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Commutators of Bilinear Hardy Operators on Two Weighted Herz Spaces with Variable Exponents

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. In this paper, we obtain the boundedness of bilinear commutators generated by the bilinear Hardy operator and BMO functions on products of two weighted Herz spaces.

Key Words: Hardy operator, commutator, Muckenhoupt, BMO, variable exponent, Herz space.

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1 Introduction

Denote by $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the set of all complex-valued locally integrable functions on $\mathbb{R}^n$. The n dimensional Hardy operator was introduced by Faris in [28] as

$$Hf(x) := \frac{1}{\Omega_n |x|^n} \int_{|y|<|x|} f(y)dy, \quad x \in \mathbb{R}^n \backslash \{0\} \quad \text{for each} \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

where and what follows Ω_n is the volume of the unit ball in $\mathbb{R}^n$. When $n = 1$, the Hardy operator was firstly considered in [10]. In [18], Christ and Grafakos showed that the Hardy operator is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$. For the boundedness of H on other functions, we refer the reader to the survey [25] by Shanzhen Lu.

For $m \in \mathbb{N}$, suppose $f_1, \cdots, f_m$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the m-linear Hardy operator is defined by

$$H(f_1,\cdots,f_m) := \frac{1}{\Omega_{mn}|x|^{mn}} \int_{|(y_1,\cdots,y_m)|<|x|} \prod_{i=1}^{m} f_i(y_i) dy_1 \cdots dy_m, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

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If $b \in L^1_{\rm loc}(\mathbb{R}^n)$, set

$$\|b\|_* := \sup_B \frac{1}{|B|}\int_B |b(x)-b_B|dx,$$

where the supremum is taken over all balls in $\mathbb{R}^n$, b_B is the mean of b on B, and what follows $|E|$ is the Lebesgue measure of measurable set E in $\mathbb{R}^n$. A function b is called bounded mean oscillation if $\|b\|_* < \infty$. Denote by BMO($\mathbb{R}^n$) the set of all bounded mean oscillation functions on $\mathbb{R}^n$.

The commutator of m-linear Hardy operator is defined by

$$H_{\vec{b}}(f_1,\cdots,f_m)(x) := \sum_{i=1}^m H^i_{b_i}(f_1,\cdots,f_m)(x),$$

where

$$H^i_{b_i}(f_1,\cdots,f_m)(x) := b_i(x)H(f_1,\cdots,f_m)(x) - H(f_1,\cdots,f_{i-1},f_ib_i,f_{i+1},\cdots,f_m)(x).$$

When $m=1$, the operator $H_b(f)(x) = b(x)Hf(x) - H(bf)(x)$. Although, our results hold for each $m \geq 2$, for brevity, we only consider $m=2$.

Fu, Liu and Lu [31] studied the boundedness of the commutators of weighted Hardy operators (with symbols in BMO($\mathbb{R}^n$)) on $L^p(\mathbb{R}^n)$, where $1<p<\infty$. Fu, Gong and Lu, ect. [34] proved the boundedness of the commutators of weighted multilinear Hardy operators (with symbols in central BMO space) on the product of central Morrey spaces $\dot{B}^{p,\lambda}$. Shi and Lu [27] introduced some characterizations of $\dot{C}^{p,\lambda}$ for $-1/p<\lambda<0$, via the boundedness of commutator operators of Hardy type. Zhao and Lu [7] precisely evaluate the operator norm of the fractional Hardy operator $\mathbb{H}^\beta$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $0<\beta<n$, $1<p<q<\infty$ and $1/p-1/q=\beta/n$. Zhao, Fu and Lu [8] gived some new properties of M_p weight functions on $\mathbb{R}^n$ and used them to characterize the boundedness of bilinear Hardy inequalities on the weighted Lebesgue spaces. Yu and Lu [29] studied the H^1-boundedness of the generalized commutators of Hardy operator with a homogeneous kernel $\mathcal{H}^m_{\Omega,A,\beta}$. Lu, Yan and Zhao [26] explicitly worked out the bounds of the operator $\mathbb{H}$ fromL^p to L^q and from L^1 to $L^{\frac{n}{n-\beta},\infty}$. Fu, Wu and Lu proved the boundedness of commutators generated by the p-adic Hardy operators (Hardy-Littlewood-Pólya operators) and the central BMO functions on $L^q(|x|^\alpha_p dx)$ and Herz spaces $K^{\alpha,q}_r$. Zhao, Fu and Lu [9] proved that the higher dimensional Hardy operator is bounded from Hardy spaces to Lebesgue spaces, and discussed the endpoint estimate for the commutators generated by the Hardy operator and (central) BMO functions. Fu, Grafakos and Lu, etc. [32] obtained norms of m-linear Hardy operators and m-linear Hilbert operators on Lebesgue spaces with power weights.

In [19], Izuki proved the boundedness of commutators generated by singular integrals and BMO functions on Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$. Shu, Wang and Meng [17] obtained the boundedness of commutators of Hardy type operators and BMO functions on Herz spaces $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$ with variable exponents $\alpha(\cdot)$ and $p(\cdot)$. Izuki and Noi [24] proved the boundedness of fractional integral operators on weighted Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(w)$ with variable exponent. Almeida and Drihem [1] obtained the boundedness of sublinear operator on variable Herz spaces $K^{\alpha(\cdot),q}_{p(\cdot)}(w)$.

In 2020, Izuki and Noi introduced two weighted Herz spaces with variable exponents in [23]. Motivated by the mentioned works, we will consider the boundedness of commutators generated by bilinear Hardy operator and BMO functions on two weighted Herz spaces with variable exponents. The plan of the paper is as follows. In Section 2, we collect some notations and state the main result. The proof of the main result will be given in Section 3.

2 Notations and main result

In this section, we firstly recall some definitions and notations, then we state our result. Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ and take values in $[1,\infty)$. The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable: } \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The space $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\}.$$

The space $L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n) := \{ f : f_{\chi_K} \in L^{p(\cdot)}(\mathbb{R}^n) \ \text{ for all compact subsets } K \subset \mathbb{R}^n \},$$

where and what follows, χ_S denotes the characteristic function of a measurable set $S \subset \mathbb{R}^n$. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, we denote

$$p_- := \operatorname{ess} \inf_{x\in\mathbb{R}^n} p(x), \quad p_+ := \operatorname{ess} \sup_{x\in\mathbb{R}^n} p(x).$$

The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$; $\mathcal{P}^0(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 0$ and $p_+ < \infty$. $L^{p(\cdot)}$ can be similarly defined as above for $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$. $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$, that means $1/p(\cdot) + 1/p'(\cdot) = 1$.

Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$. Then the standard Hardy-Littlewood maximal function of f is defined by

$$Mf(x) := \sup_{B\ni x} \frac{1}{|B|} \int_B |f(y)| dy, \quad \forall x \in \mathbb{R}^n,$$

where B is a ball. In general, the Hardy-littlewood maximal operator M is not bounded on variable Lebesgue spaces. But if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies the following global log-Hölder continuous, then M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, see [6]. For more details see [2,14–16].

Definition 2.1. *Let $\alpha(\cdot)$ be a real-valued measurable function on $\mathbb{R}^n$.*

(i) The function $\alpha(\cdot)$ is locally log*-Hölder continuous if there exists a constant C_1 such that*

$$|\alpha(x) - \alpha(y)| \leq \frac{C_1}{\log(e + 1/|x-y|)}, \quad x, y \in \mathbb{R}^n, \quad |x-y| < \frac{1}{2}.$$

(ii) The function $\alpha(\cdot)$ is log*-Hölder continuous at the origin if*

$$|\alpha(x) - \alpha(0)| \leq \frac{C_2}{\log(e + 1/|x|)}, \quad \forall x \in \mathbb{R}^n.$$

Denote by $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ the set of all log*-Hölder continuous functions at the origin.*

(iii) The function $\alpha(\cdot)$ is log-Hölder continuous at infinity if there exists a constant C_3 such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_3}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n.$$

Denote by $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ the set of all log*-Hölder continuous functions at infinity.*

(iv) The function $\alpha(\cdot)$ is global log*-Hölder continuous if $\alpha(\cdot)$ are both locally* log*-Hölder continuous and* log*-Hölder continuous at infinity. Denote by $\mathcal{P}^{\log}(\mathbb{R}^n)$ the set of all global* log*-Hölder continuous functions.*

Definition 2.2. *Fix $p \in (1, \infty)$. A positive measurable function w is said to be in the Muckenhoupt class A_p, if there exists a positive constant C for all balls B in $\mathbb{R}^n$ such that*

$$\left(\frac{1}{B}\int_B w(x)dx\right)\left(\frac{1}{B}\int_B w(x)^{1-p'}dx\right)^{p-1} \leq C.$$

We say $w \in A_1$, if $Mw(x) \leq Cw(x)$ for a.e. x. If $1 \leq p < q < \infty$, then $A_p \subset A_q$. We denote $A_\infty = \cup_{p>1} A_p$. The Muckenhoupt A_p class with constant exponent $p \in (1, \infty)$ was firstly proposed by Muckenhoupt in [5].

Definition 2.3. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, a positive measurable function w is said to be in $A_{p(\cdot)}$, if there exists a positive constant C for all balls B in $\mathbb{R}^n$ such that*

$$\frac{1}{|B|}\|w^{1/p(\cdot)}\chi_B\|_{L^{p(\cdot)}}\|w^{-1/p(\cdot)}\chi_B\|_{L^{p'(\cdot)}} \leq C.$$

Definition 2.4. *Let $p \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight. The weight variable Lebesgue space $L^{p(\cdot)}(w)$ is defined by*

$$L^{p(\cdot)}(w) := \big\{ f \text{ is measurable on } \mathbb{R}^n \text{ and } \|f\|_{L^{p(\cdot)}(w)} := \|fw^{1/p(\cdot)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty \big\}.$$

Then $L^{p(\cdot)}(w)$ is a Banach space.

To give the definitions of the weighted Herz space with variable exponents, we use the following notations. For each $k \in \mathbb{Z}$ we define $B_k := \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $D_k := B_k \backslash B_{k-1} = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$, $\chi_k := \chi_{D_k}$, $\widetilde{\chi}_m = \chi_m$, $m \geq 1$, $\widetilde{\chi}_0 = \chi_{B_0}$. We also need the notation of the variable mixed sequence space $\ell^{q(\cdot)}(L^{p(\cdot)}(w))$. Let w be a positive measurable function. Given a sequence of functions $\{f_j\}_{j\in\mathbb{Z}}$, define the modular

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}((f_j)_j) := \sum_{j\in\mathbb{Z}} \inf \left\{ \lambda_j : \int_{\mathbb{R}^n} \left(\frac{|f_j(x)|}{\lambda_j^{\frac{1}{q(x)}}} \right)^{p(x)} w(x)dx \leq 1 \right\},$$

where $\lambda^{1/\infty} = 1$. If $q^+ < \infty$ or $q(\cdot) \leq p(\cdot)$, the above can be written as

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}((f_j)_j) = \sum_{j\in\mathbb{Z}} \big\| |f_j w^{1/p(\cdot)}|^{q(\cdot)} \big\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

The norm is

$$\|(f_j)_j\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} := \inf\{\mu > 0 : \ \rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}((f_j/\mu)_j) \leq 1\}.$$

Definition 2.5. *Let w_1, w_2 be weights on $\mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous two weighted variable Herz space $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w_1,w_2)$ and non-homogeneous two weighted variable Herz space $K^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w_1,w_2)$ are defined, respectively, by*

$$\begin{aligned}
\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w_1,w_2) &:= \Big\{ f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n \setminus \{0\}, w_2) : \|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w_1,w_2)} < \infty \Big\}, \\
K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w_1,w_2) &:= \Big\{ f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n, w_2) : \|f\|_{K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w_1,w_2)} < \infty \Big\},
\end{aligned}$$

where

$$\begin{aligned}
\|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w_1,w_2)} &:= \|\{w_1(B_k)^{\alpha(\cdot)/n} f\chi_k\}_{k\in\mathbb{Z}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w_2))}, \\
\|f\|_{K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w_1,w_2)} &:= \|\{w_1(B_k)^{\alpha(\cdot)/n} f\widetilde{\chi}_k\}_{k\geq 0}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w_2))}.
\end{aligned}$$

For any quantities A and B, if there exists a constant $C > 0$ such that $A \leq CB$, we write $A \lesssim B$. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$.

