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Inside the Light Boojums: a Journey to the Land of Boundary Defects

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Abstract. We consider minimizers of the energy

$$E_\varepsilon(u) =: \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2-1)^2\right] dx + \frac{1}{2\varepsilon^s}\int_{\partial\Omega} W(u,g)ds,\ \ u:\Omega\to\mathbb{C},\ \ 0<s<1,$$

in a two-dimensional domain Ω, with weak anchoring potential

$$W(u,g) =: \frac{1}{2}(|u|^2-1)^2 + (\langle u,g\rangle - \cos\alpha)^2, \quad 0<\alpha<\frac{\pi}{2}.$$

This functional was previously derived as a thin-film limit of the Landau-de Gennes energy, assuming weak anchoring on the boundary favoring a nematic director lying along a cone of fixed aperture, centered at the normal vector to the boundary.
In the regime where $s[\alpha^2+(\pi-\alpha)^2]<\pi^2/2$, any limiting map $u_*:\Omega\to\mathbb{S}^1$ has only boundary vortices, where its phase jumps by either 2α (light boojums) or $2(\pi-\alpha)$ (heavy boojums). Our main result is the fine-scale description of the light boojums.

Key Words: Nematics, thin-film limit, Ginzburg-Landau type energy, weak anchoring, boundary vortices, asymptotic profile.

AMS Subject Classifications: 35J20, 35B25, 35J61, 49K20

1 Introduction

In this paper, we consider minimizers of the following two-dimensional variational functional of Ginzburg-Landau type:

$$E_\varepsilon(u) =: \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2-1)^2\right] dx + \frac{1}{2\varepsilon^s}\int_{\partial\Omega} W(u,g)\, ds. \tag{1.1}$$

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Here,

1. $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, supposed for simplicity simply connected.
2. $u : \Omega \to \mathbb{R}^2 \simeq \mathbb{C}$ belongs to the energy space $H^1(\Omega;\mathbb{C})$.
3. $W(u,g)$ is of the form
$$W(u,g) =: \frac{1}{2}(|u|^2-1)^2 + (\langle u,g\rangle - \cos\alpha)^2, \tag{1.2}$$
with $g : \partial\Omega \to \mathbb{S}^1$ is smooth, $\alpha \in (0,\pi/2)$ and $\langle\ ,\ \rangle$ the scalar product in $\mathbb{R}^2$.
4. $0 < s < 1$ is a parameter indicating the strength of the anchoring term $W(u,g)$.

This problem was derived in [2] as a thin-film limit of the Landau-de Gennes (Q-tensorial) model of nematic liquid crystals. Assuming the physical sample occupies a very thin cylinder over a planar domain $\Omega \subset \mathbb{R}^2$, and restricting to Q-tensors with a fixed eigenvalue in the vertical direction, Golovaty, Montero, & Sternberg [7] proved that the three-dimensional Landau-de Gennes energy Gamma converges to a two-dimensional Ginzburg-Landau functional. While the connection between nematics and the Ginzburg-Landau energy has been well established, the allure of (1.1) arises from the boundary condition imposed on minimizers, via energy penalization (or weak anchoring, in the parlance of liquid crystals). Instead of imposing a Dirichlet condition on the Q-tensor which forces the nematic director to align with the normal vector to the boundary, we may instead assume that it is energetically favorable for the director to lie along a cone of prescribed aperture to the normal. This may be modeled at the Landau-de Gennes level via a Rapini-Papoular surface energy [15], which in the thin-film limit appears as the boundary integral present in E_ε. The given function $g : \partial\Omega \to \mathbb{S}^1$, which may be chosen arbitrarily in the mathematical analysis of minimizers of E_ε, in the physical derivation of the model is given by the square $g = \nu^2$ of the complex representation of the outward unit normal vector $\nu = \nu_1 + \imath\nu_2$ to $\partial\Omega$. As $W(u,g) \geq 0$, with equality precisely when $u = g\,e^{\pm\imath\alpha}$, energy minimization favors u's which lie, on $\partial\Omega$, in the cone of aperture α around the vector g, and we expect to have $u \simeq ge^{\pm\imath\alpha}$ on $\partial\Omega$ most of the time.

The asymptotic analysis of the energy of minimizers of E_ε was undertaken in [2]. Using the bad discs construction as in [3, 14], adapted to problems with boundary penalization (see also [1, 12]), the authors derived a uniform upper bound on the energy of the minimizers u_ε in the complement of a finite number of small discs containing the defects. It follows from this preliminary analysis that any weak limit u_* of u_ε is smooth away from a finite defect set $\mathfrak{S}$, and satisfies $|u_*| = 1$ in $\overline{\Omega} \setminus \mathfrak{S}$ and $u_* = ge^{\pm\imath\alpha}$ on $\partial\Omega \setminus \mathfrak{S}$.

The novelty of this problem is that there are four classes of defects $\zeta \in \mathfrak{S}$ which might occur. As in the Dirichlet problem for Ginzburg-Landau, one may observe interior vortices, of integer degree. For boundary defects, there are three possibilities. First, u_* may jump from $u_* = ge^{+\imath\alpha}$ to $u_* = ge^{-\imath\alpha}$ (or from $ge^{-\imath\alpha}$ to $ge^{+\imath\alpha}$) across a defect $\zeta \in \partial\Omega$, by following the shortest path on $\mathbb{S}^1$, of length 2α. This type of defect is termed a light boojum in [2]. Around ζ, minimizers u_ε satisfy, in a fixed annulus $A_{r,R}(\zeta) =: [B_R(\zeta) \setminus \overline{B}_r(\zeta)] \cap \Omega$, the estimate

$$E_\varepsilon(u_\varepsilon; A_{r,R}(\zeta)) \geq 2\pi \left(\frac{\alpha}{\pi}\right)^2 \ln \frac{R}{r} + \mathcal{O}(1). \tag{1.3}$$

Here and in what follows, we set, for any open set $G \subset \Omega$,

$$E_\varepsilon(u; G) =: \int_G \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2-1)^2\right] dx + \frac{1}{2\varepsilon^s} \int_{\partial\Omega \cap \overline{G}} W(u,g) ds.$$

The second class of boundary defect are heavy boojums, for which u_* jumps from $u_* = ge^{-\imath\alpha}$ to $u_* = ge^{+\imath\alpha}$ by following the longer route on $\mathbb{S}^1$, of length $2(\pi-\alpha)$. The energy cost of such a defect is estimated by

$$E_\varepsilon(u_\varepsilon; A_{r,R}(\zeta)) \geq 2\pi \left(1-\frac{\alpha}{\pi}\right)^2 \ln \frac{R}{r} + \mathcal{O}(1). \tag{1.4}$$

By topological considerations, such heavy (or even heavier) boundary defects have to exist when the boundary datum g has non zero winding number (degree).

The third possibility is that of a boundary vortex for which the phase of u_* jumps by an integer multiple of 2π. However it is easily shown that one of these vortices are more costly than a pair of boojums, one light and one heavy, and so these defects are never present in energy minimizers.

Define, for each $\alpha \in (0, \pi/2)$,

$$C_\alpha =: \left(\frac{\alpha}{\pi}\right)^2 + \left(1-\frac{\alpha}{\pi}\right)^2 < 1.$$

The value of $s\,C_\alpha$ governs the nature of defects: there are only interior defects when $s\,C_\alpha > 1/2$ and only boundary defects when $s\,C_\alpha < 1/2$. In particular, we have the following

Theorem 1.1 ([2]). *Assume $s\,C_\alpha < 1/2$ and $D =: \deg(g, \partial\Omega) > 0$. Let u_ε minimize E_ε in $H^1(\Omega; \mathbb{C})$. Then*

$$E_\varepsilon(u_\varepsilon) = 2\pi s C_\alpha D |\ln \varepsilon| + \mathcal{O}(1), \tag{1.5}$$

and there exist a subsequence $\varepsilon = \varepsilon_n \to 0$ and $2D$ points $\zeta_1, \cdots, \zeta_{2D} \in \partial\Omega$, ordered along the boundary curve, such that:

1. *For every $0 < \sigma < \frac{1}{8} \min_{i \neq j} |\zeta_i - \zeta_j|$, there exists K_σ such that*

$$E_\varepsilon\left(u_\varepsilon; \Omega \setminus \bigcup_j \overline{B}_\sigma(\zeta_j)\right) \leq K_\sigma. \tag{1.6}$$

2. *$u_\varepsilon \rightharpoonup u_*$ weakly in $H^1_{loc}(\Omega \setminus \{\zeta_1, \cdots, \zeta_{2D}\})$, where $u_* \in C^1(\overline{\Omega} \setminus \{\zeta_1, \cdots, \zeta_{2D}\}; \mathbb{S}^1)$ is a harmonic map satisfying $W(u_*, g) = 0$ on $\partial\Omega \setminus \{\zeta_1, \cdots, \zeta_{2D}\}$.*

3. *Each* ζ_{2j-1} *is a light boojum jumping from* $ge^{+\imath\alpha}$ *to* $ge^{-\imath\alpha}$*, and each* ζ_{2j} *is a heavy boojum jumping from* $ge^{-\imath\alpha}$ *to* $ge^{+\imath\alpha}$*.*

In particular, we may write $u_* = e^{\imath\varphi_*}$, where the phase φ_* is smooth in $\overline{\Omega}\setminus\{\zeta_1,\cdots,\zeta_{2D}\}$, and its restriction to $\partial\Omega\setminus\{\zeta_1,\cdots,\zeta_{2D}\}$ has jump discontinuities of amplitude 2α at light boojums and $2(\pi-\alpha)$ at heavy boojums.

