

## Borderline Weighted Estimates for Commutators of Fractional Integrals

Zhidan Wang<sup>1</sup>, Huoxiong Wu<sup>2</sup> and Qingying Xue<sup>1,\*</sup>

<sup>1</sup> School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems Ministry of Education, Beijing 100875, China

<sup>2</sup> School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, China

Received 8 August 2020; Accepted (in revised version) 21 July 2021

Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

---

**Abstract.** Let  $I_{\alpha, \vec{b}}$  be the multilinear commutators of the fractional integrals  $I_{\alpha}$  with the symbol  $\vec{b} = (b_1, \dots, b_k)$ . We show that the constant of borderline weighted estimates for  $I_{\alpha}$  is  $\frac{1}{\varepsilon}$ , and for  $I_{\alpha, \vec{b}}$  is  $\frac{1}{\varepsilon^{k+1}}$  with each  $b_i$  belongs to the Orlicz space  $Osc_{\exp L^i}$ .

**Key Words:** Commutators, fractional integrals, borderline weighted estimates, Fefferman-Stein inequality.

**AMS Subject Classifications:** 42B25, 47G10

---

### 1 Introduction

Let  $M$  be the Hardy-Littlewood maximal function, which is defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes  $Q$  containing  $x$  in  $\mathbb{R}^n$  with the sides parallel to the coordinate axes. Since the 1930s, there have been many outstanding works in the study of the Hardy-Littlewood maximal function. Among such achievements are the celebrated works of Hardy, Littlewood and Wiener, Fefferman and Stein [9], and Muckenhoupt [13]. Recall that the Hardy-Littlewood-Wiener theorem states that  $M$  is bounded

---

\*Corresponding author. Email addresses: zdwang@mail.bnu.edu.cn (Z. Wang), huoxwu@xmu.edu.cn (H. Wu), qyxue@bnu.edu.cn (Q. Xue)

from  $L^p(\mathbb{R}^n)$  to  $L^p(\mathbb{R}^n)$  ( $1 < p \leq \infty$ ) and from  $L^1(\mathbb{R}^n)$  to  $L^{1,\infty}(\mathbb{R}^n)$ , and the Fefferman-Stein inequality [9] can be expressed in the way that

$$\|Mf\|_{L^{1,\infty}(\omega)} \leq C \int_{\mathbb{R}^n} |f| M\omega dx. \tag{1.1}$$

The question whether one can extend inequality (1.1) to other type of operators, such as the Hilbert transforms and the Calderón-Zygmund singular intergrals, is known as Muckenhoupt and Wheeden conjecture. In 2012, Reguera and Thiele [22] surprisingly showed that the Muckenhoupt and Wheeden conjecture was not true for the Hilbert transform, which fully indicates that the Hilbert transform does not enjoy the similar weak type inequality as in (1.1). In 1994, Pérez [17] obtained the following less fine inequality for the Calderón-Zygmund singular intergrals.

$$\|Tf\|_{L^{1,\infty}(\omega)} \leq C_{\epsilon,T} \int_{\mathbb{R}^n} |f| M_{L(\log L)^\epsilon} \omega dx, \quad \omega \geq 0, \quad \epsilon > 0. \tag{1.2}$$

Since then, efforts have been made to clarify and separate the constant  $C_{\epsilon,T}$ . It was Hytönen and Pérez [10] who first showed that the constant can be gained is  $\epsilon^{-1}$  for  $T$  and its corresponding maximal singular integral operators  $T^*$ . Recently, Domingo-Salazar, Lacey, Rey [8] generalized the results in [10] and further proved that  $T^*$  is bounded as a map from  $L^1(M_{L \log \log L (\log \log \log L)^\alpha} w)$  into weak- $L^1(w)$  for  $1 < \alpha < 2$  and the constant can be obtained is  $(\alpha - 1)^{-1}$ .

Now we turn to the background of the commutators of  $T$ , which can be traced back to the celebrated works of Coifman, Rochberg and Weiss [3]. For a suitable smooth function  $f$ , the commutator of  $T$  is defined as  $[b, T]f = T(bf) - bT(f)$ . In [3], the authors proved that if  $b$  belongs to  $BMO(\mathbb{R}^n)$ , then  $[b, T]$  is bounded from  $L^p(\mathbb{R}^n)$  onto itself ( $1 < p < \infty$ ). Conversely, if all commutators of Riesz transform  $[R_j, b]$ ,  $1 \leq j \leq n$ , are  $L^p$  bounded, then  $b \in BMO(\mathbb{R}^n)$ .

In 1995, Pérez [18] pointed out that the commutators of CZOs are not weak type (1, 1) operators. As a replacement, he gave the following  $L \log L$  endpoint estimate:

$$\omega(\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}) \leq C_{\|b\|_{BMO}} [w]_{A_1} \int_{\mathbb{R}^n} \Phi(|f|/\lambda) \omega dx, \tag{1.3}$$

where  $\Phi(t) = t(1 + \log^+ t)$ ,  $\omega \in A_1$ .

Quite naturally, one may ask whether the commutators  $[b, T]$  still enjoy the similar inequality as in (1.2) or not. In 2001, Pérez and Pradolín [19] established the following inequality for arbitrary non negative weights  $w$ .

$$\omega(\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}) \leq C_{T,\epsilon} \int_{\mathbb{R}^n} \Phi(|f| \|b\|_{BMO} / \lambda) M_{L(\log L)^{1+\epsilon}} \omega dx. \tag{1.4}$$

In 2017, Pérez et al [20] further figured out that the constant in (1.4) is  $\frac{C_T}{\epsilon^2}$ , that is

$$\omega(\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}) \leq \frac{C_T}{\epsilon^2} \int_{\mathbb{R}^n} \Phi(|f| \|b\|_{BMO} / \lambda) M_{L(\log L)^{1+\epsilon}} \omega dx. \tag{1.5}$$

In this paper, our object of investigation is the fractional integral and its commutators. It is well known that the study of weighted estimates of fractional integrals originated from the works of Muckenhoupt and Wheeden [14] in 1974. They proved that the fractional integral operator was of strong type  $(L^p(w^p), L^q(w^q))$  if  $p > 1$  and of weak type  $(L^1(w), L^{\frac{n}{n-\alpha}, \infty}(w^{\frac{n}{n-\alpha}}))$  if  $p = 1$ . In 2001, endpoint  $L \log L$  type estimates for the commutators of the fractional integral were studied by Ding, Lu and Zhang [7]. Later on, Carro [1] considered borderline weighted estimates for fractional integrals and he proved that

$$\omega(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_{\Psi, \alpha} u(x) dx, \quad (1.6)$$

where  $\Psi(t) = t \log(e + t)^{1+\epsilon}$ . Some other weighted results can be found in the works of Cruz-Uribe and Fiorenza [5], Chen and Xue [2].

This paper is concerned with the borderline weighted estimates of the fractional integral and its commutators, for such particular inequalities as (1.2) and (1.5). We show that the constants in the norm inequalities of  $I_\alpha$  and  $I_{\alpha, b}$  are still  $\frac{1}{\epsilon}$  and  $\frac{1}{\epsilon^2}$  (just take  $k = 1$  in Theorem 1.2), respectively.

The main results of this paper are:

**Theorem 1.1.** *Let  $0 < \epsilon < 1$  and  $\omega$  be a weight, then for any  $\alpha \in (0, \frac{n\epsilon}{2(2+\epsilon)})$ ,  $c = \frac{2n}{2n+\alpha(\epsilon+2)}$ , there exists a constant  $C_{I_\alpha}$  such that*

$$\|I_\alpha f\|_{L^{1, \infty}(\omega)} \leq \frac{C_{I_\alpha}}{\epsilon} \left[ \left( \int_{\mathbb{R}^n} |f(x)| (M_{L(\log L)^\epsilon}^c \omega(x) dx) \right)^{\frac{1}{c}} + \int_{\mathbb{R}^n} |f(x)| M\omega(x) dx \right].$$

**Theorem 1.2.** *Let  $w$  be a weight,  $\vec{b} = (b_1, \dots, b_k)$ ,  $0 < \epsilon < 1$ ,  $\phi_\rho(t) = t(1 + \log^+ t)^\rho$ ,  $\rho > 0$ . For  $\alpha \in (0, \frac{5n\epsilon}{24(1+\frac{1}{s})+10\epsilon})$ ,*

$$c = \frac{12n(1 + \frac{1}{s})}{12(n + \alpha) \left(1 + \frac{1}{s}\right) + 5\alpha\epsilon}, \quad u = M_{L(\log L)^{\frac{1}{s}+\epsilon}, \alpha} \omega(x), \quad v = M_{L(\log L)^{\frac{1}{s}+\epsilon}} \omega(x),$$

there exists a constant  $C_{I_\alpha}$  such that

$$\begin{aligned} & \omega\left(\left\{x \in \mathbb{R}^n : |I_{\alpha, \vec{b}} f(x)| > \lambda\right\}\right) \\ & \leq \frac{C_{I_\alpha}}{\epsilon^{k+1}} \left\{ \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}}(\|\vec{b}\| \|f(x)\|/\lambda) (u + v) dx + \left( \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}}(\|\vec{b}\| \|f(x)\|/\lambda^c) v^c dx \right)^{\frac{1}{c}} \right. \\ & \quad \left. + \left( (1 + \lambda^{1-c}) \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}}(\|\vec{b}\| \|f(x)\|/\lambda) v^c dx \right)^{\frac{1}{c}} \right\}. \end{aligned}$$

**Remark 1.1.** When  $p = q$ , the above results are consistent with the result in [20].

The article is organized as follows. In Section 2, some definitions and basic lemmas will be given. Section 3 will be devoted to demonstrate Theorem 1.1 and Theorem 1.2.