Lemma 2.1 ([23, Theorem 3]). *Let $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$, $p(\cdot),q(\cdot)\in\mathcal{P}_0(\mathbb{R}^n)$, $w_1\in A_r$ for some $r\in[1,\infty)$ and w_2 be a weight. If $\alpha(\cdot)$ and $q(\cdot)$ are* log*-Hölder continuous at infinity, then*

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1,w_2)=K_{p(\cdot)}^{\alpha_\infty,q_\infty}(w_1,w_2).$$

Additionally, if $\alpha(\cdot)$ and $q(\cdot)$ are log*-Hölder continuous at the origin, then*

$$\|f\|_{\dot K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1,w_2)}\approx\Big(\sum_{k\le 0}\big\|w_1(B_k)^{\alpha(0)/n}f\chi_k\big\|_{L^{p(\cdot)}(w_2)}^{q(0)}\Big)^{\frac{1}{q(0)}}+\Big(\sum_{k>0}\big\|w_1(B_k)^{\alpha_\infty/n}f\chi_k\big\|_{L^{p(\cdot)}(w_2)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}.$$

The following Lemma 2.2 comes from the result of [3,11,12].

Lemma 2.2. *If $q\in[1,\infty)$ and $w\in A_q$, then there exist constants $\delta\in(0,1)$ and $C>0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S\subset B$,*

$$\frac{w(B)}{w(S)}\le C\Big(\frac{B}{S}\Big)^q, \tag{2.1a}$$

$$\frac{w(S)}{w(B)}\le C\Big(\frac{S}{B}\Big)^\delta. \tag{2.1b}$$

Lemma 2.3 ([23, (21) and (22)]). *If $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n)$ and $w\in A_{p(\cdot)}$, then there exist constants δ_1, $\delta_2\in(0,1)$ and $C>0$ such that for all $k,l\in\mathbb{Z}$ with $k\le l$,*

$$\frac{\|\chi_k\|_{L^{p(\cdot)}(w^{p(\cdot)})}}{\|\chi_l\|_{L^{p(\cdot)}(w^{p(\cdot)})}}\le C\Big(\frac{|D_k|}{|D_l|}\Big)^{\delta_1}, \tag{2.2a}$$

$$\frac{\|\chi_k\|_{(L^{p(\cdot)}(w^{p(\cdot)}))'}}{\|\chi_l\|_{(L^{p(\cdot)}(w^{p(\cdot)}))'}}\le C\Big(\frac{|D_k|}{|D_l|}\Big)^{\delta_2}. \tag{2.2b}$$

Remark 2.1. If $w\in A_{p(\cdot)}$, then $w^{-p'(\cdot)/p(\cdot)}\in A_{p'(\cdot)}$. Therefore, the Hardy-littlewood maximal operator M is bounded on $L^{p'(\cdot)}(w^{-p'(\cdot)/p(\cdot)})$.

Our main result is as follows.

Theorem 2.1. *Suppose that H is the bilinear Hardy operator, $p_1(\cdot)$, $p_2(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n)$. Let $v\in A_r$ for some $r\in[1,\infty)$, $w_i\in A_{p_i(\cdot)}$ and $w=\prod_{i=1}^2 w_i^{p(\cdot)/p_i(\cdot)}$. Assume that $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, $\alpha(\cdot)=\alpha_1(\cdot)+\alpha_2(\cdot)$, $q(\cdot)\in\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, $1/q(0)=1/q_1(0)+1/q_2(0)$, $1/q_\infty=1/q_{1\infty}+1/q_{2\infty}$, δ_{i1}, $\delta_{i2}\in(0,1)$ are the constants in Lemma* 2.3 *for exponents $p_i(\cdot)$ and weights w_i, $i=1,2$. If $w^+\alpha^+<n\delta_{i2}$ for $i=1$, 2, $1/p(\cdot)=1/p_1(\cdot)+1/p_2(\cdot)$, then $H_{\vec b}^1$ is bounded from $\dot K_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(v,w_1)\times\dot K_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(v,w_2)$ into $\dot K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(v,w)$, where $\vec b=(b_1,b_2)$, $b_1,b_2\in\mathrm{BMO}(\mathbb{R}^n)$.*

3 Proof of Theorem 2.1

To prove Theorem 2.1, we need a series of Lemmas.

Lemma 3.1 ([23, Lemma 7]). *Let $k,l\in\mathbb{Z}$, $w\in A_q$ with $q\in[1,\infty)$, $\delta\in(0,1)$ be the constants in Lemma 2.2,*

$$w^-=\begin{cases}\delta, & \text{if } \alpha^-\geq 0,\\ q, & \text{if } \alpha^-<0,\end{cases}\quad w^+=\begin{cases}q, & \text{if } \alpha^+\geq 0,\\ \delta, & \text{if } \alpha^+<0.\end{cases}$$

If $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$ is log-*Hölder continuous both at the origin and infinity, then for any $x\in D_k$ and $y\in D_l$,*

$$w(B_k)^{\alpha(x)}\leq Cw(B_l)^{\alpha(y)}\times\begin{cases}2^{(k-l)nw^+\alpha^+}, & \text{if } l\leq k-1,\\ 1, & \text{if } k-1<l\leq k+1,\\ 2^{(k-l)nw^-\alpha^-}, & \text{if } l>k+1.\end{cases}$$

Lemma 3.2 ([23, Lemma 8]). *Let $w\in A_q$ with $q\in[1,\infty)$. If $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$ is* log-*Hölder continuous both at the origin and infinity. then for all $k\in\mathbb{Z}$ and $x\in D_k$,*

$$\begin{aligned} &w(D_k)^{\alpha(x)}\approx w(D_k)^{\alpha_\infty}, && \text{if } k\geq 0,\\ &w(D_k)^{\alpha(x)}\approx w(D_k)^{\alpha(0)}, && \text{if } k\leq -1.\end{aligned}$$

Lemma 3.3 below have been proved by Izuki in [21,22]

Lemma 3.3. *Let X be a Banach function space on $\mathbb{R}^n$. If the Hardy-littlewood maximal operator M is weakly bounded on X, that is,*

$$\|\chi_{\{Mf>\lambda\}}\|_X\lesssim\lambda^{-1}\|f\|_X$$

holds for all $f\in X$ and $\lambda>0$. Then we have

$$\sup_{\text{ball } B}\frac{1}{|B|}\|\chi_B\|_X\|\chi_B\|_{X'}<\infty,$$

where X' denotes the associated space of X.