Remark 1.1. A word about degree of g and the jumps at the boojums. We choose on $\partial\Omega$ the positive, conterclockwise, orientation. The degree and the jumps are considered with respect to this orientation. At a light boojum ζ, the limit of $u_{*|\partial\Omega}$ on the left of ζ (with respect to the positive orientation) is $g(\zeta)e^{+\imath\alpha}$; on its right, $g(\zeta)e^{-\imath\alpha}$. As we will see in the proof of Lemma 2.2, on small arcs $C_R(\zeta)\cap\overline{\Omega}$, φ_* essentially increases, linearly in the polar angle, from $\theta_0-\alpha+o(1)$ to $\theta_0+\alpha+o(1)$, for an appropriate constant $\theta_0=\theta_0(\zeta)$ such that $e^{\imath\theta_0}=g(\zeta)$. Similarly, at a heavy boojum, the limits of $u_{*|\partial\Omega}$ are respectively $g(\zeta)e^{-\imath\alpha}$ on the left of ζ and $g(\zeta)e^{+\imath\alpha}$ on the right of ζ, and the phase φ_* essentially increases on small arcs $C_R(\zeta)\cap\overline{\Omega}$, linearly, from $\theta_0+\alpha+o(1)$ to $\theta_0+2\pi-\alpha+o(1)$. This monotonicity is determined by our assumption $\deg g>0$. If $\deg g<0$, then the phase decreases on small arcs, and the side limits are reversed. If $\deg g=0$, then there are no boojums at all.

A natural question concerning the boojums is to describe the local behavior of minimizers u_ε in a neighborhood of each defect. From the analysis in [2], the natural scale of boundary defects seems to be ε^s, determined by the strength of the penalization appearing with the boundary energy term. (Intuitively, the scale of a defect is the smallest scale at which one sees, after blowup, non constant functions.) This suggests that boundary defects are significantly larger than interior vortices, which (as always) have characteristic length ε. The goal of this paper is to confirm this intuition, via a fine-scale asymptotic analysis near a singularity. More specifically, our main result identifies the profile of u_ε in a suitable $\mathcal{O}(\varepsilon^s)$ blowup limit around a light boojum. As we will show in Theorem 1.2 below, to each choice of the aperture $\alpha\in(0,\pi/2)$ corresponds a unique non constant profile function, which satisfies an elliptic boundary value problem in the half-plane.

To state the result, we first describe the blowup procedure. For any $y\in\overline{\Omega}$ and $r>0$ we denote

$$\omega_r(y)=:B_r(y)\cap\Omega,^{\dagger} \tag{1.7}$$

so that, for small r, we have $\overline{\omega_r(y)}=\overline{B}_r(y)\cap\overline{\Omega}$. Since $\partial\Omega$ is smooth, there exists $r_0>0$ such that, for every point $y\in\partial\Omega$, there exist a smooth diffeomorphism

$$\Psi=\Psi_y:\ \overline{\omega_{r_0}(y)}\to\overline{B}_1^+(0)=:\{z=(z_1,z_2)\in\overline{B}_1(0);\ z_2\geq 0\} \tag{1.8}$$

and a rotation $\mathfrak{R}=\mathfrak{R}_y$ such that:

1. $\Psi(y)=0$.

†Here and in what follows, $B_r(y)$ (respectively $C_r(y)$) is the open disc (respectively circle) of center y and radius r.

2. $D\Psi(y) = \mathfrak{R}$.

3. Ψ is orientation preserving, and thus $\mathfrak{R}\tau = (1,0)$, $\mathfrak{R}\nu = (0,-1)$, where τ is the unit tangent to $\partial\Omega$ at y for the direct orientation on $\partial\Omega$ (respectively ν is the unit outward normal at y to Ω).

4. $\|D\Psi(z) - \mathfrak{R}\| \le C|z-y|$, $\forall z \in \overline{\omega_r(y)}$, uniformly in y.

Using Ψ, we may blowup a map $u : \overline{\Omega} \to \mathbb{C}$ around a point $y \in \partial\Omega$ at a scale $\lambda \ll 1$ by setting

$$(u)^{y,\lambda}(z) =: u\left(\Psi^{-1}(\lambda z)\right), \quad \forall z \in \overline{B}^{+}_{1/\lambda}(0) =: \{z = (z_1, z_2) \in \overline{B}_{1/\lambda}(0);\, z_2 \ge 0\}. \tag{1.9}$$

We may now state our main result.

Theorem 1.2. *Let u_ε be minimizers of E_ε in $H^1(\Omega;\mathbb{C})$ such that $u_\varepsilon \to u_*$ (possibly along a sequence $\varepsilon = \varepsilon_n \to 0$), and assume that $\zeta \in \partial\Omega$ is a light boojum defect of u_*. Then there exists $y_\varepsilon \in \partial\Omega$ with $y_\varepsilon \to \zeta$ such that (with the notation in* (1.9)*)*

$$v_\varepsilon =: (u_\varepsilon)^{y_\varepsilon, \varepsilon^s} \to g(\zeta)\, v \tag{1.10}$$

locally uniformly on compact sets of $\overline{\mathbb{R}^2_+}$, where $v =: e^{\imath\psi}$, and $\psi \in C^\infty(\overline{\mathbb{R}^2_+};\mathbb{R})$ is the unique (up to a horizontal translation) locally minimizing solution of

$$\begin{cases} \Delta\psi = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial\psi}{\partial\nu} = \sin\psi(\cos\psi - \cos\alpha) & \text{on } \mathbb{R} \simeq \partial\mathbb{R}^2_+, \end{cases} \tag{1.11}$$

satisfying

$$\lim_{x_1 \to \pm\infty} \psi(x_1, 0) = \mp\alpha. \tag{1.12}$$

Here, locally minimizing means

$$\begin{aligned} \int_{\mathbb{R}^2_+} [|\nabla(\psi+\xi)|^2 - |\nabla\psi|^2] + \int_{\mathbb{R}} [(\cos(\psi+\xi) - \cos\alpha)^2 \\ - (\cos\psi - \cos\alpha)^2] \ge 0, \quad \forall \xi \in C^\infty_c(\overline{\mathbb{R}^2_+}). \end{aligned} \tag{1.13}$$

Equivalently, ψ minimizes, with respect to its own boundary value, the local energy, defined on bounded open sets $U \subset \mathbb{R}^2_+$ by

$$E(\psi; U) =: \frac{1}{2}\int_U [|\nabla\psi|^2] + \frac{1}{2}\int_{\mathbb{R}\cap\overline{U}} (\cos\psi - \cos\alpha)^2. \tag{1.14}$$

The uniqueness statement in Theorem 1.2 is due to Cabré & Solà-Morales [6, Theorem 1.2, Theorem 1.5]: the system (1.11)–(1.12) has, up to horizontal translations, exactly one locally minimizing solution.

The idea of the proof is the following. We first show that $|u_\varepsilon|$ is bounded away from zero in a neighborhood of a light boojum. In order to prove this, we rely on the minimality of u_ε. More specifically, for light boojums ζ, the total oscillation of the phase around ζ is expected to be strictly less than π; this is verified along appropriate circular arcs in Lemma 2.2. The fact that u_ε does not vanish in a neighborhood of ζ is then obtained by means of a projection of the image of u_ε; see Lemmas 3.2 and 3.3.

Writing $u_\varepsilon = \rho_\varepsilon e^{\imath\varphi_\varepsilon}$, we derive uniform a priori estimates near the boundary for the rescaled maps $u^{y_\varepsilon,\varepsilon^s}$, the crucial estimate being

$$\rho_\varepsilon \to 1 \text{ uniformly near } \zeta \text{ as } \varepsilon \to 0. \tag{1.15}$$

These estimates rely on a variant of Boyarskiĭ's regularity theorem for elliptic systems (see [4, 5, 11] and Lemma 3.4). They enable the passage to the limit after blowup. The limiting map is of the form $e^{\imath\psi}$, with ψ as in (1.11). The existence of blowup points y_ε such that (1.12) holds requires some work (see Lemma 3.9.) The proof of Theorem 1.2 is then completed in Section 4.

Once the existence of a boundary bad half-disc, of characteristic size ε^s, as above, is proved, natural questions concern its uniqueness, and the possible existence of interior bad discs. It turns out that, at the characteristic scale ε^s, there are no other bad half-discs, and no interior bad discs.

Theorem 1.3. *Let u_ε, ζ and y_ε be as in Theorem* 1.2.

1. *Let $z_\varepsilon \in \partial\Omega$ be such that $z_\varepsilon \to \zeta$ and $|z_\varepsilon - y_\varepsilon| \gg \varepsilon^s$. Then there exists $t \in \{-1, +1\}$ such that, up to a subsequence,*
$$(u_\varepsilon)^{z_\varepsilon,\varepsilon^s} \to g(\zeta)e^{t\imath\alpha}$$
uniformly on compacts of $\overline{\mathbb{R}^2_+}$.

2. *Let $z_\varepsilon \in \Omega$ be such that $z_\varepsilon \to \zeta$ and* $\operatorname{dist}(z_\varepsilon, \partial\Omega) \gg \varepsilon^s$. *Then there exists a constant $\xi \in \mathbb{S}^1$ such that, up to a subsequence,*
$$u_\varepsilon(\varepsilon^s(\cdot - z_\varepsilon)) : \frac{1}{\varepsilon^s}(\Omega - z^\varepsilon) \to \mathbb{C} \textit{ converges to } \xi \textit{ uniformly on compacts of } \mathbb{R}^2.$$

With more effort, it is possible to improve the conclusion of item 2 and obtain the convergence to a constant at scales λ^ε, with $\varepsilon^s \ll \lambda^\varepsilon \ll \operatorname{dist}(z_\varepsilon, \partial\Omega) \ll 1$; however, we do not follow this route here.

