}|z| \right\}.$$

After the observation  $|b(x) - b(y)| \leq C|x - y|$  for  $|x - y| < 1$ , we can estimate  $K_3^b$  as

$$|K_3^b f| \leq \frac{C}{|x|^{n-\alpha}} \int_{E_4} |\Omega(x-y)f(y)|x-y||dy \leq \frac{C}{|x|^{n-\alpha-1}} \int_{E_4} |\Omega(x-y)f(y)|dy.$$

Hence, a further application of the Minkoski inequality and the Hölder inequality show that

$$\begin{aligned} \|K_3^b f\|_{L^q(\mathbb{R}^n)} &\leq C \left( \int_{x:|x-y|\leq e^{\frac{1}{\varepsilon}}|z|} \left| \frac{1}{|x|^{n-\alpha-1}} \int_{E_4} |\Omega(x-y)f(y)|dy \right|^q dx \right)^{1/q} \\ &\leq C(e^{\frac{1}{\varepsilon}}|z|)^q. \end{aligned}$$

Since

$$|x - y + z| \leq (e^{\frac{1}{\varepsilon}} + 1)|z| < 1, \quad |b(x+z) - b(y)| \leq C|x+z-y|,$$

we can estimate  $K_4^b f$  as follows,

$$\begin{aligned} \|K_4^b f\|_{L^q(\mathbb{R}^n)} &\leq C \left( \int_{\mathbb{R}^n} \left| \int_{E_2} \frac{|\Omega(x+z-y)f(y)|}{|x+z-y|^{n-\alpha-1}} dy \right|^q dx \right)^{1/q} \\ &\leq C \|f\|_{L^p(\mathbb{R}^n)} \left( \int_{E_2} dy \right)^{\frac{1}{p'}} \left( \int_{\{x:|x-y| < e^{\frac{1}{\varepsilon}}|z|\}} \frac{|\Omega(x+z-y)|^q}{|x+z-y|^{(n-\alpha-1)q}} dx \right)^{\frac{1}{q}} \\ &\leq C \left( (e^{\frac{1}{\varepsilon}} + 1)|z| \right)^q. \end{aligned}$$

So, (3.7) is obtained thanking to

$$\lim_{|z| \rightarrow 0} \|[b, H_{\Omega, \alpha}]f(x) - [b, H_{\Omega, \alpha}]f(x+z)\|_{L^q(\mathbb{R}^n)} = 0 \quad \text{uniformly in } f \in Q.$$

Next, we finish the consideration of  $S_1$  by showing (3.6c). To do so, we first choose  $\beta$  large enough such that

$$\left( \int_{\beta}^{\infty} \frac{1}{t^{(n-\alpha)q-n+1}} dt \right)^{\frac{1}{s}} < \varepsilon, \quad \forall \varepsilon > 0, \quad s > 1,$$

and denote by  $U := \text{supp}(b) \subset \{x : |x| < r\}$  for some  $r > 0$ . Then for  $|x| > \max\{\beta, 4r\}$  and  $f \in Q$ , apply the Hölder inequality to  $\frac{1}{s} + \frac{1}{p} + \frac{1}{q} = 1$ , one has

$$\begin{aligned} |[b, H_{\Omega, \alpha}]f(x)| &\leq \frac{C}{|x|^{n-\alpha}} \int_U |b(y)\Omega(x-y)f(y)| dy \\ &\leq \frac{C\|f\|_{L^p(\mathbb{R}^n)}}{|x|^{n-\alpha}} \left( \int_U |\Omega(x-y)|^q dy \right)^{\frac{1}{q}}. \end{aligned}$$

Thereby reaching (3.6c) by the Minkoskin inequality and the fact  $|x-y| > 3|x|/4$  as

$$\begin{aligned} \left( \int_{|x|>\beta} |[b, H_{\Omega, \alpha}]f(x)|^q dx \right)^{1/q} &\leq C \left( \int_{|x|>\beta} \left| \frac{1}{|x|^{n-\alpha}} \int_U |\Omega(x-y)|^q dy \right| dx \right)^{1/q} \\ &\leq C \left( \int_{\beta}^{\infty} \frac{dt}{t^{(n-\alpha)q-n+1}} \int_{S^{n-1}} |\Omega(y')|^q d\sigma(y') \right)^{\frac{1}{q}} \\ &\leq C\varepsilon. \end{aligned}$$

Similar arguments apply to  $S_2$ , we have

$$\sup_{f \in Q} \|[b, H_{\Omega, \alpha}^*]f\|_{L^q(\mathbb{R}^n)} \leq C \sup_{f \in Q} \|f\|_{L^p(\mathbb{R}^n)} \leq C < \infty,$$

whence finding (3.6a). Since  $\{y : |y| \geq |x|\} \cap \{y : |y| < R\} = \emptyset$  if  $U := \text{supp}(b) \subset \{y : |y| < r\}$  for some  $r > 0$  and  $x$  satisfying  $|x| > \max\{\beta, 4r\}$  in this case, (3.6c) is obviously.

It is sufficient to prove (3.6b) for  $S_2$ . We are about to show that for any  $\varepsilon > 0$ ,  $f \in Q$  and  $|z|$  small enough,

$$\| [b, H_{\Omega, \alpha}^*]f(\cdot + z) - [b, H_{\Omega, \alpha}^*]f(\cdot) \|_{L^q(\mathbb{R}^n)} \leq C\varepsilon.$$

The rest of the proof runs as that of  $S_1$  with a slight modification. We omit here for the similarity. We complete the proof of Theorem 3.1.  $\square$

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