Ho [13] has initially proved the following characterization of the BMO norm via Banach function spaces. Izuki [20] gave another simple proof.

Lemma 3.4. *Let X be a Banach function space on $\mathbb{R}^n$. If M is bounded on the associate sapce X', then for all $b\in$ BMO$(\mathbb{R}^n)$,*

$$\|b\|_*\approx\sup_{\text{ball } B\subset\mathbb{R}^n}\frac{1}{\|\chi_B\|_X}\|(b-b_B)\chi_B\|_X.$$

Lemma 3.5 (The generalized Hölder inequality). *Let X be a Banach function space on $\mathbb{R}^n$. If $f\in X$ and $g\in X'$, then we have*

$$\int_{\mathbb{R}^n}|f(x)g(x)|dx\le\|f\|_X\|g\|_{X'},$$

where X' denotes the associated space of X.

Lemma 3.6 ([4, Theorem 2.3]). *Let $p(\cdot),p_1(\cdot),p_2(\cdot)\in\mathcal{P}_0(\mathbb{R}^n)$ such that $1/p(x)=1/p_1(x)+1/p_2(x)$ for $x\in\mathbb{R}^n$. Then there exists a constant C_{p,p_1} independent of functions f and g such that*

$$\|fg\|_{L^{p(\cdot)}}\le C_{p,p_1}\|f\|_{L^{p_1(\cdot)}}\|g\|_{L^{p_2(\cdot)}}$$

holds for every $f\in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p_2(\cdot)}(\mathbb{R}^n)$.

If $p(\cdot)$, $p_1(\cdot)$, $p_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that $1/p(x)=1/p_1(x)+1/p_2(x)$ for $x\in\mathbb{R}^n$, and $w\in A_{p(\cdot)}$ with $w_i\in A_{p_i(\cdot)}$, $w=\prod_{i=1}^2 w_i^{p(\cdot)/p_i(\cdot)}$, then

$$\|fg\|_{L^{p(\cdot)}(w)}\le C_{p,p_1}\|f\|_{L^{p_1(\cdot)}(w_1)}\|g\|_{L^{p_2(\cdot)}(w_2)}.$$

Lemma 3.7 ([30, Proposition 1.2]). *Let $0<p\le\infty$, $\delta>0$. Then there is a positive constant C such that*

$$\left(\sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}2^{-|k-j|\delta}a_k\right)^p\right)^{1/p}\le C\left(\sum_{j=-\infty}^{\infty}a_j^p\right)^{1/p} \tag{3.1}$$

for non-negative sequences $\{a_j\}_{j=-\infty}^{\infty}$. Here, when $p=\infty$, it is understood that (3.1) *stands for*

$$\sup_{j\in\mathbb{Z}}\left(\sum_{k=-\infty}^{\infty}2^{-|k-j|\delta}a_k\right)\le C\sup_{j\in\mathbb{Z}}a_j.$$

Now we give the proof Theorem 2.1.

Proof of Theorem 2.1. Let $f_1\in\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(v,w_1)$, $f_2\in M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(v,w_2)$. Let $(b_i)_{B_k}$ express the average of b_i on the ball B_k for $i=1,2$, and $k\in\mathbb{Z}$. If $x\in D_k$, then

$$\begin{aligned}|H^1_{b_1}(f_1,f_2)(x)|\lesssim&2^{-2kn}\int_{|(y_1,y_2)|<|x|}|f_1(y_1)f_2(y_2)||b_1(x)-b_1(y_1)|dy_1dy_2\\ \lesssim&2^{-2kn}\int_{|(y_1,y_2)|<|x|}|f_1(y_1)f_2(y_2)||b_1(x)-(b_1)_{B_k}|dy_1dy_2\\ &+2^{-2kn}\int_{|(y_1,y_2)|<|x|}|f_1(y_1)f_2(y_2)||b_1(y_1)-(b_1)_{B_k}|dy_1dy_2\end{aligned}$$

$$
\begin{aligned}
&\lesssim 2^{-2kn}|b_1(x)-(b_1)_{B_k}|\sum_{i=-\infty}^{k}\int_{D_i}|f_1(y_1)|dy_1\sum_{j=-\infty}^{k}\int_{D_j}|f_2(y_2)|dy_2\\
&\quad+2^{-2kn}\sum_{i=-\infty}^{k}\int_{D_i}|f_1(y_1)||b_1(y_1)-(b_1)_{B_i}|dy_1\sum_{j=-\infty}^{k}\int_{D_j}|f_2(y_2)|dy_2\\
&\quad+2^{-2kn}\sum_{i=-\infty}^{k}\int_{D_i}|f_1(y_1)||(b_1)_{B_k}-(b_1)_{B_i}|dy_1\sum_{j=-\infty}^{k}\int_{D_j}|f_2(y_2)|dy_2\\
&=:I_1+I_2+I_3.
\end{aligned}
$$

If $k>i$, $k>j$, then by Lemma 3.1, we have

$$
\begin{aligned}
&v(B_k)^{\alpha_1(x)/n}\int_{D_i}|f_1(y_1)|dy_1\lesssim 2^{(k-i)w^+\alpha^+}\int_{D_i}v(B_i)^{\alpha_1(y_1)/n}|f_1(y_1)|dy_1, &\text{(3.2a)}\\
&v(B_k)^{\alpha_2(x)/n}\int_{D_j}|f_2(y_2)|dy_2\lesssim 2^{(k-j)w^+\alpha^+}\int_{D_j}v(B_j)^{\alpha_2(y_2)/n}|f_2(y_2)|dy_2, &\text{(3.2b)}\\
&v(B_k)^{\alpha_1(x)/n}\int_{D_i}|f_1(y_1)||G_1|dy_1\lesssim 2^{(k-i)w^+\alpha^+}\int_{D_i}v(B_i)^{\alpha_1(y_1)/n}|f_1(y_1)||G_1|dy_1, &\text{(3.2c)}\\
&v(B_k)^{\alpha_1(x)/n}\int_{D_i}|f_1(y_1)||G_2|dy_1\lesssim 2^{(k-i)w^+\alpha^+}\int_{D_i}v(B_i)^{\alpha(y_1)/n}|f_1(y_1)||G_2|dy_1, &\text{(3.2d)}
\end{aligned}
$$

where $G_1=b_1(y_1)-(b_1)_{B_i}$ and $G_2=(b_1)_{B_k}-(b_1)_{B_i}$.