Corollary 1.1. *Let u_ε and ζ be as in Theorem* 1.2. *Consider a small fixed neighborhood V of ζ. For small ε, V contains exactly one bad disc, which is a boundary bad disc.*

As was the case for Ginzburg-Landau with $\mathbb{S}^1$-valued Dirichlet conditions, one expects that Theorem 1.1 gives the first term in an asymptotic expansion [3, 14] of the energy near its minimum, revealing a renormalized energy which determines the locations of singularities. A recent result of Ignat & Kurzke [8] proves such a lower bound expansion for a similar Ginzburg-Landau functional with boundary penalization, arising in thin film ferromagnetism. These authors work within the context of Γ-convergence and weak Jacobians, and their method applies more generally to families with bounded energy rather than minimizers, and with a more flexible relation between the boundary penalization and the length scale parameter ε. However, their work also assumes that the energy cost of boundary defects is of a much smaller scale than interior vortices, and hence it is complementary to our analysis. Moreover, our primary interest in this paper is in the fine structure of solutions in the neighborhood of a singularity.

Our methods are limited to the analysis of light boojums. *We do not know what happens near a heavy boojum.* As explained above, our proof relies heavily on a projection method (Lemma 3.2), whose conclusion is that $|u_\varepsilon|$ is far away from zero near a boundary defect ζ, and whose proof requires that the phase turns by less than π around ζ; this is indeed the case when ζ is a light boojum. If we take for granted that $|u_\varepsilon|$ is far away from zero near ζ, then actually $|u_\varepsilon| \to 1$ uniformly near ζ (see (1.15))–this part of the proof does not see the difference between light and heavy boojums. In a neighborhood of a heavy boojum, the phase turns by $2(\pi-\alpha) > \pi$, and hence our method of proof does not apply. Topologically, there is no need for u_ε to vanish at a heavy boojum, and indeed the upper bound construction in [2] is achieved with $\mathbb{S}^1$-valued maps for both species of boojums. However, numerical evidence in [2] suggests that $|u_\varepsilon|$ may not tend to 1 uniformly near a heavy boojum, by contrast with the case of light boojums. If this is indeed the case, then, as explained above, u_ε gets close to 0 near a heavy boojum, making life inside the heavy boojums of a different nature, ...

2 Refining the convergence $u_\varepsilon \to u_*$

In this section we re-examine some of the convergence arguments in [2] to obtain a more precise description of u_ε.

2.1 Basic facts

First, we review some basic estimates and constructions related to minimizers. For fixed $\varepsilon > 0$, minimizers of E_ε in $H^1(\Omega;\mathbb{C})$ exist and satisfy the Euler-Lagrange equations

$$\begin{cases} -\Delta u + \dfrac{1}{\varepsilon^2}(|u|^2-1)u = 0 & \text{in } \ \Omega, \\ \dfrac{\partial u}{\partial \nu} + \dfrac{1}{\varepsilon^s}\left((|u|^2-1)u + [\langle u, g\rangle - \cos\alpha]g\right) = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{2.1}$$

We next recall some a priori bounds obtained in [2, Lemma 3.2] for minimizers of E_ε and, more generally, for solutions of (2.1) satisfying the natural energy bound $E_\varepsilon(u_\varepsilon) \le K|\ln\varepsilon|$.

For any $0 < s < 1$, such u_ε's satisfy:

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_\infty = 1, \tag{2.2a}$$

$$\exists\, C_1 > 0 \text{ such that } |\nabla u_\varepsilon(x)| \le \frac{C_1}{\varepsilon}, \quad \forall x \in \overline{\Omega}, \quad \forall 0 < \varepsilon < 1. \tag{2.2b}$$

2.2 Refined bad discs construction

For $0 < \varepsilon \ll 1$, minimizers u_ε will satisfy $|u_\varepsilon| \simeq 1$ in Ω and $W(u_\varepsilon, g) \simeq 0$ on $\partial\Omega$, except for a finite number of bad discs, which converge to the defects which are observed in the $\varepsilon \to 0$ limit. Following the classical η-compactness techniques [3,14] for the Ginzburg-Landau functional, for any $a \in (0, 1/2)$ one constructs a finite disjoint collection $\{B^j\}_{j=1,\cdots,N_\varepsilon}$ of open discs, whose number $N_\varepsilon \le N_0(a, g, s, \alpha)$ is uniformly bounded, satisfying

$$\left\{x \in \Omega;\ |u_\varepsilon(x)|^2 < 1 - a\sqrt{2}\right\} \cup \left\{x \in \partial\Omega;\, W(u_\varepsilon(x), g(x)) > a^2\right\} \subset \bigcup_{j=1}^{N_\varepsilon} B^j. \tag{2.3}$$

The interior bad discs, centered at points in Ω, have radius $k\varepsilon$, for a constant $k = k(a, g, \Omega) > 0$. The remaining, boundary bad discs, centered at points on $\partial\Omega$, are larger, with radius $k\varepsilon^s$; see [2, Lemma 4.4] for details.

By including the interior bad discs at distance $\approx \varepsilon$ from the boundary into (artificial) boundary bad discs, merging the interior bad discs at relative distance $\approx \varepsilon$, respectively merging the boundary bad discs with the other (interior or boundary) bad discs at distance $\approx \varepsilon^s$ and increasing the value of k, we may also assume that:

1. The distance from an interior bad disc to the boundary is $\gg \varepsilon^s$.
2. The distance between any two interior bad discs is $\gg \varepsilon$.
3. The distance from a boundary bad disc to any other (interior or boundary) bad disc is $\gg \varepsilon^s$.
4. Possibly up to a subsequence $\varepsilon_n \to 0$, the number of bad discs B^j and their type for fixed j (interior or boundary) is the same for each ε.

We now briefly revisit the construction in [2] in order to define the topological invariants associated with the bad discs.

We first consider an interior bad disc $B^j = B_{k\varepsilon}(x_j^\varepsilon)$, for which we set $d_j =: \deg(u_\varepsilon, \partial B^j)$. Let us note that d_j is invariant in the following sense. If $r \ge k\varepsilon$, $B_r(x_j^\varepsilon) \subset \Omega$ and $B_r(x_j^\varepsilon)$ does not intersect any bad disc except B^j, then $d_j = \deg(u_\varepsilon, \partial B_r(x_j^\varepsilon))$. By (2.2b), d_j is uniformly bounded, and thus, up to a subsequence, it does not depend on ε.

The case of a boundary bad disc $B^j = B_{k\varepsilon^s}(x_j^\varepsilon)$ is more involved. To such a disc, we associate two invariants, $n_j \in \mathbb{N}$ and $\tau_j \in \{-1, 0, 1\}$, as follows. Assume that $x_j^\varepsilon \to$

$\zeta \in \partial\Omega$. Locally on $\partial\Omega$ near ζ, write $g = e^{\imath\gamma}$, with smooth γ. Fix $0 < r < R$ such that $B^j \subset B_r(x_j^\varepsilon)$ and such that the closed annular region $A_{r,R}(x_j^\varepsilon)$ is disjoint from any of the other bad discs. When R is sufficiently small, $A_{r,R}(x_j^\varepsilon)$ is simply connected and $\partial\Omega \cap \overline{A_{r,R}(x_j^\varepsilon)}$ consists of two arcs, $\Gamma_{r,R}^\pm$, with $\Gamma_{r,R}^+$ on the right of x_j^ε with respect to the direct orientation of $\partial\Omega$. By definition of bad discs and the choice of r and R, we have $|u_\varepsilon| \geq 1 - \sqrt{2}a > 0$ in $\overline{A_{r,R}(x_j^\varepsilon)}$ and $W(u_\varepsilon, \gamma) < a^2$ on $\Gamma_{r,R}^\pm$. Thus we may write $u_\varepsilon = f e^{\imath\psi}$ in $\overline{A_{r,R}(x_j^\varepsilon)}$, with $f \geq 0$, $f, \psi \in C^1$. Set $\psi^\pm =: \psi_{|\Gamma_{r,R}^\pm}$. Then, by [2, Lemma 5.1], for a sufficiently small, for each of the two boundary arcs $\Gamma_{r,R}^\pm$, we may find integers $t^\pm \in \{-1, 1\}$ and $m^\pm \in \mathbb{Z}$ such that

$$\left|\psi^\pm - \gamma - t^\pm\alpha - 2\pi m^\pm\right| < C_\alpha a \quad \text{on} \quad \Gamma_{r,R}^\pm, \tag{2.4}$$

this is equivalent to $u_\varepsilon \simeq g e^{\imath t^\pm \alpha}$ on $\Gamma_{r,R}^\pm$. If a is sufficiently small, then $C_\alpha a < \alpha/2$, and then $t^\pm$ and $m^\pm$ in (2.4) are unique. We may then set $n_j =: m^+ - m^- \in \mathbb{Z}$ and $\tau_j =: (1/2)[t^+ - t^-] \in \{-1, 0, 1\}$; these integers describe the winding of the phase of u_ε around a boundary bad disc centered at $x_j^\varepsilon \in \partial\Omega$.