## 2 Definitions and main lemmas

We begin by introducing some definitions and notations.

**Definition 2.1** (Orlicz spaces, [20]). *The Orlicz spaces are defined in the way that*

$$Osc_{exp L^s} = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{Osc_{exp L^s}} < \infty \right\},$$

where

$$\|f\|_{Osc_{exp L^s}} = \sup \|f - f_Q\|_{\Psi_s, Q}, \quad \Psi_s(t) = e^{t^s} - 1$$

with  $t \geq 0, s > 0$ .

From the John-Nirenberg's theorem, it is known that  $BMO = Osc_{exp L}$  and for every  $s > 1$ , it holds that  $Osc_{exp L^s} \subsetneq BMO$ .

**Definition 2.2** (Young functions, Orlicz maximal functions, [6]). *A function  $\Phi$  is called a Young function, if it is a continuous, nonnegative, strictly increasing and convex function defined on  $[0, \infty)$  such that*

$$\Phi(0) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \Phi(t) = \infty.$$

The local Luxembourg norm of a function  $f$  with respect to  $\Phi$  is defined in the way that

$$\|f\|_{\Phi, Q} = \|f\|_{\Phi(L), Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\},$$

which is equivalent to

$$\|f\|'_{\Phi, Q} = \inf_{\mu > 0} \left\{ \mu + \frac{\mu}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\mu}\right) dx \right\}.$$

The Orlicz maximal function associated to  $\Phi$  is defined by

$$M_{\Phi, \alpha} f(x) = \sup_{x \in Q} |Q|^{\frac{\alpha}{n}} \|f\|_{\Phi, Q} \quad \text{for } \alpha \geq 0.$$

Let

$$\Phi_\rho(t) = t(1 + \log^+(t))^\rho \quad \text{with } \log^+(t) = \chi_{(1, \infty)}(t) \log(t)$$

and  $\rho > 0, t \geq 0$ . Then we use the notation

$$\|f\|_{\Phi, Q} = \|f\|_{L(\log L)^\rho, Q}.$$

**Definition 2.3** (Commutators of  $I_\alpha$ ). *Let  $b_i \in Osc_{exp L^{s_i}}, s_i \geq 1, i = 1, \dots, k, k \in \mathbb{N}^+$ . The symbol-multilinear commutators of the fraction integral  $I_\alpha$  with respect to the symbol  $\vec{b} = (b_1, \dots, b_k)$  is defined as follows :*

$$I_{\alpha, \vec{b}} f(x) = \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\alpha}} \prod_{i=1}^k (b_i(x) - b_i(y)) f(y) dy.$$

**Definition 2.4** ( $A_p$  weights, [13]). *A weight  $\omega$  belongs to the class  $A_p$ ,  $1 < p < \infty$ , if*

$$[\omega]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q \omega \right) \left( \frac{1}{|Q|} \int_Q \omega^{-\frac{1}{p-1}} \right)^{p-1} < \infty.$$

*A weight  $w$  belongs to the class  $A_1$  if there is a finite constant  $C$  such that*

$$\frac{1}{|Q|} \int_Q \omega(y) dy \leq C \inf_Q \omega,$$

*and the infimum of these constants  $C$  is called the  $A_1$  constant of  $w$  denoted by  $[\omega]_{A_1}$ .*

We will need to use the following Hölder inequality in our proof later.

**Lemma 2.1** (Hölder inequality, [21]). *Let  $\Phi_0, \Phi_1, \dots, \Phi_k$  be Young functions. If*

$$\Phi_1^{-1}(t)\Phi_2^{-1}(t) \cdots \Phi_k^{-1}(t) \leq \kappa \Phi_0^{-1}(t),$$

*then the following inequality holds*

$$\|f_1 f_2 \cdots f_k\|_{\Phi_0, Q} \leq \kappa \|f_1\|_{\Phi_1, Q} \|f_2\|_{\Phi_2, Q} \cdots \|f_k\|_{\Phi_k, Q}$$

*for all functions  $f_1, \dots, f_m$  and all cubes  $Q$ .*

In Particular, if  $\sum_{i=1}^k \frac{1}{s_i} = \frac{1}{s}$  with each  $s_i \geq 1$ , then it holds that

$$\frac{1}{Q} \int_Q |f_1 f_2 \cdots f_k g| \leq C_s \|f_1\|_{\exp L^{s_1}, Q} \|f_2\|_{\exp L^{s_2}, Q} \cdots \|f_k\|_{\exp L^{s_k}, Q} \|g\|_{L(\log L)^{\frac{1}{s}}, Q}.$$

It was known that  $M_\delta^\#(I_\alpha f)(x) \leq C_\delta M_\alpha f(x)$  pointwisely (see for example [2]). For the commutators of  $I_\alpha$ , one may get

**Lemma 2.2** (Sharp Estimate). *Let  $0 < \delta < \epsilon < 1$ ,  $M^\#$  be the Fefferman-Stein sharp maximal function and  $M_\delta^\#(f) = M^\#(|f|^\delta)^{\frac{1}{\delta}}$ . Then there exists a constant  $C > 0$ , depending only on  $\delta$  and  $\epsilon$  such that*

$$M_\delta^\#(I_{\alpha, \vec{b}})(x) \leq C \|\vec{b}\| \left( M_{L(\log L)^{\frac{1}{\delta}, \alpha}} f(x) + \sum_{i=1}^m \sum_{\sigma \in C_j^m} \|\vec{\sigma}\| M_\epsilon(I_{\alpha, b_{\sigma'}} f)(x) \right). \tag{2.1}$$

Here  $b = \sigma \cup \sigma'$ , where  $\sigma$  and  $\sigma'$  are pairwise disjoint sets be a splitting of  $b$ .

This lemma can be obtained with small and straightforward modifications in the proof of Lemma 3.1 in [21].

**Lemma 2.3** ([15,16]). *Let  $0 < p < \infty$ ,  $0 < \delta < 1$  and  $w \in A_\infty$ ,  $M^d$  denotes the dyadic maximal function. Then for any function  $f$  and  $t > 0$  satisfying  $|\{x : |f(x)| > t\}| < \infty$ , it holds that*

$$\|f\|_{L^p(w)} \leq C_p [w]_{A_\infty} \|M_\delta^{\#,d} f\|_{L^p(w)}.$$

**Lemma 2.4** ([20]). *Let  $0 < p < \infty$ ,  $0 < \epsilon \leq 1$  and  $w \in A_\infty$ . Suppose that  $|\{x : |f(x)| > t\}| < \infty$  for all  $t > 0$ . Then there is a constant  $C = C_{n,\epsilon}$  such that*

$$\|M_\epsilon^d f\|_{L^p(w)} \leq C_p[w]_{A_\infty} \|M_\epsilon^{\#,d} f\|_{L^p(w)}.$$

**Lemma 2.5** ([11]). *Let  $0 < \alpha < mn$ ,  $1 < p_i < \infty$ ,  $i = 1, \dots, m$ ,  $m \in \mathbb{N}^+$ ,  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ ,  $\frac{1}{m} < p < \frac{n}{\alpha}$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ . Suppose that  $X_i$  is Banach space and  $M_{X_i}$  is bounded on  $L^{p_i}(\mathbb{R}^n)$ ,  $u$  and  $v_1, \dots, v_m$  are weights satisfying*

$$K = \sup_Q \left( \frac{u(Q)}{|Q|} \right)^{\frac{1}{q}} \prod_{i=1}^m \|v_i^{-1}\|_{X_i, Q} < \infty,$$

then

$$\|M_\alpha(\vec{f})\|_{L^q(u)} \leq CK \prod_{i=1}^m \|M_{X_i}\|_{L^{p_i}(\mathbb{R}^n)} \|f_i\|_{L^{p_i}(v_i^{p_i})}.$$

Next we will give a key lemma which will be useful in the proof of Lemma 2.8.