Since

$$
w=\prod_{i=1}^{2}w_i^{p(\cdot)/p_i(\cdot)}\quad\text{and}\quad 1/p(\cdot)=1/p_1(\cdot)+1/p_2(\cdot),
$$

by Hölder's inequality, Lemmas 3.3 and 2.3, we have

$$
\begin{aligned}
&2^{-2kn}\|\chi_i\|_{L^{p_1'(\cdot)}\left(w_1^{-p_1'(\cdot)/p_1(\cdot)}\right)}\|\chi_j\|_{L^{p_2'(\cdot)}\left(w_2^{-p_2'(\cdot)/p_2(\cdot)}\right)}\|\chi_k\|_{L^{p(\cdot)}(w)}\\
\lesssim&2^{-2kn}\|\chi_i\|_{L^{p_1'(\cdot)}\left(w_1^{-p_1'(\cdot)/p_1(\cdot)}\right)}\|\chi_j\|_{L^{p_2'(\cdot)}\left(w_2^{-p_2'(\cdot)/p_2(\cdot)}\right)}\|\chi_k\|_{L^{p_1(\cdot)}(w_1)}\|\chi_k\|_{L^{p_2(\cdot)}(w_2)}\\
\lesssim&\frac{\|\chi_i\|_{L^{p_1'(\cdot)}\left(w_1^{-p_1'(\cdot)/p_1(\cdot)}\right)}}{\|\chi_k\|_{L^{p_1'(\cdot)}\left(w_1^{-p_1'(\cdot)/p_1(\cdot)}\right)}}\times\frac{\|\chi_j\|_{L^{p_2'(\cdot)}\left(w_2^{-p_2'(\cdot)/p_2(\cdot)}\right)}}{\|\chi_k\|_{L^{p_2'(\cdot)}\left(w_2^{-p_2'(\cdot)/p_2(\cdot)}\right)}}\\
\lesssim&2^{-(k-i)n\delta_{12}}\times 2^{-(k-j)n\delta_{22}}. \qquad\qquad (3.3)
\end{aligned}
$$

Since

$$
w=\prod_{i=1}^{2}w_i^{p(\cdot)/p_i(\cdot)}\quad\text{and}\quad \alpha(\cdot)=\alpha_1(\cdot)+\alpha_2(\cdot),
$$

by (3.2a)-(3.2d) and Hölder's inequality, Lemma 3.4 and (3.3), we have

$$
\begin{aligned}
&\|v(B_k)^{\alpha(\cdot)/n} I_1\chi_k\|_{L^{p(\cdot)}(w)}\\
\lesssim&2^{-2kn}\|(b_1-(b_1)_{B_k})\cdot\chi_k\|_{L^{p(\cdot)}(w)}\\
&\quad\times\sum_{i=-\infty}^{k}2^{(k-i)w^+\alpha^+}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}\|\chi_i\|_{L^{p_1'(\cdot)}(w_1^{-p_1'(\cdot)/p_1(\cdot)})}\\
&\quad\times\sum_{j=-\infty}^{k}2^{(k-j)w^+\alpha^+}\|v(B_j)^{\alpha_2(\cdot)/n}f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}\|\chi_j\|_{L^{p_2'(\cdot)}(w_2^{-p_2'(\cdot)/p_2(\cdot)})}\\
\lesssim&2^{-2kn}\|b_1\|_*\sum_{i=-\infty}^{k}2^{(k-i)w^+\alpha^+}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}\|\chi_i\|_{L^{p_1'(\cdot)}(w_1^{-p_1'(\cdot)/p_1(\cdot)})}\\
&\quad\times\sum_{j=-\infty}^{k}2^{(k-j)w^+\alpha^+}\|v(B_j)^{\alpha_2(\cdot)/n}f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}\|\chi_j\|_{L^{p_2'(\cdot)}(w_2^{-p_2'(\cdot)/p_2(\cdot)})}\|\chi_k\|_{L^{p(\cdot)}(w^{p(\cdot)})}\\
\lesssim&\|b_1\|_*\sum_{i=-\infty}^{k}2^{(k-i)(w^+\alpha^+-n\delta_{12})}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}\\
&\quad\times\sum_{j=-\infty}^{k}2^{(k-j)(w^+\alpha^+-n\delta_{22})}\|v(B_j)^{\alpha_2(\cdot)/n}f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned}
$$

Since

$$
w=\prod_{i=1}^{2}w_i^{p(\cdot)/p_i(\cdot)}\quad\text{and}\quad\alpha(\cdot)=\alpha_1(\cdot)+\alpha_2(\cdot),
$$

by (3.2a), (3.2b), (3.2c), (3.2d) and Hölder's inequality, Lemma 3.4 and (3.3), we have

$$
\begin{aligned}
&\|v(B_k)^{\alpha(\cdot)/n} I_2\chi_k\|_{L^{p(\cdot)}(w)}\\
\lesssim&2^{-2kn}\sum_{i=-\infty}^{k}2^{(k-i)w^+\alpha^+}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}\|(b_1-(b_1)_{B_i})\chi_i\|_{L^{p_1'(\cdot)}\left(w_1^{-p_1'(\cdot)/p_1(\cdot)}\right)}\\
&\quad\times\sum_{j=-\infty}^{k}2^{(k-j)w^+\alpha^+}\|v(B_j)^{\alpha_2(\cdot)/n}f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}\|\chi_j\|_{L^{p_2'(\cdot)}\left(w_2^{-p_2'(\cdot)/p_2(\cdot)}\right)}\|\chi_k\|_{L^{p(\cdot)}(w)}\\
\lesssim&2^{-2kn}\|b_1\|_*\sum_{i=-\infty}^{k}2^{(k-i)w^+\alpha^+}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}\|\chi_i\|_{L^{p_1'(\cdot)}\left(w_1^{-p_1'(\cdot)/p_1(\cdot)}\right)}\\
&\quad\times\sum_{j=-\infty}^{k}2^{(k-j)w^+\alpha^+}\|v(B_j)^{\alpha_2(\cdot)/n}f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}\|\chi_j\|_{L^{p_2'(\cdot)}\left(w_2^{-p_2'(\cdot)/p_2(\cdot)}\right)}\|\chi_k\|_{L^{p(\cdot)}(w)}\\
\lesssim&\|b_1\|_*\sum_{i=-\infty}^{k}2^{(k-i)(w^+\alpha^+-n\delta_{12})}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}
\end{aligned}
$$

$$\times \sum_{j=-\infty}^{k} 2^{(k-j)(w^+\alpha^+-n\delta_{22})} \|v(B_j)^{\alpha_2(\cdot)/n} f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}.$$