It is not clear at this stage that the integers n_j are uniformly bounded. However, the analysis in [2] (see the proof of Theorem 1.1 there) shows that this is indeed the case, and thus we may assume that n_j and τ_j do not depend on ε.

We now introduce the ad hoc notion of essential bad discs: these are either interior bad discs with $d_j \neq 0$, or boundary bad discs with $(\tau_j, n_j) \neq (0, 0)$.

Lemma 2.1. *Let g, D, u_ε, u_*, $\zeta_1, \cdots, \zeta_{2D}$ be as in Theorem* 1.1. *Then*

1. *For every small $\sigma > 0$ and for every $\ell = 1, \cdots, 2D$, $\overline{B}_\sigma(\zeta_\ell) \cap \overline{\Omega}$ contains exactly one essential bad disc.*

2. *This bad disc is a boundary bad disc, of the form $B^j = B_{k\varepsilon^s}(x_j^\varepsilon)$, with $x_j^\varepsilon \in \partial\Omega$, $x_j^\varepsilon \to \zeta_\ell$.*

3. *If ζ_ℓ is a light boojum, then $\tau_j = -1$ and $n_j = 0$.*

4. *If ζ_ℓ is a heavy boojum, then $\tau_j = 1$ and $n_j = 1$.*

In other words, the limiting defects ζ_ℓ are simple, in the sense that each one is the limit of a single essential bad disc, which carries the same topological type. Thus, one cannot have interior discs with nonzero degree colliding with $\partial\Omega$ as $\varepsilon \to 0$, nor can one observe the merging of essential boundary defects in the limit. In particular, the bad disc in Corollary 1.1 is an essential bad disc.

This phenomenon is not specific to our particular problem, and the above lemma can be adapted to similar situations where the number of bad discs is bounded and the upper bound on the energy matches the lower bound up to terms of order one.

Proof. The conclusion of the lemma essentially follows from the method of discs expansion and fusion in [9, 13]. Such a procedure is followed in the proof of Lemma 7.1 of [2]: the bad discs (centered at $x_j^\varepsilon \in \overline{\Omega}$) are expanded in time $t \geq 1$, with radii $R_j(t)$, from a seed size $R_j(1)$, which is initially equal to the bad disc radius $k\varepsilon^s$ (for boundary discs) or $k\varepsilon$ (for interior discs), by setting

$$R_j(t) =: \begin{cases} t^{1/s}k\varepsilon & \text{for an interior bad disc,} \\ tk\varepsilon^s & \text{for a boundary bad disc.} \end{cases}$$

The expansion phase yields a lower bound, depending on t, for the energy of u_ε in the union of the annuli $A_{R_j(1),R_j(t)}(x_j^\varepsilon)$. This lower bound is obtained by applying the adapted analogues of the lower bounds (1.3), (1.4) around each bad disc B^j (see [2, formula (6.1)] for the precise lower bound). The (first step of this) process stops when either two expanded bad discs collide, or an expanded interior bad disc touches $\partial\Omega$. By the assumption on the mutual position of the bad discs, the collision time $t(\varepsilon)$ goes to ∞ as $\varepsilon \to 0$.

At the end of this expansion phase, the lower bound for the energy is

$$\begin{aligned} &E_\varepsilon\Big(u_\varepsilon; \bigcup_j A_{R_j(1),R_j(t(\varepsilon))}(x_j^\varepsilon)\Big) \\ \geq &\Big[\sum 2\pi\left(n_j - \tau_j\frac{\alpha}{\pi}\right)^2 + \sum\frac{\pi}{s}d_j^2\Big]\ln t(\varepsilon) - K \\ \geq &2\pi\, C_\alpha D\ln t(\varepsilon) - K, \end{aligned} \tag{2.5}$$

where the first sum on the right-hand side of (2.5) is over the boundary bad discs, and the second one over the interior bad discs (see [2, formula (7.5)] for the first inequality in (2.5), and [2, Lemma 6.3] for the second one). Furthermore, by investigating the equality case in

$$\sum 2\pi\left(n_j - \tau_j\frac{\alpha}{\pi}\right)^2 + \sum\frac{\pi}{s}d_j^2 \geq 2\pi C_\alpha D,$$

(see [2, proof of Theorem 1.1]), we find that the conclusion of (2.5) can be improved to

$$E_\varepsilon\Big(u_\varepsilon; \bigcup_j A_{R_j(1),R_j(t(\varepsilon))}(x_j^\varepsilon)\Big) \geq (2\pi\, C_\alpha D + \delta)\ln t(\varepsilon) - K, \tag{2.6}$$

for some constant $\delta > 0$, unless there are exactly $2D$ essential discs, each on the boundary, satisfying items 1–4 of the lemma.

The next step is fusion (merging). This is explained in [2, proof of Lemma 7.1] when an expanded bad disc hits an expanded boundary bad disc. When an expanded interior bad disc $B_{R_j(t(\varepsilon))}(x_j^\varepsilon)$ hits the boundary at some point y_j^ε. We then replace it by $B_{2R_j(t(\varepsilon))}(y_j^\varepsilon)$, that we treat in the next steps as a boundary bad disc. The fusion of interior bad discs was already described in [9,13].

Once the touching expanded bad discs are merged, we start a new expansion phase, then a fusion one, and so on; we end up with a single bad disc around each singularity. Assuming that Lemma 2.1 does not hold, we may keep the extra term $\delta\ln t(\varepsilon)$ during all steps and improve the conclusion of [2, Lemma 7.1] from

$$E_\varepsilon\left(u_\varepsilon;\bigcup_\ell (B_\sigma(\zeta_\ell)\cap\Omega)\right)\geq 2\pi C_\alpha\, D\,\ln\frac{\sigma}{\varepsilon^s}-K_\sigma, \tag{2.7}$$

to

$$E_\varepsilon\left(u_\varepsilon;\bigcup_\ell (B_\sigma(\zeta_\ell)\cap\Omega)\right)\geq 2\pi C_\alpha\, D\,\ln\frac{\sigma}{\varepsilon^s}+\delta\,\ln t(\varepsilon)-K_\sigma. \tag{2.8}$$

However, for small ε the lower bound (2.8) contradicts the global upper bound (1.5). This contradiction shows that items 1–4 hold and completes the proof of the lemma. □

Remark 2.1. Using a similar argument, one may prove that the boundary vortices considered in [10] satisfy the conclusion of [10, Theorem 4.2]. The final estimate in [10, proof of Theorem 4.2] seems too optimistic. However, a weaker estimate, in the spirit of the above lemma, suffices to derive that result.

2.3 Variation of the phase

From the convergence stated in Theorem 1.1, along a sequence $\varepsilon=\varepsilon_n\to 0$, we have $u_\varepsilon\to u_*$ weakly in $H^1_{loc}(\Omega\setminus\{\zeta_1,\cdots,\zeta_{2D}\})$, where $u_*\in C^1(\overline{\Omega}\setminus\{\zeta_1,\cdots,\zeta_{2D}\};\mathbb{S}^1)$ is an $\mathbb{S}^1$-valued harmonic map in Ω, and on $\partial\Omega$, we have $u_*=ge^{\pm\imath\alpha}$ except at the defects ζ_j, $j=1,\cdots,2D$. In particular, we may write

$$u_*=e^{\imath\varphi_*}\quad\text{in}\quad\overline{\Omega}\setminus\{\zeta_1,\cdots,\zeta_{2D}\}),\quad\text{with}\ \ \varphi_*\in C^1(\overline{\Omega}\setminus\{\zeta_1,\cdots,\zeta_{2D}\}). \tag{2.9}$$

At any point $\zeta\in\partial\Omega$, we may define the jump of φ_* at ζ by

$$J(\zeta)=:\varphi_*^+(\zeta)-\varphi_*^-(\zeta),\quad\text{where}\quad\varphi_*^\pm(\zeta)=:\lim_{\substack{x\to\zeta^\pm\\ x\in\partial\Omega}}\varphi_*(x), \tag{2.10}$$

as discussed in Remark 1.1, the side limits are calculated with respect to the positive orientation on $\partial\Omega$.

If ζ is not a boojum, then we have $J(\zeta)=0$. By Theorem 1.1 item 3, across a light boojum we have $J(\zeta)=-2\alpha$ and thus $|J(\zeta)|<\pi$, while $J(\zeta)=-2(\pi-\alpha)$ and $|J(\zeta)|>\pi$ at a heavy boojum.

Away from fixed small neighborhoods of the boojums, we have, for small ε, $|u_\varepsilon|>0$, and thus we may write, locally (and actually globally), $u_\varepsilon=\rho_\varepsilon e^{\imath\varphi_\varepsilon}$, with $\rho_\varepsilon=:|u_\varepsilon|$ and $\varphi_\varepsilon\in C^1$.

Lemma 2.2. *Let $\zeta\in\partial\Omega$ and let $\theta_0\in\mathbb{R}$ be such that $g(\zeta)=e^{\imath\theta_0}$.*

1. *For all sufficiently small $\sigma>0$, there exists $R=R_\sigma\in(\sigma/2,\sigma)$ such that, possibly along a further subsequence $\varepsilon=\varepsilon_n\to 0$, we have $u_\varepsilon\to u_*$ uniformly on $C_R(\zeta)\cap\overline{\Omega}$.*

2. *Let $\varepsilon=\varepsilon_n\to 0$ be as in item 1. For every $\overline{\delta}>0$, there exist $\overline{\sigma}>0$ and $\overline{\varepsilon}>0$ such that, for $\sigma<\overline{\sigma}$ and $\varepsilon<\overline{\varepsilon}$, we may write $u_\varepsilon=\rho_\varepsilon e^{\imath\varphi_\varepsilon}$ on $C_{R_\sigma}(\zeta)\cap\overline{\Omega}$, with*

$$\frac{1}{2}|J(\zeta)|-\overline{\delta}\le\max_{C_{R_\sigma}(\zeta)\cap\overline{\Omega}}|\varphi_\varepsilon-\theta_0-2k\pi|\le\frac{1}{2}|J(\zeta)|+\overline{\delta}, \tag{2.11}$$

for some appropriate integer $k=k_\varepsilon\in\mathbb{Z}$.