**Lemma 2.6.** *Let  $\omega \geq 0$  be a weight,*

$$v = M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}^{p/q}(\omega), \quad s \geq 1 \quad \text{and} \quad 0 < \delta < 1.$$

Then there exists  $C > 0$  such that for every  $p \in (1, \infty)$

$$\left\| \frac{M_{L(\log L)^{\frac{1}{s}, \alpha}} f}{v} \right\|_{L^{p'}(v)} \leq C q^{1+\frac{1}{s}} \left( \frac{q-1}{\delta} q' \right)^{\frac{1}{p'}} \left\| \frac{f}{\omega} \right\|_{L^{q'}(\omega)}. \tag{2.2}$$

*Proof.* To prove inequality (2.2), it is equivalent to show that

$$\begin{aligned} & \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}, \alpha}} (f \omega^{\frac{1}{q}})^{p'} \left( M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}^{\frac{p}{q}}(\omega) \right)^{1-p'} \\ & \leq C_n^{p'} \left( q^{1+\frac{1}{s}} \right)^{p'} \left( \frac{q-1}{\delta} q' \right) \left( \int_{\mathbb{R}^n} |f|^{q'} \right)^{\frac{p'}{q'}}. \end{aligned} \tag{2.3}$$

We now introduce the notations

$$A_{\frac{1}{s}}(t) = t(1 + \log^+(t))^{\frac{1}{s}}, \quad X_{\frac{1}{s}}(t) = \frac{t}{(1 + \log^+ t)^{\frac{1}{s}}}.$$

Then by [20], it holds that

$$A_{\frac{1}{s}}^{-1}(t) \geq X_{\frac{1}{s}}(t).$$

Moreover, one can see that

$$\begin{aligned} X_{\frac{1}{s}}(t) &= \left( \frac{t}{(1 + \log^+(t))^{(1+\frac{1}{s})q-1+\delta}} \right)^{\frac{1}{q}} (t(1 + \log^+(t))^{1+\delta(q'-1)})^{\frac{1}{q'}} \\ &=: F_1(t)^{\frac{1}{q}} F_2(t)^{\frac{1}{q'}}, \end{aligned}$$

where

$$F_1(t) = X_{(1+\frac{1}{s})q-1+\delta}(t), \quad F_2(t) = A_{1+\delta(q'-1)}(t).$$

Again, by [20], one may get

$$M_{L(\log L)^{\frac{1}{s}}, \alpha}(f w^{\frac{1}{q}}) \leq C q^{1+\frac{1}{s}} M_{\tilde{X}_{1+\delta(q'-1)}(L^{q'}), \alpha}(f) \left( M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}(w)} \right)^{\frac{1}{q}}.$$

Now we give the proof of inequality (2.3).

$$\begin{aligned} &\int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}}, \alpha}(f \omega^{\frac{1}{q}})^{p'} \left( M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}(\omega)} \right)^{1-p'} dx \\ &\leq \int_{\mathbb{R}^n} \left( C q^{1+\frac{1}{s}} M_{\tilde{X}_{1+\delta(q'-1)}(L^{q'}), \alpha}(f) \left( M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}(w)} \right)^{\frac{1}{q}} \right)^{p'} \\ &\quad \times \left( M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}(w)} \right)^{1-p'} dx \\ &= (C q^{1+\frac{1}{s}})^{p'} \int_{\mathbb{R}^n} M_{\tilde{X}_{1+\delta(q'-1)}(L^{q'}), \alpha}(f)^{p'} dx. \end{aligned}$$

Let  $B = \tilde{X}_{1+\delta(p-1)}(L^p)$ , by [4] we know

$$\tilde{X}_{1+\delta(p-1)}(L^p) \preccurlyeq t^p.$$

Then

$$|Q|^{\frac{\alpha}{n}} \|f\|_{B,Q} = |Q|^{\frac{\alpha}{n}} \|f\|_{B,Q}^{\frac{\alpha p}{n}} \|f\|_{B,Q}^{1-\frac{\alpha p}{n}} \leq C \|f\|_p^{1-\frac{p}{q}} \|f\|_{B,Q}^{\frac{p}{q}}$$

which yields that

$$M_{\alpha,B} f(x) \leq C \|f\|_p^{1-\frac{p}{q}} M_{0,B} f(x)^{\frac{p}{q}}.$$

Hence, Lemma 2.1 in [10] gives that

$$\int_{\mathbb{R}^n} (M_{\alpha,B} f(x))^q dx \leq C \|f\|_p^{q-p} p \int_1^\infty \frac{B(t) dt}{t^p} \|f\|_{L^p}^p \leq C p \frac{cp-1}{\delta} \|f\|_p^q.$$

Therefore,

$$\int_{\mathbb{R}^n} M_{\tilde{X}_{1+\delta(q'-1)}(L^{q'}), \alpha}(f)^{p'} dx \leq C q' \frac{q-1}{\delta} \|f\|_{q'}^{p'}.$$

Consequently, we have shown that

$$\|M_{L(\log L)^{\frac{1}{s}}}(f w^{\frac{1}{q}})\|_{L^{p'}(v^{1-p'})} \leq C q^{1+\frac{1}{s}} \left( \frac{q-1}{\delta} q' \right)^{\frac{1}{p'}} \|f\|_{L^{q'}(\mathbb{R}^n)}.$$

Thus, we complete the proof.  $\square$

**Lemma 2.7.** *Let  $\omega$  be a weight and  $A$  be a Young function,  $p \in (1, \infty)$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ , then there exists a constant  $C_{I_\alpha}$  such that*

$$\|I_\alpha(f)\|_{L^q(\omega)} \leq C_{I_\alpha} p' \|M_{\bar{A}}\|_{L^{q'}(\mathbb{R}^n)} \|f\|_{L^p(M_{A_q}^{\frac{p}{q}}(\omega))},$$

where  $\bar{A}$  is the complementary function of  $A$ ,  $A_q(t) = A(t^{\frac{1}{q}})$ . Moreover, if

$$A(t) = t^q(1 + \log^+ t)^{q-1+\delta},$$

it holds that

$$\|I_\alpha(f)\|_{L^q(\omega)} \leq C_{I_\alpha} p' q^2 \left(\frac{1}{\delta}\right)^{\frac{1}{q'}} \|f\|_{L^p(M_{L(\log L)^{q-1+\delta}}^{\frac{p}{q}}(\omega))}, \quad 0 < \delta < 1.$$

*Proof.* By Proposition 2.2 and Proposition 2.3 of [12], it suffices to show that

$$\|I_\alpha^S(f)\|_{L^q(\omega)} \leq C_{I_\alpha} p' \|M_{\bar{A}}\|_{L^{q'}(\mathbb{R}^n)} \|f\|_{L^p(M_{A_q}^{\frac{p}{q}}(\omega))}, \tag{2.4}$$

where  $S \in D$  is a sparse set and  $D$  is a standard dyadic grid,

$$I_\alpha^S f = \sum_{Q \in S} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q f dx \cdot \chi_Q.$$

Let  $v = M_{A_q}^{\frac{p}{q}}(\omega)$ . By duality, there exists  $\|g\|_{L^{p'}(\omega)} = 1$  such that

$$\|I_\alpha^S(f)\|_{L^q(\omega)} = \int_{\mathbb{R}^n} \frac{I_\alpha^S g \omega}{v} f v dx \leq \left\| \frac{I_\alpha^S g \omega}{v} \right\|_{L^{p'}(v)} \|f\|_{L^p(v)}.$$

Observe that the adjoint of  $I_\alpha^S$  is itself. Then, to prove inequality (2.4), it suffices to show that

$$\left\| \frac{I_\alpha^S g \omega}{v} \right\|_{L^{p'}(v)} \leq C_{I_\alpha} p' \|M_{\bar{A}}\|_{L^{q'}(\mathbb{R}^n)}. \tag{2.5}$$

First, we consider to prove

$$\left\| \frac{I_\alpha^S g}{v} \right\|_{L^{p'}(v)} \leq p' \left\| \frac{M_\alpha g}{v} \right\|_{L^{p'}(v)}. \tag{2.6}$$

By duality, there exists a nonnegative function  $h$  with  $\|h\|_{L^p(v)} = 1$  such that

$$I = \left\| \frac{I_\alpha^S g}{v} \right\|_{L^{p'}(v)} = \int_{\mathbb{R}^n} I_\alpha^S g(x) h(x) dx.$$

Now we consider the operator:

$$S(h) = \frac{M(hv^{\frac{1}{p}})}{v^{\frac{1}{p}}},$$

and use  $S$  to build the Rubio de Francia algorithm

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{\|S\|_{L^p(v)}^k},$$

where  $R$  satisfies

- (1)  $0 \leq h \leq R(h)$ ;
- (2)  $\|R(h)\|_{L^p(v)} \leq 2\|h\|_{L^p(v)}$ ;
- (3)  $R(h)v^{\frac{1}{p}} \in A_1$ ,  $[R(h)v^{\frac{1}{p}}]_{A_1} \leq Cp'$ .