Since

$$w = \prod_{i=1}^{2} w_i^{p(\cdot)/p_i(\cdot)} \quad \text{and} \quad \alpha(\cdot) = \alpha_1(\cdot) + \alpha_2(\cdot),$$

by (3.2a)-(3.2d) and Hölder's inequality, Lemma 3.4 and (3.3), we have

$$\begin{aligned}
&\|v(B_k)^{\alpha(\cdot)/n} I_3\chi_k\|_{L^{p(\cdot)}(w)} \\
\lesssim &2^{-2kn} \sum_{i=-\infty}^{k} |(b_1)_{B_k} - (b_1)_{B_i}| 2^{(k-i)w^+\alpha^+} \\
&\times \|v(B_i)^{\alpha_1(\cdot)/n} f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)} \|\chi_i\|_{L^{p_1'(\cdot)}\left(w_1^{-p_1'(\cdot)/p_1(\cdot)}\right)} \\
&\times \sum_{j=-\infty}^{k} 2^{(k-j)w^+\alpha^+} \|v(B_j)^{\alpha_2(\cdot)/n} f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)} \|\chi_j\|_{L^{p_2'(\cdot)}\left(w_2^{-p_2'(\cdot)/p_2(\cdot)}\right)} \|\chi_k\|_{L^{p(\cdot)}(w)} \\
\lesssim &2^{-2kn} \|b_1\|_* \sum_{i=-\infty}^{k} (k-i) 2^{(k-i)w^+\alpha^+} \\
&\times \|v(B_i)^{\alpha_1(\cdot)/n} f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)} \|\chi_i\|_{L^{p_1'(\cdot)}\left(w_1^{-p_1'(\cdot)/p_1(\cdot)}\right)} \\
&\times \sum_{j=-\infty}^{k} 2^{(k-j)w^+\alpha^+} \|v(B_j)^{\alpha_2(\cdot)/n} f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)} \|\chi_j\|_{L^{p_2'(\cdot)}\left(w_2^{-p_2'(\cdot)/p_2(\cdot)}\right)} \|\chi_k\|_{L^{p(\cdot)}(w)} \\
\lesssim &\|b_1\|_* \sum_{i=-\infty}^{k} (k-i) 2^{(k-i)(w^+\alpha^+-n\delta_{12})} \|v(B_i)^{\alpha_1(\cdot)/n} f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)} \\
&\times \sum_{j=-\infty}^{k} 2^{(k-j)(w^+\alpha^+-n\delta_{22})} \|v(B_j)^{\alpha_2(\cdot)/n} f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\|v(B_k)^{\alpha(\cdot)/n} H_{b_1}^1(f_1, f_2)\chi_k\|_{L^{p(\cdot)}(w)} \\
\lesssim &\|b_1\|_* \sum_{i=-\infty}^{k} (k-i) 2^{(k-i)(w^+\alpha^+-n\delta_{12})} \|v(B_i)^{\alpha_1(\cdot)/n} f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)} \\
&\times \sum_{j=-\infty}^{k} 2^{(k-j)(w^+\alpha^+-n\delta_{22})} \|v(B_j)^{\alpha_2(\cdot)/n} f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned} \tag{3.4}$$

By Lemmas 2.1 and 3.2, we have

$$\begin{aligned}\|H^1_{b_1}(f_1,f_2)\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(v,w)} \approx &\bigg(\sum_{k=-\infty}^{-1}\big\|v(B_k)^{\alpha(\cdot)/n}H^1_{b_1}(f_1,f_2)\chi_k\big\|^{q(0)}_{L^{p(\cdot)}(w)}\bigg)^{\frac{1}{q(0)}}\\ &+\bigg(\sum_{k=0}^{\infty}\big\|v(B_k)^{\alpha(\cdot)/n}H^1_{b_1}(f_1,f_2)\chi_k\big\|^{q_\infty}_{L^{p(\cdot)}(w)}\bigg)^{\frac{1}{q_\infty}}\\ =:&\,E+F.\end{aligned}$$

Estimate E. Since $1/q(0)=1/q_1(0)+1/q_2(0)$, $\alpha(\cdot)=\alpha_1(\cdot)+\alpha_2(\cdot)$, by (3.4) and Hölder's inequality, we have

$$\begin{aligned}E=&\bigg(\sum_{k=-\infty}^{-1}\big\|v(B_k)^{\alpha(\cdot)/n}H^1_{b_1}(f_1,f_2)\chi_k\big\|^{q(0)}_{L^{p(\cdot)}(w)}\bigg)^{\frac{1}{q(0)}}\\ \lesssim&\|b_1\|_*\bigg\{\sum_{k=-\infty}^{-1}\bigg(\sum_{i=-\infty}^{k}(k-i)2^{(k-i)(w^+\alpha^+-n\delta_{12})}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}\\ &\times\sum_{j=-\infty}^{k}2^{(k-j)(w^+\alpha^+-n\delta_{22})}\|v(B_j)^{\alpha_2(\cdot)/n}f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}\bigg)^{q(0)}\bigg\}^{\frac{1}{q(0)}}\\ \lesssim&\|b_1\|_*\bigg\{\sum_{k=-\infty}^{-1}\bigg(\sum_{i=-\infty}^{k}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}(k-i)2^{(k-i)(w^+\alpha^+-n\delta_{12})}\bigg)^{q_1(0)}\bigg\}^{\frac{1}{q_1(0)}}\\ &\times\bigg\{\sum_{k=-\infty}^{-1}\bigg(\sum_{j=-\infty}^{k}\|v(B_j)^{\alpha_2(\cdot)/n}f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)(w^+\alpha^+-n\delta_{22})}\bigg)^{q_2(0)}\bigg\}^{\frac{1}{q_2(0)}}\\ =:&\,\|b_1\|_*E_1\times E_2,\end{aligned}$$

where

$$\begin{aligned}E_1&:=\bigg\{\sum_{k=-\infty}^{-1}\bigg(\sum_{i=-\infty}^{k}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}(k-i)2^{(k-i)(w^+\alpha^+-n\delta_{12})}\bigg)^{q_1(0)}\bigg\}^{\frac{1}{q_1(0)}},\\ E_2&:=\bigg\{\sum_{k=-\infty}^{-1}\bigg(\sum_{j=-\infty}^{k}\|v(B_j)^{\alpha_2(\cdot)/n}f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)(w^+\alpha^+-n\delta_{22})}\bigg)^{q_2(0)}\bigg\}^{\frac{1}{q_2(0)}}.\end{aligned}$$