Proof. 1. Choose $\sigma>0$ for which the annulus $\overline{A_{\sigma/2,\sigma}(\zeta)}$ does not contain any of the boojums ζ_k, with $k=1,\cdots,2D$. By the global bound (1.5) on the energy, Tonelli's theorem and Fatou's lemma, there exists $R=R_\sigma\in(\sigma/2,\sigma)$ with

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_{C_R(\zeta)\cap\overline{\Omega}}\left[\frac{1}{2}|\nabla u_\varepsilon|^2+\frac{1}{4\varepsilon^2}(|u_\varepsilon|^2-1)^2\right]ds_R\\ \le&\limsup_{\varepsilon\to 0}\int_{C_R(\zeta)\cap\overline{\Omega}}\frac{E_\varepsilon(u_\varepsilon;A_{\sigma/2,\sigma}(\zeta))}{\sigma/2}\le C_\sigma\end{aligned} \tag{2.12}$$

along a sequence $\varepsilon=\varepsilon_n\to 0$.

Hence, u_ε is uniformly bounded in $H^1(C_R(\zeta)\cap\Omega)$ and thus, along a further subsequence, we have $u_\varepsilon\to u_*$ uniformly on $C_R(\zeta)\cap\overline{\Omega}$.

2. In view of item 1, it suffices to prove that, with φ_* as in (2.9), there exists some fixed integer $k\in\mathbb{Z}$ such that

$$\lim_{R\to 0}\max_{C_R(\zeta)\cap\overline{\Omega}}|\varphi_*-2k\pi-\theta_0|=\frac{1}{2}|J(\zeta)|,\quad\forall\zeta\in\partial\Omega. \tag{2.13}$$

When ζ is not a defect, this is clear. Assume next that ζ is a defect. For the sake of concreteness, we consider the case of a light boojum; the case of heavy boojums is similar. With no loss of generality, we may assume that $\zeta=0$ and that the oriented unit tangent to $\partial\Omega$ at $\zeta=0$ is $\tau=(1,0)$. Denote $J=:J(0)=-2\alpha$. Let $\Phi(r,\theta)=:(r\cos\theta,r\sin\theta)$. For sufficiently small σ, we have

$$\begin{aligned}&B_\sigma(0)\cap\Omega=\omega_\sigma(0)=\Phi\left(\widetilde{\omega}_\sigma(0)\right),\\ &\quad\text{with}\ \ \widetilde{\omega}_\sigma(0)=:\left\{(r,\theta);\,0<r<\sigma,\,\theta_-(r)<\theta<\theta_+(r)\right\},\end{aligned} \tag{2.14}$$

here, $\theta_\pm(r)$ are differentiable and satisfy

$$\theta_-(r)=\mathcal{O}(r),\quad\theta_+(r)=\pi+\mathcal{O}(r). \tag{2.15}$$

For $0<r\le\sigma$, $\theta_-(r)\le\theta\le\theta_+(r)$ and $x=:\Phi(r,\theta)\in\overline{\omega_\sigma(0)}\setminus\{0\}$, set

$$h(x)=:-\frac{J}{\pi}\theta+\varphi_*^+(0)\quad\text{and}\quad\psi=:\varphi_*-h. \tag{2.16}$$

We next note that ψ has the following properties.

$$\psi \text{ is harmonic in } \omega_\sigma(0) \text{ and continuous in } \overline{\omega_\sigma(0)}\setminus\{0\}, \tag{2.17a}$$

$$\psi \text{ restricted to } [\partial\omega_\sigma(0)]\setminus\{0\} \text{ has a continuous extension (still denoted } \psi) \text{ to } \partial\omega_\sigma(0), \tag{2.17b}$$

$$\text{and this extension satisfies } \psi(0)=0. \tag{2.17c}$$

Property (2.17b) follows from the fact that, thanks to Definition (2.16), h makes the same jump across $\zeta=0$ as φ_*.

We will see below that, under the assumptions (2.17a)–(2.17b), ψ has a removable singularity at the origin, i.e., ψ can be extended as a continuous function to $\overline{\omega_\sigma(0)}$. Assuming this, we conclude the proof of item 2 as follows. On the one hand, since $e^{\imath\varphi_*^+(0)}=g(0)e^{-\imath\alpha}$(see Theorem 1.1), there exists some $k\in\mathbb{Z}$ such that $\varphi_*^+(0)=\theta_0+2k\pi-\alpha$, and therefore (again, by Theorem 1.1) $\varphi_*^-(0)=\theta_0+2k\pi+\alpha$. We next observe that h is increasing in θ for fixed r, and, by (2.15), (2.16) and the (granted) continuity of ψ, the restriction of h to $C_R(\zeta)\cap\overline{\Omega}$ takes values in an interval of the form

$$\begin{aligned}&[\varphi_*^+(0)+o(1),\varphi_*^-(0)+o(1)]=[\theta_0+2k\pi-\alpha+o(1),\theta_0+2k\pi+\alpha+o(1)]\\ =&\left[\theta_0+2k\pi+\frac{1}{2}J+o(1),\theta_0+2k\pi-\frac{1}{2}J+o(1)\right]\quad\text{as}\quad R\to 0.\end{aligned}$$

This implies (2.13) and thus (via item 1) (2.11). It remains to show that the function ψ above is continuous at the boundary point $\zeta=0$. First, we observe that, by Corollary 7.2 of [2], there exists a constant $c_0=c_0(g,\alpha,\Omega)$ such that for, all fixed r, $0<r<\sigma$, we have the upper bound

$$\begin{aligned}&\frac{1}{2}\int_{A_{r,\sigma}(0)}|\nabla u_*|^2\le\liminf_{\varepsilon\to 0}\ E_\varepsilon(u_\varepsilon;A_{r,\sigma}(0))\\ \le&\liminf_{\varepsilon\to 0}\ E_\varepsilon(u_\varepsilon;\Omega\setminus\cup_j\overline{B}_r(\zeta_j))\le 2\pi sC_\alpha\ln\frac{1}{r}+c_0.\end{aligned}\tag{2.18}$$

Note that, by (2.18), (2.16) and the fact that $|\nabla\theta|=1/r$, the function ψ satisfies an upper bound of the same form,

$$\int_{A_{r,\sigma}(0)}|\nabla\psi|^2\le c_2\ln\frac{1}{r}+c_1,\tag{2.19}$$

with constants c_1,c_2 which are independent of $r\in(0,\sigma)$.

Consider a conformal bi-Lipschitz transformation $\Xi:\overline{\Omega}\to\Xi(\overline{\Omega})$, which maps the origin into itself, $\overline{\Omega}$ into a region contained in the upper half plane, and straightens the boundary near the origin. With no loss of generality, we may assume that $\Xi(\partial\Omega\cap\overline{B}_\sigma(0))\subset\mathbb{R}\times\{0\}\simeq\mathbb{R}$. Choose $r_1>0$ such that $B_{r_1}^+(0)=:B_{r_1}(0)\cap\{\operatorname{Im}z>0\}\subset\Xi(\omega_\sigma(0))$. Since Ξ is bi-Lipschitz, there exists a constant $c<1$, independent of $r<r_1$, such that

$$\Xi^{-1}(B_{r_1}^+(0)\setminus\overline{B}_r^+(0))\subset A_{cr,\sigma}(0)\subset\omega_\sigma(0).\tag{2.20}$$

For $z \in \Xi\left(\overline{\omega_\sigma(0)}\right) \setminus \{0\}$, define $v(z) =: \psi(\Xi^{-1}(z))$. Using (2.17a)–(2.17b), (2.20) and the conformal invariance of the Dirichlet integral, we find that

$$v \text{ is harmonic in } B^+_{r_1}(0) \text{ and continuous in } \overline{B^+_{r_1}(0)} \setminus \{0\}; \tag{2.21a}$$

$$v \text{ restricted to } [\partial B^+_{r_1}(0)] \setminus \{0\} \text{ has a continuous extension to } \partial B^+_{r_1}(0); \tag{2.21b}$$

$$\int_{B^+_{r_1} \setminus B^+_r} |\nabla v|^2 \le c_2 \ln \frac{1}{r} + c_3, \tag{2.21c}$$

with c_3 independent of r. In order to complete the proof of the lemma, it suffices to show that, under the assumptions (2.21a)–(2.21c), v has an extension continuous up to the origin. By (2.21c), there exists a sequence $\{\rho_j\}$ such that $0 < \rho_j < r_1$, $\forall j$, $\rho_j \to 0$ and

$$\int_{C_{\rho_j}(0) \cap \{\operatorname{Im} z \ge 0\}} |\nabla v|^2 \le \frac{2c_2}{\rho_j}, \quad \forall j. \tag{2.22}$$

(2.22) is easily obtained, via (2.21c), by arguing by contradiction. By (2.22) and Cauchy-Schwarz, the oscillation osc of v on $C_{\rho_j}(0) \cap \{\operatorname{Im} z \ge 0\}$ satisfies

$$\begin{aligned} \operatorname{osc}(v, C_{\rho_j}(0) \cap \{\operatorname{Im} z \ge 0\}) &\le \int_{C_{\rho_j}(0) \cap \{\operatorname{Im} z \ge 0\}} |\nabla v| \\ &\le (\pi\rho_j)^{1/2} \left(\int_{C_{\rho_j}(0) \cap \{\operatorname{Im} z \ge 0\}} |\nabla v|^2 \right)^{1/2} \le (2\pi c_2)^{1/2}. \end{aligned} \tag{2.23}$$

Combining (2.23) with the fact that v is uniformly bounded at the endpoints of $C_{\rho_j}(0) \cap \{\operatorname{Im} z \ge 0\}$, we find that v is uniformly bounded on $C_{\rho_j}(0) \cap \{\operatorname{Im} z \ge 0\}$. The maximum principle applied to v in $B^+_{r_1}(0) \setminus \overline{B}_{\rho_j}(0)$ implies that v is bounded in the closure of this set, with bounds independent of j. Letting $j \to \infty$, we find that v is bounded in $\overline{B^+_{r_1}(0)} \setminus \{0\}$.