Note that  $[v^{\frac{1}{2p}}]_{A_1}^2 \leq C_n$ , by Lemma 4.2 in [10]. Taking this into account yields that

$$[Rh]_{A_3} = [R(h)v^{\frac{1}{p}}(v^{-\frac{1}{p(1-q)}})^{1-q}]_{A_3} \leq [Rhv^{\frac{1}{p}}]_{A_1} [v^{\frac{1}{2p}}]_{A_1}^2 \leq C_n p'.$$

Therefore  $[Rh]_{A_\infty} \leq [Rh]_{A_3} \leq C_n p'$  gives that

$$I \leq \int_{\mathbb{R}^n} I_\alpha^S g(x) Rh(x) dx \leq [Rh]_{A_\infty} \int_{\mathbb{R}^n} M_\alpha g \cdot Rh dx \leq Cp' \left\| \frac{M_\alpha g}{v} \right\|_{L^{p'}(v)}. \quad (2.7)$$

Eq. (2.7) can be obtained by similar reasoning as in Lemma 4.1 of [10] with slight modifications. So we have shown that inequality (2.6) holds. In order to prove (2.5), it suffices to show that

$$\left\| \frac{M_\alpha f}{v} \right\|_{L^{p'}(v)} \leq C \|M_{\bar{A}}\|_{L^{q'}(\mathbb{R}^n)} \left( \int_{\mathbb{R}^n} \left( \frac{f}{\omega^{\frac{1}{q}}} \right)^{q'} dx \right)^{\frac{1}{q'}}.$$

Note that Lemma 2.5 and Theorem 6.4 in [6] yield that

$$\begin{aligned} K &= \sup_Q \left( \frac{1}{|Q|} \int_Q v^{1-p'} \right)^{\frac{1}{p'}} \|\omega^{\frac{1}{q}}\|_{A,Q} \\ &= \sup_Q \left( \frac{1}{|Q|} \int_Q M_{A_q}^{\frac{p}{q}}(\omega)^{1-p'} dy \right)^{\frac{1}{p'}} \|\omega^{\frac{1}{q}}\|_{A,Q} \\ &\leq \|\omega\|_{A_q, Q}^{-\frac{1}{q}} \|\omega^{\frac{1}{q}}\|_{A,Q} = \|\omega^{\frac{1}{q}}\|_{A,Q}^{-1} \|\omega^{\frac{1}{q}}\|_{A,Q} = 1. \end{aligned}$$

Moreover, if  $A(t) = t^q(1 + \log^+ t)^{q-1+\delta}$ , from the inequality (25) in [10], one may get

$$\|M_{\bar{A}}\|_{L^{q'}(\mathbb{R}^n)} \leq C_n \left( \int_1^\infty \left( \frac{t}{A(t)} \right)^{q'} A'(t) dt \right)^{\frac{1}{q'}} \leq q^2 \left( \frac{1}{\delta} \right)^{\frac{1}{q'}}.$$

Therefore

$$\|I_\alpha(f)\|_{L^q(\omega)} \leq C_{I_\alpha} p' q^2 \left(\frac{1}{\delta}\right)^{\frac{1}{q'}} \|f\|_{L^p\left(M_{L(\log L)^{q-1+\delta}}^{\frac{p}{q}}(\omega)\right)}.$$

Thus, we complete the proof. □

**Lemma 2.8.** Let  $\vec{b} = (b_1, \dots, b_k)$ ,  $k \in \mathbb{N}$ ,  $\omega$  be a weight, for  $\delta \in (0, 1)$ ,

$$p \in (1, \infty), \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \quad v = M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}^{\frac{p}{q}} \omega,$$

there exists a constant  $C_{I_\alpha}$  such that

$$\|I_{\alpha, \vec{b}} f\|_{L^q(\omega)} \leq C_{I_\alpha} p^{k+1} q^{(1+\frac{1}{s})} \left(\frac{q-1}{\delta} q'\right)^{\frac{1}{p'}} \|\vec{b}\| \|f\|_{L^p(v)}.$$

*Proof.* We will divide the proof in two cases, as follows.

Case 1:  $k = 1$ . We denote

$$v = M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}^{\frac{p}{q}} \omega, \quad m = C_{I_\alpha} (p')^2 q^{1+\frac{1}{s}} \left(\frac{q-1}{\delta} q'\right)^{\frac{1}{p'}}.$$

By duality, there exists  $\|g\|_{L^{q'}(\omega)} = 1$  such that

$$\|I_{\alpha, b}(f)\|_{L^q(\omega)} = \int_{\mathbb{R}^n} I_{\alpha, b} f(x) g(x) \omega dx = \int_{\mathbb{R}^n} \frac{I_{\alpha, b} g \omega}{v} f v dx \leq \left\| \frac{I_{\alpha, b} g \omega}{v} \right\|_{L^{p'}(v)} \|f\|_{L^p(v)}.$$

We only need to show that

$$\left\| \frac{I_{\alpha, b}^t(f)}{v} \right\|_{L^{p'}(v)} \leq m \left\| \frac{f}{\omega} \right\|_{L^{q'}(\omega)},$$

where  $I_{\alpha, b}^t$  is the adjoint of  $I_{\alpha, b}$ . The method of duality allows us to find a non-negative function  $h \in L^p(v)$  with  $\|h\|_{L^p(v)} = 1$  such that

$$\left\| \frac{I_{\alpha, b}^t(f)}{v} \right\|_{L^{p'}(v)} = \int_{\mathbb{R}^n} \frac{|I_{\alpha, b}^t(f)|}{v} h v dx = \int_{\mathbb{R}^n} |I_{\alpha, b}^t(f)| h dx = I.$$

Now we consider the same operator  $S$  as we used in Theorem 2.7 and build the Rubio de Francia algorithm  $R$  using the operator  $S$ . From the properties of  $R$ , we have

$$I \leq \int_{\mathbb{R}^n} |I_{\alpha, b}^t f| R h dx \leq C_n [Rh]_{A_\infty} \int_{\mathbb{R}^n} M_r^\#(I_{\alpha, b}^t f) R h dx \leq C_n p' \int_{\mathbb{R}^n} M_r^\#(I_{\alpha, b}^t f) R h dx.$$

Observing that  $[b, T]^t = -[b, T^t]$ , so  $[b, T]^t$  is also a commutator. By Lemma 2.2, it holds that

$$I \leq C_n p' \|b\|_{Osc_{expL^s}} \left[ \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}, \alpha}} f(x) Rh(x) dx + \int_{\mathbb{R}^n} M_\epsilon(I_\alpha^t f)(x) Rh(x) dx \right] \\ = C_n p' \|b\|_{Osc_{expL^s}} (I_1 + I_2).$$

Applications of Hölder inequality and the second property of operator  $R$  lead to the inequalities that

$$I_1 \leq \left( \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}, \alpha}} f^{p'}(x) v^{1-p'}(x) dx \right)^{\frac{1}{p'}} \left( \int_{\mathbb{R}^n} Rh^p(x) v(x) dx \right)^{\frac{1}{p}} \\ \leq 2 \left\| \frac{M_{L(\log L)^{\frac{1}{s}, \alpha}} f}{v} \right\|_{L^{p'}(v)}.$$

Now, Lemma 2.4 will be used to estimate  $I_2$ .

$$I_2 \leq C_n [Rh]_{A_\infty} \int_{\mathbb{R}^n} M_\epsilon^\#(I_\alpha^t f)(x) Rh(x) dx \leq C_n p' \int_{\mathbb{R}^n} M_\epsilon^\#(I_\alpha^t f)(x) Rh(x) \\ \leq c_{n,\epsilon} p' \int_{\mathbb{R}^n} M_\alpha f(x) Rh(x) dx \leq C_n p' \left\| \frac{M_\alpha f}{v} \right\|_{L^{p'}(v)}.$$

The last inequality can be obtained the same as what we have done to deal  $I_1$ . Consequently, when

$$v = M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}^{\frac{p}{q}} \omega,$$

it holds that

$$\left\| \frac{I_{\alpha,b}^t(f)}{v} \right\|_{L^{p'}(v)} \leq C_n (p')^2 \|b\|_{Osc_{expL^s}} \left\| \frac{M_{L(\log L)^{\frac{1}{s}, \alpha}} f}{v} \right\|_{L^{p'}(v)}.$$

Therefore by Lemma 2.6, there exists  $C_n > 0$  such that

$$\left\| \frac{I_{\alpha,b}^t(f)}{v} \right\|_{L^{p'}(v)} \leq C_n (p')^2 q^{1+\frac{1}{s}} \left( \frac{q-1}{\delta} q' \right)^{\frac{1}{p'}} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\omega)}.$$

Case2:  $k > 1$ . The following argument is essentially taken from [20], and we only prepare to give an outline of the proof. We will divide it in two steps.