Estimate E_1. Since $w^+\alpha^+-n\delta_{12}<0$, we write

$$(k-i)2^{-|k-i|(n\delta_{12}-w^+\alpha^+)}\lesssim 2^{-|k-i|\varepsilon_1}$$

for some $\varepsilon_1\in(0,n\delta_{12}-w^+\alpha^+)$, in Lemma 3.7 replacing δ by ε_1, we obtain that

$$E_1\lesssim\bigg(\sum_{i=-\infty}^{-1}\|[v(B_i)]^{\alpha_1(\cdot)/n}f_1\chi_i\|^{q_1(0)}_{L^{p_1(\cdot)}(w_1)}\bigg)^{\frac{1}{q_1(0)}}\lesssim\|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(v,w_1)}.$$

Estimate E_2. Since $w^+\alpha^+ - n\delta_{22} < 0$, in Lemma 3.7 replacing δ by $n\delta_{22} - w^+\alpha^+$, we obtain that

$$E_2 \lesssim \left(\sum_{j=-\infty}^{-1} \|v(B_j)^{\alpha_2(\cdot)/n} f_2\chi_j\|^{q_2(0)}_{L^{p_2(\cdot)}(w_2)} \right)^{\frac{1}{q_2(0)}} \lesssim \|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(v,w_2)}.$$

Thus, we get

$$E \lesssim \|b_1\|_* \|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(v,w_1)} \|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(v,w_2)}.$$

Estimate F. Since $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, $\alpha(\cdot) = \alpha_1(\cdot) + \alpha_2(\cdot)$, by (3.4) and Hölder's inequality, we have

$$\begin{aligned}
F =& \left(\sum_{k=0}^{\infty} \|v(B_k)^{\alpha(\cdot)/n} H^1_{b_1}(f_1,f_2)\chi_k\|^{q_\infty}_{L^{p(\cdot)}(w)} \right)^{\frac{1}{q_\infty}} \\
\lesssim& \|b_1\|_* \Bigg\{ \sum_{k=0}^{\infty} \Bigg(\sum_{i=-\infty}^{k} \|v(B_i)^{\alpha_1(\cdot)/n} f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)} (k-i) 2^{(k-i)(w^+\alpha^+ - n\delta_{12})} \\
&\times \sum_{j=-\infty}^{k} \|v(B_j)^{\alpha_2(\cdot)/n} f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(w^+\alpha^+ - n\delta_{22})} \Bigg)^{q_\infty} \Bigg\}^{\frac{1}{q_\infty}} \\
\lesssim& \|b_1\|_* \Bigg\{ \sum_{k=0}^{\infty} \Bigg(\sum_{i=-\infty}^{k} \|v(B_i)^{\alpha_1(\cdot)/n} f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)} (k-i) 2^{(k-i)(w^+\alpha^+ - n\delta_{12})} \Bigg)^{q_{1\infty}} \Bigg\}^{\frac{1}{q_{1\infty}}} \\
&\times \Bigg\{ \sum_{k=0}^{\infty} \Bigg(\sum_{j=-\infty}^{k} \|v(B_j)^{\alpha_2(\cdot)/n} f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(w^+\alpha^+ - n\delta_{22})} \Bigg)^{q_{2\infty}} \Bigg\}^{\frac{1}{q_{2\infty}}} \\
=:& \|b_1\|_* F_1 \times F_2,
\end{aligned}$$

where

$$\begin{aligned}
F_1 :=& \Bigg\{ \sum_{k=0}^{\infty} \Bigg(\sum_{i=-\infty}^{k} \|v(B_i)^{\alpha_1(\cdot)/n} f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)} (k-i) 2^{(k-i)(w^+\alpha^+ - n\delta_{12})} \Bigg)^{q_{1\infty}} \Bigg\}^{\frac{1}{q_{1\infty}}}, \\
F_2 :=& \Bigg\{ \sum_{k=0}^{\infty} \Bigg(\sum_{j=-\infty}^{k} \|v(B_j)^{\alpha_2(\cdot)/n} f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(w^+\alpha^+ - n\delta_{22})} \Bigg)^{q_{2\infty}} \Bigg\}^{\frac{1}{q_{2\infty}}}.
\end{aligned}$$

Estimate F_1. we obtain that

$$\begin{aligned}
F_1 \lesssim& \Bigg\{ \sum_{k=0}^{\infty} \Bigg(\sum_{i=-\infty}^{-1} \|v(B_i)^{\alpha_1(\cdot)/n} f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)} (k-i) 2^{(k-i)(w^+\alpha^+ - n\delta_{12})} \\
&+ \sum_{i=0}^{k} (k-i) \|v(B_i)^{\alpha_1(\cdot)/n} f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)} 2^{(k-i)(w^+\alpha^+ - n\delta_{12})} \Bigg)^{q_{1\infty}} \Bigg\}^{\frac{1}{q_{1\infty}}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim\Big\{\sum_{k=0}^{\infty}\Big(\sum_{i=-\infty}^{-1}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}(k-i)2^{(k-i)(w^+\alpha^+-n\delta_{12})}\Big)^{q_{1\infty}}\Big\}^{\frac{1}{q_{1\infty}}}\\
&\quad+\Big\{\sum_{k=0}^{\infty}\Big(\sum_{i=0}^{k}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}(k-i)2^{(k-i)(w^+\alpha^+-n\delta_{12})}\Big)^{q_{1\infty}}\Big\}^{\frac{1}{q_{1\infty}}}\\
&=:F_{1,1}+F_{1,2}.
\end{aligned}
$$