Next, let $\widetilde{v}$ be the continuous function on $\partial B^+_{r_1}(0)$ which agrees with v outside the origin, and let w be the harmonic extension of $\widetilde{v}$ to $B^+_{r_1}(0)$ (so that w is continuous up to the boundary). In order to conclude, it suffices to prove that $V =: v - w$ vanishes everywhere in $\overline{B^+_{r_1}(0)} \setminus \{0\}$. In turn, the equality $V = 0$ is a consequence of the following straightforward properties of V:

$$V \text{ is harmonic in } B^+_{r_1}(0) \text{ and continuous and bounded in } \overline{B^+_{r_1}(0)} \setminus \{0\}, \tag{2.24a}$$

$$V \text{ vanishes everywhere on } \partial B^+_{r_1}(0) \setminus \{0\}. \tag{2.24b}$$

Indeed, let $\delta > 0$ and set

$$V^\delta(x) =: V(x) - \delta \ln \frac{|x|}{r_1}, \quad \forall x \in B^+_{r_1}(0) \setminus \{0\}.$$

By (2.24a)–(2.24b), V^δ is harmonic in $B^+_{r_1}(0)$ and continuous in $\overline{B^+_{r_1}(0)} \setminus \{0\}$, $\lim_{x\to 0} V^\delta(x) = \infty$ and $V^\delta \geq 0$ on $\partial B^+_{r_1}(0) \setminus \{0\}$. By the maximum (or rather minimum) principle, we have $V^\delta \geq 0$ in $\overline{B^+_{r_1}(0)} \setminus \{0\}$. By letting $\delta \to 0$, we find that $V \geq 0$. Similarly, by considering

$$x \mapsto V(x) + \delta \ln \frac{|x|}{r_1}, \quad \forall x \in B^+_{r_1}(0) \setminus \{0\}, \quad \forall \delta > 0,$$

we find that $V \leq 0$ in $\overline{B^+_{r_1}(0)} \setminus \{0\}$, whence the conclusion $V = 0$ in $\overline{B^+_{r_1}(0)} \setminus \{0\}$. The proof of Lemma 2.2 is completed. □

Remark 2.2. The argument used to establish item 2 proves a property of u_* near the defects which is common in the Ginzburg-Landau theory. More specifically, assume e.g., that ζ is a light boojum. Assume moreover that the unit tangent vector to $\partial\Omega$ at ζ is $(1,0)$. Then the proof of Lemma 2.2 implies that, near ζ, we may write

$$u_* = e^{\imath\left[-\frac{J}{\pi}\theta - \alpha\right]} e^{\imath(\theta_0 + \psi)} = \left(\frac{x-\zeta}{|x-\zeta|}\right)^{\frac{2\alpha}{\pi}} e^{-\imath\alpha} e^{\imath\widetilde{\psi}},$$

where θ is the polar angle as in (2.16) and $\widetilde{\psi}$ is harmonic in $\omega_\sigma(\zeta)$ and continuous in $\overline{\omega_\sigma(\zeta)}$, with $\widetilde{\psi}(\zeta) = \theta_0$. This is the analogue of the canonical harmonic map in [3]. While we have only stated an estimate on the oscillation of the phase φ_ε on circular arcs around a defect, in fact we have shown that the minimizer u_ε closely resembles (up to a constant multiplicative factor) $e^{\imath h}$, which interpolates the phase linearly around the jump.

3 Blowup analysis

In this section, we derive the sharp estimates needed to justify the blowup at scale ε^s near a light boojum ζ.

3.1 Localizing the image of u_ε near a light boojum

Our first task in the proof of Theorem 1.2 will be to show that u_ε is bounded away from zero in a neighborhood of ζ. This is achieved by constructing, in Lemma 3.2, a convenient projection Π which reduces the energy inside $\omega_R(\zeta)$, for some convenient small R. We start with a preliminary remark.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^m$, $U \subset \mathbb{R}^n$, $\Pi : U \to \mathbb{R}^k$ be an L-Lipschitz function*‡ *and $u \in H^1_{loc}(\Omega, U)$. Then $\Pi \circ u \in H^1_{loc}(\Omega, \mathbb{R}^k)$ and*

$$|\nabla(\Pi \circ u)| \leq L\,|\nabla u| \quad a.e.\ in \quad \Omega. \tag{3.1}$$

‡Here and in what follows, Lipschitz constants are calculated with respect to the standard Euclidean norms.

Proof. We extend Π to an L-Lipschitz function on $\mathbb{R}^n$, still denoted Π. We are then in position to smooth both u and Π (without increasing the Lipschitz constant of Π) and thus it suffices to prove (3.1) when both u and Π are smooth. In this case, we have, with (e_j) the canonical basis of $\mathbb{R}^m$,

$$|\nabla u(x)|^2 = \sum_{j=1}^m |\partial_j u(x)|^2 = \lim_{h\to 0}\sum_{j=1}^m \frac{|u(x+he_j)-u(x)|^2}{h^2},$$

and similarly for $\Pi\circ u$. We find that

$$\begin{aligned}|\nabla(\Pi\circ u)(x)|^2 =& \lim_{h\to 0}\sum_{j=1}^m \frac{|(\Pi\circ u)(x+he_j)-(\Pi\circ u)(x)|^2}{h^2}\\ \le& L^2\lim_{h\to 0}\sum_{j=1}^m \frac{|u(x+he_j)-u(x)|^2}{h^2} = L^2|\nabla u(x)|^2.\end{aligned}$$

Thus, we complete the proof. □

Lemma 3.2. *Consider $\alpha,\beta,\gamma,\delta>0$ such that*

$$\alpha+\delta<\beta<\gamma<\frac{\pi}{2}, \tag{3.2a}$$

$$\cos\gamma+\sin\delta\le\cos\alpha. \tag{3.2b}$$

Let

$$\mathbb{S}^1_\delta =: \{e^{\imath\varphi};\ |\varphi|\le\delta\}. \tag{3.3}$$

For $0<\lambda<1$, set

$$U=U_\lambda =: \{z\in\mathbb{C};\ |z|\le 1+\lambda\},\ V=V_{\lambda,\beta} =: \{\rho\, e^{\imath\theta};\ 1-\lambda\le\rho\le 1+\lambda,\ |\theta|\le\beta\}, \tag{3.4a}$$

$$W=W_{\lambda,\gamma} =: \{z=\rho e^{\imath\theta}\in U;\ \mathrm{Re}\, z\ge\cos\gamma,\ |\theta|\le\gamma\}\subset\{z\in\mathbb{C};\ |z|\ge\cos\gamma\}. \tag{3.4b}$$

Then, for sufficiently small λ (depending on α, β, γ and δ), there exists a function $\Pi: U\to W$ such that:

$$\Pi \text{ is } 1\text{-Lipschitz}, \tag{3.5a}$$

$$\Pi=\mathrm{Id}\quad \text{on } V, \tag{3.5b}$$

$$(\langle w,\Pi(z)\rangle-\cos\alpha)^2\le(\langle w,z\rangle-\cos\alpha)^2,\quad \forall z\in U,\quad w\in\mathbb{S}^1_\delta,^{\S} \tag{3.5c}$$

$$(1-|\Pi(z)|^2)^2\le(1-|z|^2)^2,\quad \forall z\in U. \tag{3.5d}$$

[§] Here and in what follows, $\langle\ ,\ \rangle$ denotes the real scalar product of complex numbers.

Proof. Set

$$U^1 =: \{z \in U; \operatorname{Re} z \geq 0\}, \quad U^2 =: \{\rho e^{\imath\theta}; 0 \leq \rho \leq 1+\lambda, |\theta| \leq \gamma\} \subset U^1.$$

We will construct $\Pi_1 : U \to U^1$, $\Pi_2 : U^1 \to U^2$, $\Pi_3 : U^2 \to W$ satisfying (3.5a)–(3.5d), for small λ and for z in the respective domain of definition for each map Π_j, $j = 1,2,3$. Then clearly $\Pi =: \Pi_3 \circ \Pi_2 \circ \Pi_1$ has all the required properties.

Step 1. Construction and properties of Π_1. We let

$$\Pi_1(x+\imath y) =: \begin{cases} x+\imath y, & \text{if } \ x \geq 0, \\ -x+\imath y, & \text{if } x < 0. \end{cases}$$

Clearly, $\Pi_1 : U \to U^1$ satisfies (3.5a), (3.5b) and (3.5d).