Step 1. Due to the homogeneity of the operator we may assume that

$$\|b_1\|_{Osc_{expL^{s_1}}} = \|b_2\|_{Osc_{expL^{s_2}}} = \dots \|b_k\|_{Osc_{expL^{s_k}}} = 1.$$

Repeated reasoning as the case  $k = 1$  yields that

$$\begin{aligned} I &= \int_{\mathbb{R}^n} |I_{\alpha, \vec{b}}^t f| h dx \\ &\leq C_n p' \left[ \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}, \alpha}} f Rh dx + \sum_{j=1}^m \sum_{\sigma \in C_j^m} \int_{\mathbb{R}^n} M_{\epsilon}(I_{\alpha, \vec{\sigma}}^t f) Rh dx \right] \\ &= C_n p'(I_1 + I_2). \end{aligned}$$

Similarly as the case for  $k = 1$ , one can get

$$I_1 \leq 2 \left\| \frac{M_{L(\log L)^{1/s, \alpha}} f}{v} \right\|_{L^{p'}(v)}.$$

Step 2. Now we consider the contribution of  $I_2$ . Let

$$\Gamma(j) = \begin{cases} 1, & j = 0, \\ 2, & j = 1, \\ 2 + \sum_{i=0}^{j-1} \binom{j}{i} \Gamma(i), & j > 1. \end{cases} \tag{2.8}$$

For every  $\epsilon \in (0, 1)$ , we claim that:

$$\int_{\mathbb{R}^n} M_{\epsilon}(I_{\alpha, \vec{\sigma}}^t f)(x) Rh(x) dx \leq C \Gamma(\#\sigma')(p')^{\#\sigma'+1} \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \sigma'} \frac{1}{s_i}, \alpha}} f(x) Rh(x) dx.$$

Now we turn to the proof of this claim. It will be proved by induction on the number of symbols of  $I_{\alpha, \vec{\sigma}}^t$ . We will use the notation  $I_{\alpha, \vec{\sigma}}$  instead of  $I_{\alpha, \vec{\sigma}}^t$  and denote  $k = \#\sigma'$ .

If  $\#\sigma' = 0$ , since

$$\sum_{i \in \emptyset} \frac{1}{s_i} = 0,$$

then

$$\int_{\mathbb{R}^n} M_{\epsilon}(I_{\alpha} f)(x) Rh(x) dx \leq C p' M_{\alpha} f(x) Rh(x) dx.$$

If  $\#\sigma' = 1$ , then

$$\int_{\mathbb{R}^n} M_{\epsilon}(I_{\alpha, b} f) Rh dx \leq 2C(p')^2 \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s_1}, \alpha}} f(x) Rh(x) dx.$$

Suppose that the claim holds for  $0 \leq \#\tau < k$ , then for every  $\epsilon \in (0, 1)$ :

$$\int_{\mathbb{R}^n} M_{\epsilon}(I_{\alpha, \vec{\tau}} f)(x) Rh(x) dx \leq C \Gamma(\#\tau)(p')^{\#\tau+1} \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \tau} \frac{1}{s_i}, \alpha}} f(x) Rh(x) dx.$$

We only need to show the case for  $\#\tau' = k$ . With Lemma 2.4 and the inductive hypothesis in hand, one may get

$$\begin{aligned} & \int_{\mathbb{R}^n} M_\epsilon(I_{\alpha, \vec{\sigma}'} f)(x) Rh(x) dx \\ & \leq \left(1 + \sum_{j=1}^k \sum_{\tau \in C_j^k} \Gamma(\#\tau')\right) C p'^{k+1} \int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i \in \sigma'} \frac{1}{s_i}}, \alpha} f(x) Rh(x) dx. \end{aligned}$$

Denoting

$$\left(1 + \sum_{j=1}^k \sum_{\tau \in C_j^k} \Gamma(\#\tau')\right) = \Gamma(k),$$

then

$$I_2 \leq C_{n, \delta, \epsilon} (p')^{m+1} \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}}, \alpha} f Rh dx.$$

Therefore, the same reasoning as what we have done in dealing with  $I_1$  yields that

$$I_2 \leq C_{n, \delta, \epsilon} (p')^{m+1} \left\| \frac{M_{L(\log L)^{\frac{1}{s}}, \alpha} f}{v} \right\|_{L^{p'}(v)}.$$

Consequently, by Lemma 2.6, as already noted, it holds that

$$\left\| \frac{I_{\alpha, \vec{b}'}^t f}{v} \right\|_{L^{p'}(v)} \leq C_n (p')^{k+1} q^{(1+\frac{1}{s})} \left( \frac{q-1}{\delta} q' \right)^{\frac{1}{p'}} \left\| \frac{M_{L(\log L)^{\frac{1}{s}}, \alpha} f}{\omega} \right\|_{L^{p'}(\omega)}.$$

Thus, we complete the proof.  $\square$

### 3 The proof of borderline estimates

Now we turn to prove Theorem 1.1. The following argument is essentially taken from [10].

*Proof of Theorem 1.1.* By Proposition 2.2 and 2.3 in [12], it suffices to show that

$$\|I_\alpha^S f\|_{L^{1,\infty}(\omega)} \leq \frac{C_{I_\alpha}}{\epsilon} \left[ \int_{\mathbb{R}^n} |f(x)| (M_{L(\log L)^\epsilon, \alpha}^c \omega(x) dx)^{\frac{1}{\epsilon}} + \int_{\mathbb{R}^n} |f(x)| M \omega(x) dx \right],$$

where  $S \in D$  is the sparse set,  $D$  is the dyadic grid,

$$I_\alpha^S f = \sum_{Q \in S} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q f dx \cdot \chi_Q.$$

Thanks to the homogeneity of the operator, it is enough to show

$$\begin{aligned} & \omega(\{x \in \mathbb{R}^n : |I_\alpha^S f| > 2\}) \\ & \leq \frac{C_{I_\alpha}}{\epsilon} \left[ \int_{\mathbb{R}^n} |f(x)| (M_{L(\log L)^\epsilon, \alpha}^c \omega(x) dx)^{\frac{1}{c}} + \int_{\mathbb{R}^n} |f(x)| M\omega(x) dx \right]. \end{aligned}$$

Decompose  $f = g + b$  at height  $\lambda = 1$  with

$$b = \sum_j (f - f_{Q_j}) \chi_{Q_j}.$$

Introduce the notations

$$\Omega = \bigcup_j Q_j, \quad \tilde{\Omega} = \bigcup_j 3Q_j,$$

then  $\|g\|_{L^\infty} \leq 2^n$  and

$$\begin{aligned} & \omega(\{x \in \mathbb{R}^n : |I_\alpha^S f(x)| > 2\}) \\ & \leq \omega(\tilde{\Omega}) + \omega(\{(\tilde{\Omega})^c : I_\alpha^S g(x) > 1\}) + \omega(\{(\tilde{\Omega})^c : I_\alpha^S b(x) > 1\}) \\ & = : I + II + III. \end{aligned}$$

Consider first the contribution made by  $I$ . The Chebyshev inequality gives that

$$I \leq \sum_j |3Q_j| \frac{1}{|3Q_j|} \int_{3Q_j} \omega(x) dx \leq \sum_j |3Q_j| \inf_{z \in Q_j} M\omega(z) \leq \int_{\mathbb{R}^n} f(y) M\omega(y) dy.$$

By [10], it is easy to see  $III=0$ . Moreover, by the fact that  $\|g\|_{L^\infty} \leq 2^n$ , Then

$$\begin{aligned} II & \leq \|I_\alpha^S g\|_{L^q(\omega_{\chi_{(\tilde{\Omega})^c})}}^q \\ & \leq C \left( q^2 p' \left( \frac{1}{\delta} \right)^{\frac{1}{q'}} \right)^q \left( \int_{(\tilde{\Omega})^c} |g|^{\frac{p}{q}} M_{L(\log L)^{q-1+\delta}}^{\frac{p}{q}} \right)^{\frac{q}{p}} \\ & \leq C q^{2q} \left( \frac{1}{\delta} \right)^{\frac{q}{q'}} (p')^q \left( \int_{(\mathbb{R}^n)^c} |f|^{\frac{p}{q}} M_{L(\log L)^{q-1+\delta}}^{\frac{p}{q}} \right)^{\frac{q}{p}}. \end{aligned}$$

Let  $q - 1 = \frac{\epsilon}{2} = \delta < 1$ , then

$$q = \frac{\epsilon}{2} + 1, \quad q^{2q} \left( \frac{1}{\delta} \right)^{\frac{q}{q'}} (p')^q \leq \frac{1}{\epsilon}, \quad c = \frac{p}{q} = \frac{2n}{2n + \alpha(\epsilon + 2)}.$$

Therefore in all,

$$\|I_\alpha^S f\|_{L^{1,\infty}(\omega)} \leq \frac{C_{I_\alpha}}{\epsilon} \left[ \int_{\mathbb{R}^n} |f(x)| (M_{L(\log L)^\epsilon}^c \omega(x) dx)^{\frac{1}{c}} + \int_{\mathbb{R}^n} |f(x)| M\omega(x) dx \right].$$

Thus, we complete the proof. □

We now begin the proof of Theorem 1.2.