Estimate $F_{1,1}$. Since $w^+\alpha^+-n\delta_{12}<0$, we obtain that

$$
\begin{aligned}
F_{1,1}=&\Big\{\sum_{k=0}^{\infty}\Big(\sum_{i=-\infty}^{-1}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}(k-i)2^{(k-i)(w^+\alpha^+-n\delta_{12})}\Big)^{q_{1\infty}}\Big\}^{\frac{1}{q_{1\infty}}}\\
\lesssim&\|f_1\|_{\dot K^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(v,w_1)}\Big\{\sum_{k=0}^{\infty}\Big(\sum_{i=-\infty}^{-1}(k-i)2^{(k-i)(w^+\alpha^+-n\delta_{12})}\Big)^{q_{1\infty}}\Big\}^{\frac{1}{q_{1\infty}}}\\
\lesssim&\|f_1\|_{\dot K^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(v,w_1)}\Big\{\sum_{k=0}^{\infty}k^{q_{1\infty}}2^{kq_{1\infty}(w^+\alpha^+-n\delta_{12})}\Big(\sum_{i=-\infty}^{-1}(-i)2^{(-i)(w^+\alpha^+-n\delta_{12})}\Big)^{q_{1\infty}}\Big\}^{\frac{1}{q_{1\infty}}}\\
\lesssim&\|f_1\|_{\dot K^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(v,w_1)}.
\end{aligned}
$$

Estimate $F_{1,2}$. Since $w^+\alpha^+-n\delta_{12}<0$, we write

$$
(k-i)2^{-|k-i|(n\delta_{12}-w^+\alpha^+)}\lesssim 2^{-|k-i|\eta_1}
$$

for some $\eta_1\in(0,n\delta_{12}-w^+\alpha^+)$, then in Lemma 3.7 replacing δ by η_1, we have

$$
\begin{aligned}
F_{1,2}=&\Big\{\sum_{k=0}^{\infty}\Big(\sum_{i=0}^{k}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|_{L^{p_1(\cdot)}(w_1)}(k-i)2^{(k-i)(w^+\alpha^+-n\delta_{12})}\Big)^{q_{1\infty}}\Big\}^{\frac{1}{q_{1\infty}}}\\
\lesssim&\Big(\sum_{i=0}^{\infty}\|v(B_i)^{\alpha_1(\cdot)/n}f_1\chi_i\|^{q_{1\infty}}_{L^{p_1(\cdot)}(w_1)}\Big)^{\frac{1}{q_{1\infty}}}\\
\lesssim&\|f_1\|_{M\dot K^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}.
\end{aligned}
$$

For F_2, we obtain that

$$
\begin{aligned}
F_2\lesssim&\Big\{\sum_{k=0}^{\infty}\Big(\sum_{j=-\infty}^{-1}\|v(B_j)^{\alpha_2(\cdot)/n}f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)(w^+\alpha^+-n\delta_{22})}\\
&\quad+\sum_{j=0}^{k}\|v(B_j)^{\alpha_2(\cdot)/n}f_2\chi_j\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)(w^+\alpha^+-n\delta_{22})}\Big)^{q_{2\infty}}\Big\}^{\frac{1}{q_{2\infty}}}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \left\{ \sum_{k=0}^{\infty} \left(\sum_{j=-\infty}^{-1} \|v(B_j)^{\alpha_2(\cdot)/n} f_2 \chi_j\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(w^+\alpha^+ - n\delta_{22})} \right)^{q_{2\infty}} \right\}^{\frac{1}{q_{2\infty}}} \\
&\quad + \left\{ \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \|v(B_j)^{\alpha_2(\cdot)/n} f_2 \chi_j\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(w^+\alpha^+ - n\delta_{22})} \right)^{q_{2\infty}} \right\}^{\frac{1}{q_{2\infty}}} \\
&=: F_{2,1} + F_{2,2}.
\end{aligned}
$$

Since $w^+\alpha^+ - n\delta_{22} < 0$, we have

$$
\begin{aligned}
F_{2,1} &= \left\{ \sum_{k=0}^{\infty} \left(\sum_{j=-\infty}^{-1} \|v(B_j)^{\alpha_2(\cdot)/n} f_2 \chi_j\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(w^+\alpha^+ - n\delta_{22})} \right)^{q_{2\infty}} \right\}^{\frac{1}{q_{2\infty}}} \\
&\lesssim \|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(v,w_2)} \left\{ \sum_{k=0}^{\infty} \left(\sum_{j=-\infty}^{-1} 2^{(k-j)(w^+\alpha^+ - n\delta_{22})} \right)^{q_{2\infty}} \right\}^{\frac{1}{q_{2\infty}}} \\
&\lesssim \|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(v,w_2)}.
\end{aligned}
$$

Since $w^+\alpha^+ - n\delta_{22} < 0$, where wrote

$$
2^{-|k-j|(n\delta_{22} - w^+\alpha^+)} \lesssim 2^{-|k-j|\eta_2}
$$

for some $\eta_2 \in (0, n\delta_{22} - w^+\alpha^+)$, then in Lemma 3.7 replacing δ by η_2, we have

$$
\begin{aligned}
F_{2,2} &= \left\{ \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \|[v(B_j)]^{\alpha_2(\cdot)/n} f_2 \chi_j\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(w^+\alpha^+ - n\delta_{22})} \right)^{q_{2\infty}} \right\}^{\frac{1}{q_{2\infty}}} \\
&\lesssim \left(\sum_{j=0}^{\infty} \|v(B_j)^{\alpha_2(\cdot)/n} f_2 \chi_i\|^{q_{2\infty}}_{L^{p_2(\cdot)}(w_2)} \right)^{\frac{1}{q_{2\infty}}} \\
&\lesssim \|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(v,w_2)}.
\end{aligned}
$$

Thus, we get

$$
H \lesssim \|b_1\|_* \|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(v,w_1)} \|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(v,w_2)}.
$$

Combining E and F, we obtain that

$$
\|H^1_{b_1}(f_1,f_2)\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(v,w)} \lesssim \|b_1\|_* \|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(v,w_1)} \|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(v,w_2)}.
$$

Similarly, we have

$$
\|H^1_{b_2}(f_1,f_2)\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(v,w)} \lesssim \|b_2\|_* \|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(v,w_1)} \|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(v,w_2)}.
$$

Therefore, the proof of Theorem 2.1 is completed. $\square$

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