We claim that (3.5c) holds for Π_1 provided

$$(1+\lambda)\sin\delta \leq \cos\alpha \tag{3.6}$$

which, for small λ, follows from (3.2b).

Indeed, we have to check (3.5c) only for $z = x+\imath y \in U$, with $x \leq 0$. For such z and for $w = w_1 + \imath w_2 \in \mathbb{S}^1_\delta$, we have

$$(\langle w, -x+\imath y\rangle - \cos\alpha)^2 - (\langle w, x+\imath y\rangle - \cos\alpha)^2 = \underbrace{-2w_1 x}_{\geq 0}[2w_2 y - 2\cos\alpha], \tag{3.7}$$

and thus (3.5c) amounts to the following:

$$[x+\imath y \in U, \ x \leq 0] \implies w_2 y \leq \cos\alpha. \tag{3.8}$$

In turn, (3.8) follows from

$$w_2 y \leq |w_2|\,|x+\imath y| \leq (1+\lambda)\sin\delta \leq \cos\alpha \text{ (by (3.6))}.$$

Step 2. Construction and properties of Π_2. If $x+\imath y \in U^1$, then we may write $x+\imath y = \rho e^{\imath\theta}$, with $0 \leq \rho \leq 1+\lambda$ and $-\pi/2 \leq \theta \leq \pi/2$. Set

$$\Phi : [-\pi/2, \pi/2] \to [-\gamma, \gamma], \quad \Phi(\theta) =: \begin{cases} \theta, & \text{if } \ |\theta| \leq \gamma, \\ \gamma, & \text{if } \ \gamma < \theta \leq \pi/2, \\ -\gamma, & \text{if } \ -\pi/2 \leq \theta < -\gamma. \end{cases}$$

We let $\Pi_2(\rho e^{\imath\theta}) =: \rho e^{\imath\Phi(\theta)}$, $\forall 0 \leq \rho \leq 1+\lambda$, $\forall -\pi/2/ \leq \theta \leq \pi/2$. Clearly, Π_2 maps U^1 into U^2 and satisfies (3.5b) (since $\gamma > \beta$, by (3.2a)) and (3.5d).

We check (3.5a) (for Π_2). We actually claim that

$$\left|\rho e^{\imath\Phi(\theta)} - r e^{\imath\Phi(\varphi)}\right|^2 \leq \left|\rho e^{\imath\theta} - r e^{\imath\varphi}\right|^2, \quad \forall r, \rho \geq 0, \quad |\theta| \leq \pi/2, \quad |\varphi| \leq \pi/2. \tag{3.9}$$

Indeed, with θ, φ as above, (3.9) amounts to

$$\cos(\theta-\varphi)\le\cos\left(\Phi(\theta)-\Phi(\varphi)\right). \tag{3.10}$$

Assuming, with no loss of generality, that $\theta\ge\varphi$, (3.10) follows from

$$0\le\Phi(\theta)-\Phi(\varphi)\le\theta-\varphi\le\pi.$$

We next turn to the proof of (3.5c) for Π_2. We have to prove that, for small λ, we have

$$\begin{aligned}&(\rho\cos(\Phi(\theta)-\varphi)-\cos\alpha)^2\\ \le&(\rho\cos(\theta-\varphi)-\cos\alpha)^2,\quad\forall 0\le\rho\le 1+\lambda,\quad\gamma\le|\theta|\le\pi/2,\quad|\varphi|\le\delta.\end{aligned} \tag{3.11}$$

Note that (3.11) becomes an equality when $|\theta|\le\gamma$. We may thus assume that $|\theta|\ge\gamma$.

With ρ, θ, φ as above, we have $\Phi(\varphi)=\varphi$ (since $\delta<\gamma$, by (3.2a)), and thus

$$\begin{aligned}&(\rho\cos(\Phi(\theta)-\varphi)-\cos\alpha)^2-(\rho\cos(\theta-\varphi)-\cos\alpha)^2\\ =&\underbrace{\rho\left[\cos(\Phi(\theta)-\varphi)-\cos(\theta-\varphi)\right]}_{\ge 0,\text{ by (3.10)}}\times[\rho\cos(\Phi(\theta)-\varphi)+\rho\cos(\theta-\varphi)-2\cos\alpha].\end{aligned} \tag{3.12}$$

Therefore, (3.11) will follow from

$$\begin{aligned}&\rho\cos(\Phi(\theta)-\varphi)+\rho\cos(\theta-\varphi)\\ \le&2\cos\alpha,\quad\forall 0\le\rho\le 1+\lambda,\quad\gamma\le|\theta|\le\pi/2,\quad|\varphi|\le\delta.\end{aligned} \tag{3.13}$$

In turn, (3.13) is, via (3.10), a consequence of

$$\begin{aligned}\rho\cos(\Phi(\theta)-\varphi)+\rho\cos(\theta-\varphi)\le&2\rho\cos(\Phi(\theta)-\varphi)\le 2(1+\lambda)\cos(\gamma-\delta)\\ <&2\cos\alpha\ \text{ for small }\lambda\text{ (by (3.2a)).}\end{aligned}$$

In conclusion, Π_2 satisfies (3.5a)–(3.5d).

Step 3. Construction and properties of Π_3. If $x+\imath y\in U^2$, we set

$$\Pi_3(x+\imath y)=:\begin{cases}x+\imath y, & \text{if }\ x>\cos\gamma,\\ \cos\gamma+\imath y, & \text{if }\ x\le\cos\gamma.\end{cases}$$

Then Π_3 is 1-Lipschitz, since it is the nearest point projection on the convex set W.

On the other hand, we note that

$$x+\imath y\in V\implies x\ge(1-\lambda)\cos\beta>\cos\gamma\ \text{ for small }\lambda\text{ (by (3.2a)),}$$

and thus (3.5b) holds for small λ.

We check (3.5d) for Π_3. It suffices to consider $x+\imath y\in U^2$, with $x\le\cos\gamma$. We write $x+\imath y=\rho e^{\imath\theta}$, with $0\le\rho\le 1+\lambda$ and $|\theta|\le\gamma$. Since $x\le\cos\gamma$, we have $\rho\le 1$, and thus

$$|x+\imath y|^2=x^2+y^2\le|\Pi_3(x+\imath y)|^2=\cos^2\gamma+\rho^2\sin^2\theta\le\cos^2\gamma+\sin^2\gamma=1, \tag{3.14}$$

whence (3.5d).

Finally, we check (3.5c) for Π_3. Again, it suffices to consider $x+\imath y\in U^2$, with $x\le\cos\gamma$ (and thus, by the above, $|y|\le 1$). Arguing as for (3.7), it suffices to have

$$w_1(x+\cos\gamma)+2w_2y\le 2\cos\alpha,\quad \forall w=w_1+\imath w_2\in\mathbb{S}^1_\delta. \tag{3.15}$$

In order to prove (3.15), we note that

$$w_1(x+\cos\gamma)+2w_2y\le 2w_1\cos\gamma+2|w_2|\le 2\cos\gamma+2\sin\delta\le 2\cos\alpha\ \ \text{(by (3.2b))}.$$

The proof of Lemma 3.2 is completed. □

As a consequence of Lemma 3.2, a minimizer u_ε of E_ε which lives on the boundary of a domain $D\subset\Omega$, near an arc of $\mathbb{S}^1$ of length $<\pi$ is far away from zero in all the domain. More precisely, we have the following result.

Lemma 3.3. *Let α, β, γ and δ satisfy* (3.2a)–(3.2b). *Let $\lambda>0$ be such that* (3.5a)–(3.5d) *hold. Let $D\subset\Omega$ be a Lipschitz domain. Assume that:*

$$u=u_\varepsilon \text{ is a minimizer of (1.1)}, \tag{3.16a}$$
$$\partial\Omega\cap\overline{D}\ne\emptyset \text{ and } \partial D\cap\Omega\ne\emptyset, \tag{3.16b}$$
$$\overline{D} \text{ is simply connected}, \tag{3.16c}$$
$$g(\partial\Omega\cap\overline{D})\subset\{e^{\imath\theta};\ |\theta|\le\delta\}, \tag{3.16d}$$
$$u(\partial D\cap\Omega)\subset\{\rho\, e^{\imath\theta};\ 1-\lambda\le\rho\le 1+\lambda,\ |\theta|\le\beta\}. \tag{3.16e}$$

Then, for sufficiently small ε, we have

$$\begin{aligned}u(\overline{D})\subset&\{\rho e^{\imath\theta};\ 0\le\rho\le 1+\lambda,\ |\theta|\le\gamma,\ \rho\cos\theta\ge\cos\gamma\}\\ \subset&\{\rho e^{\imath\theta};\ \cos\gamma\le\rho\le 1+\lambda,\ |\theta|\le\gamma\}.\end{aligned} \tag{3.17}$$

In particular, we may write $u=\rho e^{\imath\varphi}$ in $\overline{D}$, with

$$\rho\in H^1(D)\cap C(\overline{D}),\quad \cos\gamma\le\rho\le 1+\lambda, \tag{3.18a}$$
$$\varphi\in H^1(D)\cap C(\overline{D}),\quad |\varphi|\le\gamma. \tag{3.18b}$$

Proof. Recall that u is continuous, and even C^1, in $\overline{\Omega}$; see (2.2b). We consider, as a competitor in (1.1),

$$v=:\begin{cases}u, & \text{in }\ \overline{\Omega}\setminus\overline{D},\\ \Pi\circ u, & \text{in }\ \overline{D},\end{cases}$$

with Π as in the proof of Lemma 3.2. The fact that v is a competitor follows from (3.5b) and (3.16e).