*Proof of Theorem 1.2.* We will divide it in two cases.

Case 1:  $k = 1$ . Due to the homogeneity of the operator, we may assume that  $\|b_1\|_{Osc_{exp} L^s} = 1$ . Consider the Calderón-Zygmund decomposition of  $f$  at height  $\lambda$ . It bears emphasis that there exists a family of dyadic cubes  $Q_j$  which are pairwise disjoint such that

$$\lambda \leq \frac{1}{|Q_j|} \int_{Q_j} |f| \leq 2^n \lambda.$$

Denote  $\Omega = \cup_j Q_j$ , and write  $f = g + h$ , where  $g$  is denoted to be the "good" part of  $f$ , defined by

$$g(x) = \begin{cases} f(x), & x \in \Omega^c, \\ f_{Q_j}, & x \in Q_j, \end{cases}$$

with the property that  $|g(x)| \leq 2^n \lambda$  a.e., and  $h = \sum h_j$  with

$$h_j = (f - f_{Q_j})\chi_{Q_j} \quad \text{and} \quad f_{Q_j} = \frac{1}{|Q_j|} \int_{Q_j} |f| dx.$$

Now we introduce the notations

$$w^*(x) = w(x)\chi_{\mathbb{R}^n \setminus \tilde{\Omega}}(x) \quad \text{and} \quad w_j(x) = w(x)\chi_{\mathbb{R}^n \setminus \tilde{Q}_j}(x),$$

where  $\tilde{Q}_j = 5\sqrt{n}Q_j$  and  $\tilde{\Omega} = \cup_j \tilde{Q}_j$ . The Calderón-Zygmund decomposition allows us to write

$$\begin{aligned} & w(\{x \in \mathbb{R}^n : |I_{\alpha,b} f(x)| > \lambda\}) \\ & \leq w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |I_{\alpha,b} g(x)| > \frac{\lambda}{2}\right\}\right) + w(\tilde{\Omega}) + w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |I_{\alpha,b} h(x)| > \frac{\lambda}{2}\right\}\right) \\ & =: I + II + III. \end{aligned}$$

Consider first the contribution of  $I$ , Chebyshev's inequality gives that

$$I \leq \frac{2^q}{\lambda^q} \int_{\mathbb{R}^n} |I_{\alpha,b} g(x)|^q w^*(x) dx.$$

Now we choose

$$q = 1 + \frac{5\epsilon}{12(1 + \frac{1}{s})} \quad \text{and} \quad \delta = \epsilon - \left(1 + \frac{1}{s}\right)(q - 1).$$

Then it follows that

$$(p')^{2q} p^{(1+\frac{1}{s})q} \left(\frac{q-1}{\delta}\right)^{\frac{q}{p'}} \leq C_s \frac{1}{\epsilon^2}, \quad \left(1 + \frac{1}{s}\right)q - 1 + \delta = \frac{1}{s} + \epsilon,$$

$$c = \frac{p}{q} = \frac{12n(1 + \frac{1}{s})}{12(n + \alpha)(1 + \frac{1}{s}) + 5\alpha\epsilon}.$$

By Lemma 2.8, one may obtain

$$\begin{aligned} & \frac{2^q}{\lambda^q} \int_{\mathbb{R}^n} |I_{\alpha,b}g(x)|^q \omega^*(x) dx \\ & \leq \frac{2^q}{\lambda^q} C \left( (p')^2 q^{(1+\frac{1}{s})} \left(\frac{q-1}{\delta}\right)^{\frac{1}{p'}} \right)^q \left( \int_{\mathbb{R}^n} |g(x)|^p M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}^{\frac{p}{q}} \omega^*(x) dx \right)^{\frac{q}{p}} \\ & \leq C \frac{1}{\epsilon^2} \frac{1}{\lambda^{\frac{q}{p}}} \left( \int_{\mathbb{R}^n \setminus \Omega} |f(x)| M_{L(\log L)^{\frac{1}{s}+\epsilon}}^{\frac{p}{q}} \omega^*(x) dx \right)^{\frac{q}{p}}. \end{aligned}$$

This means, of course, that there exists  $C > 0$  such that

$$I \leq C \frac{1}{\epsilon^2} \left( \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M_{L(\log L)^{\frac{1}{s}+\epsilon}}^c w(y) dy \right)^{\frac{1}{c}}.$$

We have the following standard estimate for II

$$II \leq \sum_j \int_{5\sqrt{n}Q_j} w(x) dx \leq \sum_j (5\sqrt{n})^n |Q_j| \inf_{z \in Q_j} M w(x) \leq (5\sqrt{n})^n \int_{\mathbb{R}^n} \frac{f(y)}{\lambda} M w(y) dy.$$

Now we turn to the discussion of III. Since

$$I_{\alpha,b}h = \sum_j I_{\alpha,b}h_j = \sum_j (b(x) - b_{Q_j}) I_{\alpha}h_j(x) - I_{\alpha}(b - b_{Q_j})h_j,$$

we may split III into two parts

$$\begin{aligned} III & \leq w \left( \left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| \sum_j (b(x) - b_{Q_j}) I_{\alpha}h_j(x) \right| > \frac{\lambda}{4} \right\} \right) \\ & \quad + w \left( \left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| \sum_j I_{\alpha}[b - b_{Q_j}]h_j(x) \right| > \frac{\lambda}{4} \right\} \right) \\ & := A + B. \end{aligned}$$

The Chebyshev’s inequality and the cancellation of  $h_j$  yield that

$$\begin{aligned} A &\leq \frac{C}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus \tilde{Q}_j} |b(x) - b_{Q_j}| w(x) \int_{Q_j} |h_j(y)| \left| \frac{1}{|x - y|^{n-\alpha}} - \frac{1}{|x - y'|^{n-\alpha}} \right| dy dx \\ &\leq \frac{C}{\lambda} \sum_j \int_{Q_j} |h_j(y)| \int_{\mathbb{R}^n \setminus \tilde{Q}_j} \frac{|y - y'|^{n-\alpha}}{|x - y|^{n-\alpha} |x - y'|^{n-\alpha}} |b(x) - b_{Q_j}| w_j(x) dx dy \\ &\leq \frac{C}{\lambda} \sum_j \int_{Q_j} |h_j| \sum_{k=1}^{\infty} \int_{2^k l(Q_j) \leq |x - y'| \leq 2^{k+1} l(Q_j)} \frac{(2l(Q_j))^{n-\alpha} |b - b_{Q_j}| w_j}{(2^{k-1} l(Q_j))^{n-\alpha} (2^k l(Q_j))^{n-\alpha}} dx dy \\ &\leq \frac{C}{\lambda} \sum_j \int_{Q_j} |h_j(y)| dy \sum_{k=1}^{\infty} 2^{-k(n-\alpha)} \frac{1}{|2^{k+1} Q_j|^{n-\alpha}} \int_{2^{k+1} Q_j} |b(x) - b_{Q_j}| w_j(x) dx. \end{aligned}$$

By Lemma 2.1 and

$$|b(x) - b_{Q_j}| \leq |b(x) - b_{2^{k+1} Q_j}| + |b_{2^{k+1} Q_j} - b_{Q_j}|,$$

it follows that

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{2^{-k(n-\alpha)}}{|2^{k+1} Q_j|^{n-\alpha}} \int_{2^{k+1} Q_j} |b(x) - b_{Q_j}| w_j(x) dx \\ &\leq \sum_{k=1}^{\infty} 2^{-k(n-\alpha)} \inf_{z \in Q_j} M_{L \log L^{\frac{1}{s}, \alpha}} w_j(z) + \sum_{k=1}^{\infty} 2^{-k(n-\alpha)} (k+1) \inf_{z \in Q_j} M_{L \log L^{\frac{1}{s}, \alpha}} w_j(z) \\ &\leq \inf_{z \in Q_j} M_{L \log L^{\frac{1}{s}, \alpha}} w_j(z). \end{aligned}$$

Therefore,

$$A \leq \frac{C}{\lambda} \sum_j \int_{Q_j} M_{L \log L^{\frac{1}{s}, \alpha}} w_j(y) |h_j(y)| dy \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| M_{L \log L^{\frac{1}{s}, \alpha}} w(y) dy.$$

On the other hand,  $B$  may be split into four parts

$$\begin{aligned} B &\leq \frac{C}{\lambda \epsilon} \sum_j \inf_{z \in Q_j} (M_{L \log L^\epsilon} w_j(z)) \left( \int_{Q_j} |b(x) - b_{Q_j}| |f(x)| dx + \int_{Q_j} |b(x) - b_{Q_j}| |f_{Q_j}| dx \right)^{\frac{1}{c}} \\ &\quad + \frac{C}{\lambda \epsilon} \sum_j \inf_{z \in Q_j} M w_j(z) \left( \int_{Q_j} |b(x) - b_{Q_j}| |f(x)| dx + \int_{Q_j} |b(x) - b_{Q_j}| |f_{Q_j}| dx \right) \\ &\leq \frac{1}{\epsilon} (B_1 + B_2 + B_3 + B_4). \end{aligned}$$

By the generalized Hölder inequality and definition of Orlicz maximal function, one may obtain

$$\begin{aligned} B_1 &\leq C \sum_j \inf_{z \in Q_j} (M_{L \log L^\epsilon} w_j(z)) \frac{1}{\lambda} |Q_j|^{\frac{1}{c}} \|f\|_{L(\log L)^{\frac{1}{s}, Q_j}}^{\frac{1}{c}} \\ &\leq \left( \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}} \left( \frac{|f(x)|}{\lambda^c} \right) M_{L \log L^\epsilon}^c w(x) dx + \lambda^{1-c} \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}} \left( \frac{|f(x)|}{\lambda} \right) M_{L \log L^\epsilon}^c w(x) dx \right)^{\frac{1}{c}}. \end{aligned}$$

Similarly, it holds that

$$B_2 \leq \left( \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}} \left( \frac{|f(y)|}{\lambda^c} \right) M_{L \log L^\epsilon}^c w_j(y) dy \right)^{\frac{1}{c}},$$

$$B_3 + B_4 \leq C \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}} \left( \frac{|f(y)|}{\lambda} \right) M w(x) dx.$$

This completes the proof of Case 1.

Case 2 :  $k \geq 2$ . We will use inductive discussion here, i.e., suppose that the inequality holds for  $l \leq k - 1$  symbols, we need to show it holds for  $l = k$ . Due to the homogeneity of the operator, we may assume that

$$\|b_1\|_{Osc_{\exp L^{s_1}}} = \dots = \|b_1\|_{Osc_{\exp L^{s_k}}} = 1.$$

We decompose  $f$  like in the same way as the case  $k = 1$ . Then

$$w \left( \left\{ x \in \mathbb{R}^n : |I_{\alpha, \vec{b}} f(x)| > \lambda \right\} \right)$$

$$\leq w \left( \left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |I_{\alpha, \vec{b}} g(x)| > \frac{\lambda}{2} \right\} \right) + w(\tilde{\Omega}) + w \left( \left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |I_{\alpha, \vec{b}} h(x)| > \frac{\lambda}{2} \right\} \right)$$

$$=: I + II + III.$$

The Chebyshev’s inequality gives that

$$I \leq \frac{2^q}{\lambda^q} \int_{\mathbb{R}^n} |I_{\alpha, \vec{b}} g(x)|^q \omega^*(x) dx. \tag{3.1}$$

Choosing

$$q = 1 + \frac{5\epsilon}{12(1 + \frac{1}{s})}, \quad \delta = \epsilon - \left(1 + \frac{1}{s}\right)(q - 1),$$

and therefore

$$c := \frac{p}{q} = \frac{12n(1 + \frac{1}{s})}{12(n + \alpha)(1 + \frac{1}{s}) + 5\alpha\epsilon}, \quad (p')^{(k+1)q} p^{(1+\frac{1}{s})q} \left(\frac{q-1}{\delta}\right)^{\frac{q}{p'}} \leq C_s \frac{1}{\epsilon^{k+1}},$$

$$\left(1 + \frac{1}{s}\right)q - 1 + \delta = \frac{1}{s} + \epsilon.$$

Then (3.1) is controlled by

$$C_n (p')^{(k+1)q} p^{(1+\frac{1}{s})q} \left(\frac{q-1}{\delta}\right)^{\frac{q}{p'}} \left( \int_{\mathbb{R}^n} |g(x)|^q M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta, \alpha}}^c \omega^*(x) dx \right)^{1/c}.$$

Proceeding as  $k = 1$  yields that

$$I \leq C_n \frac{1}{\epsilon^{k+1}} \left( \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M_{L(\log L)^{\frac{1}{s}+\epsilon, \alpha}}^c \omega(y) dy \right)^{\frac{1}{c}},$$

$$II \leq (5\sqrt{n})^n \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M_\omega(y) dy.$$

We continue to estimate *III*. It can be written in the following way

$$\begin{aligned}
 III &\leq \omega \left( \left\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| \sum_j (b_1(x) - (b_1)_{Q_j}) \cdots (b_k(x) - (b_k)_{Q_j}) I_\alpha h_j(x) \right| > \frac{\lambda}{6} \right\} \right) \\
 &\quad + \omega \left( \left\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| C_k I_\alpha ((b_1 - (b_1)_{Q_j}) \cdots (b_k - (b_k)_{Q_j}) h_j)(x) \right| > \frac{\lambda}{6} \right\} \right) \\
 &\quad + \omega \left( \left\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| \sum_j \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} C_\sigma I_\alpha^\sigma ((b - b_{Q_j}^\sigma) h_j)(x) \right| > \frac{\lambda}{6} \right\} \right) \\
 &=: L_1 + L_2 + L_3.
 \end{aligned}$$

Now we deal with  $L_1$ . Denote

$$\omega_j = \omega \cdot \chi_{\mathbb{R}^n \setminus 5\sqrt{n}Q_j} \quad \text{and} \quad B(x) = \prod_{i=1}^k |b_i(x) - (b_i)_{Q_j}|.$$

Same reasoning as what we have done in dealing with  $k = 1$  gives that

$$\begin{aligned}
 L_1 &\leq \sum_j \frac{C}{\lambda} \int_{Q_j} |h_j(y)| \sum_m \frac{8^{n-\alpha}}{2^{m(n-\alpha)}} \frac{1}{(2^{m+1}I(Q_j))^{n-\alpha}} \int_{|x-y'| \leq 2^{m+1}I(Q_j)} B(x) \omega_j(x) dx dy \\
 &\leq \frac{C_k}{\lambda} \sum_j \int_{Q_j} |h_j(y)| M_{L(\log L)^{\frac{1}{5}, \alpha}} \omega_j(y) dy \\
 &\leq \frac{C_k}{\lambda} \int_{\mathbb{R}^n} |f(y)| M_{L(\log L)^{\frac{1}{5}, \alpha}} \omega(y) dy.
 \end{aligned}$$

Consider now the contribution of  $L_2$ . By Theorem 1.1,  $L_2$  can be written as

$$\begin{aligned}
 L_2 &\leq \frac{C}{\lambda \epsilon} \left\{ \sum_j \left[ \int_{Q_j} B(x) |f - f_{Q_j}| M_{L(\log L)^\epsilon}^c \omega_j(x) dx \right]^{\frac{1}{c}} + \int_{Q_j} B(x) |f - f_{Q_j}| M \omega_j(x) dx \right\} \\
 &\leq \frac{C}{\lambda \epsilon} \sum_j \inf_{z \in Q_j} M_{L(\log L)^\epsilon} \omega_j(z) \left( \int_{Q_j} B(x) |f(x)| dx + \int_{Q_j} B(x) |f_{Q_j}| dx \right)^{\frac{1}{c}} \\
 &\quad + \frac{C}{\lambda \epsilon} \sum_j \inf_{z \in Q_j} M \omega_j(z) \left( \int_{Q_j} B(x) |f(x)| dx + \int_{Q_j} B(x) |f_{Q_j}| dx \right) \\
 &\leq \frac{1}{\epsilon} (B_1 + B_2 + B_3 + B_4).
 \end{aligned}$$

Lemma 2.1 tells us that

$$\frac{1}{|Q_j|} \int_{Q_j} B(x) dx \leq C \prod_{i=1}^m \|b_i - (b_i)_{Q_j}\|_{\text{exp}L^{s_i}, Q_j} \leq C \|\vec{b}\| = C,$$

therefore

$$B_2 \leq \frac{C}{\lambda} \left( \int_{\mathbb{R}^n} \phi \left( \frac{|f(x)|}{\lambda^c} \right) (M_{L(\log L)^\epsilon}^c \omega_j(x)) dx \right)^{\frac{1}{c}}.$$

Proceeding similarly as before, one may obtain

$$\begin{aligned} B_1 &\leq \frac{C}{\lambda} \sum_j \inf_{z \in Q_j} \left( M_{L(\log L)^\epsilon} \omega_j \right) |Q_j|^{\frac{1}{c}} \|f(x)\|_{L(\log L)^{\frac{1}{s}}, Q_j}^{\frac{1}{c}} \\ &\leq C \left( \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}} \left( \frac{|f|}{\lambda^c} \right) (M_{L(\log L)^\epsilon}^c \omega) dx + \lambda^{1-c} \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}} \left( \frac{|f|}{\lambda} \right) (M_{L(\log L)^\epsilon}^c \omega) dx \right)^{\frac{1}{c}}, \\ B_3 + B_4 &\leq C \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}} \left( \frac{|f(y)|}{\lambda} \right) M\omega(x) dx, \end{aligned}$$

which means that we have shown that the desired estimate holds for  $L_2$ .

Finally, we consider the last term  $L_3$ .

$$\begin{aligned} L_3 &\leq \omega \left( \left\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} C_\sigma I_\alpha^\sigma \left( \sum_j (b - \bar{\lambda})_{\sigma'} f \chi_{Q_j} \right) (x) \right| > \frac{\lambda}{12} \right\} \right) \\ &\quad + \omega \left( \left\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} C_\sigma I_\alpha^\sigma \left( \sum_j (b - \bar{\lambda})_{\sigma'} f_{Q_j} \chi_{Q_j} \right) (x) \right| > \frac{\lambda}{12} \right\} \right) \\ &=: L_{31} + L_{32}. \end{aligned}$$

By the inductive hypothesis and the Chebyshev’s inequality,  $L_{31}$  can be estimated as

$$\begin{aligned} L_{31} &\leq C \sum_{i=1}^{k-1} \sum_{\sigma \in C_i(b)} \sum_j \frac{1}{\epsilon^{\#\sigma+1}} \left\{ \inf_{z \in Q_j} \left( M_{L(\log L)^{i \in \sigma'}} \omega_j(z) + M_{L(\log L)^{i \in \sigma'}} \omega_j(z) \right) \right. \\ &\quad \int_{Q_j} \Phi_{\sum_{i \in \sigma} \frac{1}{s_i}} \left( \frac{|f(x)|}{\lambda} (b(x) - b_{Q_j})_{\sigma'} \right) dx \\ &\quad + \left( \inf_{z \in Q_j} M_{L(\log L)^{i \in \sigma'}}^c \omega_j(z) \int_{Q_j} \Phi_{\sum_{i \in \sigma} \frac{1}{s_i}} \left( \frac{|f(x)|}{\lambda^c} (b(x) - b_{Q_j})_{\sigma'} \right) \right)^{\frac{1}{c}} \\ &\quad \left. + \left( \inf_{z \in Q_j} M_{L(\log L)^{i \in \sigma'}}^c \omega_j(z) (1 + \lambda^{1-c}) \int_{Q_j} \Phi_{\sum_{i \in \sigma} \frac{1}{s_i}} \left( \frac{|f(x)|}{\lambda} (b(x) - b_{Q_j})_{\sigma'} \right) dx \right)^{\frac{1}{c}} \right\}. \\ &=: L_{311} + L_{312} + L_{313} + L_{314}. \end{aligned}$$

By Lemma 2 of [17], we have

$$\int_{Q_j} \Phi_{\sum_{i \in \sigma} \frac{1}{s_i}} \left( \frac{|f(x)|}{\lambda} (b(x) - b_{Q_j})_{\sigma'} \right) dx \leq Ck \int_{Q_j} \Phi_{\frac{1}{s}} \left( \frac{|f(x)|}{\lambda} \right) dx.$$

We may continue with the estimate

$$\begin{aligned}
 L_{311}0 &\leq C_k \sum_{i=1}^{k-1} \sum_{\sigma \in c_i^k} \sum_j \frac{1}{e^{\#\sigma+1}} \inf_{z \in Q_j} M_{L(\log L)^{\frac{1}{s} + \epsilon}, \alpha} \omega_j(z) \int_{Q_j} \Phi_{\frac{1}{s}} \left( \frac{|f(x)|}{\lambda} \right) dx \\
 &\leq C_k \frac{1}{e^k} \sum_j \inf_{z \in Q_j} M_{L(\log L)^{\frac{1}{s} + \epsilon}, \alpha} \omega_j(z) \int_{Q_j} \Phi_{\frac{1}{s}} \left( \frac{|f(x)|}{\lambda} \right) dx \\
 &\leq C_k \frac{1}{e^k} \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}} \left( \frac{|f(x)|}{\lambda} \right) M_{L(\log L)^{\frac{1}{s} + \epsilon}, \alpha} \omega(x) dx.
 \end{aligned}$$

The remainders of  $L_{31}$  can be treated in the similar way. Repeated reasoning as  $L_{3.1}$  may lead to our desired estimate for  $L_{3.2}$ . This completes the proof for Case 2 and therefore also finishes the proof of Theorem 1.2.  $\square$

## References

- [1] M. J. Carro, C. Pérez, F. Soria, and J. Soria, Maximal functions and the control of weighted inequalities for the fractional integral operator, *Indiana Univ. Math. J.*, 54(3) (2005), 627–644.
- [2] X. Chen and Q. Xue, Weighted estimates for a class of multilinear fractional type operators, *J. Math. Anal. Appl.*, 362(2) (2010), 355–373.
- [3] R. R. Coifman, R. Rochberg, and G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. Math.*, 103(3) (1976), 611–635.
- [4] D. Cruz-Uribe and A. Fiorenza, Endpoint estimates and weighted norm inequalities for commutators of fractional integrals, *Publ. Mat.*, 47(1) (2003), 103–131.
- [5] D. Cruz-Uribe and A. Fiorenza, Weighted endpoint estimates for commutators of fractional integrals, *Czechoslovak Math. J.*, 57 (2007), 153–160.
- [6] D. V. Cruz-Uribe, J. M. Martell, and C. Pérez, *Weights, Extrapolation and the Theory of Rubio de Francia*, volume 215 of *Operator Theory: Advances and Applications*, Birkhäuser/Springer Basel AG, Basel, 2011.
- [7] Y. Ding, S. Lu, and P. Zhang, Weak estimates for commutators of fractional integral operators, *Sci. China Ser. A*, 44(7) (2001), 877–888.
- [8] C. Domingo-Salazar, M. Lacey, and G. Rey, Borderline weak-type estimates for singular integrals and square functions, *Bull. Lond. Math. Soc.*, 48(1) (2016), 63–73.
- [9] C. Fefferman and E. M. Stein, Some maximal inequalities, *Amer. J. Math.*, 93 (1971), 107–115.
- [10] T. Hytönen and C. Pérez, The  $L(\log L)^\epsilon$  endpoint estimate for maximal singular integral operators, *J. Math. Anal. Appl.*, 428(1) (2015), 605–626.
- [11] S. Lan, T. Mei, and Q. Xue, The weighted estimates for multilinear fractional maximal operators with rough kernels, *Sci. Sinica Math.*, 45(12) (2015), 1939–1945.
- [12] K. Li and W. Sun, Two weight norm inequalities for the bilinear fractional integrals, *Manuscripta Math.*, 150 (2016), 159–175.
- [13] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.*, 165 (1972), 207–226.
- [14] B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for fractional integrals, *Trans. Amer. Math. Soc.*, 192 (1974), 261–274.
- [15] C. Ortiz-Caraballo, Quadratic  $A_1$  bounds for commutators of singular integrals with BMO functions, *Indiana Univ. Math. J.*, 60(6) (2011), 2107–2129.

- [16] C. Ortiz-Caraballo, C. Pérez, and E. Rela, Improving bounds for singular operators via sharp reverse Hölder inequality for  $A_\infty$ , in *Advances in Harmonic Analysis and Operator Theory*, volume 229 of *Oper. Theory Adv. Appl.*, pages 303–321. Birkhäuser/Springer Basel AG, Basel, 2013.
- [17] C. Pérez, Weighted norm inequalities for singular integral operators, *J. London Math. Soc.* (2), 49(2) (1994), 296–308.
- [18] C. Pérez, Endpoint estimates for commutators of singular integral operators, *J. Funct. Anal.*, 128(1) (1995), 163–185.
- [19] C. Pérez and G. Pradolini, Sharp weighted endpoint estimates for commutators of singular integrals, *Michigan Math. J.*, 49(1) (2001), 23–37.
- [20] C. Pérez and I. P. Rivera-Ríos, Borderline weighted estimates for commutators of singular integrals, *Israel J. Math.*, 217(1) (2017), 435–475.
- [21] C. Pérez and R. Trujillo-González, Sharp weighted estimates for multilinear commutators, *J. London Math. Soc.*(2), 65(3) (2002), 672–692.
- [22] M. C. Reguera and C. Thiele, The Hilbert transform does not map  $L^1(Mw)$  to  $L^{1,\infty}(w)$ , *Math. Res. Lett.*, 19(1) (2012), 1–7.