Write $\Pi_2 \circ \Pi_1 \circ u = f = f_1 + \imath f_2$ and $\Pi_1 \circ u = h = h_1 + \imath h_2$. By Lemma 3.1, (3.5a)–(3.5d) and the minimality of u_ε, we have

$$\begin{aligned}|\nabla(\Pi \circ u)| =&|\nabla(\Pi_3 \circ f)| = |\nabla f| = |\nabla(\Pi_2 \circ h)| = |\nabla h| \\ =&|\nabla(\Pi_1 \circ u)| = |\nabla u| \;\text{ a.e. in }\; D \cap \Omega.\end{aligned} \tag{3.19}$$

We claim that

$$f_1 \geq \cos\gamma \quad \text{in} \quad D \quad (\text{and thus in } \overline{D}). \tag{3.20}$$

Indeed, consider the open set $\omega =: \{z \in D;\, f_1(z) < \cos\gamma\}$, a.e. in ω, we have $|\nabla(\Pi_3 \circ f)|^2 = |\nabla f_2|^2$, while $|\nabla f|^2 = |\nabla f_1|^2 + |\nabla f_2|^2$. We find that $\nabla f_1 = 0$ a.e. in ω, which implies $\nabla[(\cos\gamma - f_1)_+] = 0$ a.e. in D. Since D is connected, we find that $(\cos\gamma - f_1)_+$ is constant a.e. (and thus everywhere) in D, and therefore also constant in $\overline{D}$. Since, on the other hand, $(\cos\gamma - f_1)_+ = 0$ on the non empty set $\partial\Omega \cap \overline{D}$ (by (3.16b) and (3.16e)), we find that $f_1 \geq \cos\gamma$ in $\overline{D}$, as claimed.

By (2.2a) and (3.20), for small ε we have $\cos\gamma \leq |h| \leq 1 + \lambda$; thus, thanks to (3.16c), we may write $h = \rho e^{\imath\varphi}$ in $\overline{D}$, with

$$\rho \in H^1(D) \cap C(\overline{D}), \quad \cos\gamma \leq \rho \leq 1 + \lambda, \tag{3.21a}$$

$$\varphi \in H^1(D) \cap C(\overline{D}), \quad -\pi/2 \leq \varphi \leq \pi/2. \tag{3.21b}$$

By (3.5b), and (3.16e), we have

$$\varphi(\partial D \cap \Omega) \subset [-\beta, \beta]. \tag{3.22}$$

We claim that

$$\varphi(\overline{D}) \subset [-\gamma, \gamma]. \tag{3.23}$$

Indeed, we prove that $\varphi \leq \gamma$ in $\overline{D}$. For this purpose, we consider the open set $\widetilde{\omega} =: \{z \in D;\, \varphi(z) > \gamma\}$, a.e. in $\widetilde{\omega}$, we have

$$|\nabla h|^2 = |\nabla\rho|^2 + \rho^2\,|\nabla\varphi|^2 \quad \text{and} \quad |\nabla f|^2 = |\nabla\rho|^2. \tag{3.24}$$

By (3.19) and (3.24), we obtain $\nabla\varphi = 0$ a.e. in $\widetilde{\omega}$. Combining this with (3.22), we obtain (as above) that $(\varphi - \gamma)_+ = 0$ in $\overline{D}$. Similarly, we have $(\varphi + \gamma)_- = 0$ in $\overline{D}$, whence (3.23).

By (2.2a), (3.20) and (3.23), for small ε we have $h(\overline{D}) \subset W$ (with W as in (3.4b)), and thus

$$u(\overline{D}) \subset Z =: (\Pi_1)^{-1}(W). \tag{3.25}$$

Now Z has two connected components, one of which is W. Since $u(\overline{D})$ is connected and, by (3.16b) and (3.16e), $u(\overline{D})$ intersects W, we find that $u(\overline{D}) \subset W$, i.e., (3.17) holds. The proof of Lemma 3.3 is completed. □

3.2 Uniform estimates at the ε^s scale

Lemma 3.5 below improves the conclusion $\rho \geq \cos\gamma$ to $\rho \to 1$ locally uniformly. This, in turn, leads to a number of uniform estimates at the ε^s scale.) This relies on the following variant of Boyarskiĭ's regularity result for elliptic systems in divergence form in the plane [4,5], in the form presented in Meyers [11].

Lemma 3.4. *Let $\mathbb{D}$ be the unit disc. Let $\lambda, M \in (0,\infty)$ be fixed. Let $A:\mathbb{D}\to M_2(\mathbb{R})$ satisfy*

$$(A(x)\xi)\cdot\xi \geq \lambda|\xi|^2, \qquad \forall x\in\mathbb{D}, \quad \forall \xi\in\mathbb{R}^2,^{\P} \tag{3.26a}$$

$$\|A(x)\| \leq M, \qquad \forall x\in\mathbb{D}. \tag{3.26b}$$

Then there exists some $2<p\leq 4$ (depending only on λ and M) such that the problem

$$\begin{cases}\operatorname{div}(A\nabla v)=0 & \text{in } \ \mathbb{D},\\ v=f\in H^1(\mathbb{S}^1) & \text{on } \ \mathbb{S}^1,\end{cases} \tag{3.27}$$

admits a (unique) solution $v\in W^{1,p}(\mathbb{D})$, and in addition we have

$$\|\nabla v\|_{L^p(\mathbb{D})} \leq C\|f'\|_{L^2(\mathbb{S}^1)}. \tag{3.28}$$

Proof. We have $H^1(\mathbb{S}^1)\hookrightarrow W^{3/4,4}(\mathbb{S}^1)$, and thus there exists $w\in W^{1,4}(\mathbb{D})$ such that $w=f$ on $\mathbb{S}^1$ and

$$\|\nabla w\|_{L^4(\mathbb{D})} \leq C\|f'\|_{L^2(\mathbb{S}^1)}. \tag{3.29}$$

Writing $v=w+u$, u then solves,

$$\begin{cases}\operatorname{div}(A\nabla u)=\operatorname{div}F & \text{in } \ \mathbb{D},\\ u=0 & \text{on } \ \mathbb{S}^1,\end{cases} \tag{3.30}$$

with $F=:-A\nabla w$ satisfying (by (3.26b) and (3.29))

$$\|F\|_{L^4(\mathbb{D})} \leq C\|f'\|_{L^2(\mathbb{S}^1)}. \tag{3.31}$$

By [11, Theorem 1], there exists some $2<p\leq 4$ (depending only on λ and M) such that (3.31) has a solution in $W^{1,p}(\mathbb{D})$, satisfying the estimate

$$\|\nabla u\|_{L^p(\mathbb{D})} \leq C\|F\|_{L^p(\mathbb{D})}. \tag{3.32}$$

The conclusion of the lemma follows from (3.29), (3.31) and (3.32). □

¶Here and in what follows, $\cdot$ denotes the standard scalar product in $\mathbb{R}^2$.

Lemma 3.5. *Let $s \in (0,1)$ and $A \in (0,\infty)$. Consider $y = y_\varepsilon \in \partial\Omega$. Set $r = r_\varepsilon =: A\varepsilon^s$, $U = U_\varepsilon =: \omega_r(y) = B_r(y) \cap \Omega$, $\omega = \omega_\varepsilon =: \omega_{r/2}(y) = B_{r/2}(y) \cap \Omega$. Assume that $u = u_\varepsilon$ is a critical point of E_ε, satisfying*

$$0 < C_1 \le |u| \le C_2 < \infty \quad \text{in} \quad U, \tag{3.33a}$$

$$u = |u|e^{\imath\varphi}, \quad \text{where} \quad \varphi = \varphi_\varepsilon \in H^1(U) \cap C(\overline{U}) \;\; \text{satisfies} \;\; |\varphi| \le C_3 < \infty, \tag{3.33b}$$

$$|\nabla u| \le \frac{C_4}{\varepsilon} \quad \text{for some} \quad C_4 < \infty, \tag{3.33c}$$

$$\frac{1}{\varepsilon^2}\int_U (1-|u|^2)^2 \le C_5(|\ln\varepsilon| + 1) \quad \text{for some} \quad C_5 < \infty. \tag{3.33d}$$

Then, for sufficiently small ε, we have

$$\|\nabla\varphi\|_{L^2(\omega)} \le C_6 < \infty, \tag{3.34a}$$

$$\|\nabla\varphi\|_{L^p(\omega)} \le C_7\varepsilon^{s(2/p-1)} \quad \text{for some} \quad 2 < p \le 4 \quad \text{and} \quad C_7 < \infty, \tag{3.34b}$$

$$|\varphi|_{C^{0,1-2/p}(\omega)} \le C_8\varepsilon^{s(2/p-1)} \quad \text{for some} \quad C_8 < \infty. \tag{3.34c}$$

The above constants C_6, C_7, C_8 depend on $C_1, \cdots, C_5$ and A, but not on small ε.

Moreover,

$$|u| \to 1 \quad \text{uniformly in} \quad \omega \quad \text{as} \quad \varepsilon \to 0. \tag{3.35}$$

Proof. Set $\Gamma = \Gamma_\varepsilon =: \partial\Omega \cap U$. In what follows, $C, C', \cdots$, denote generic finite positive constants independent of (possibly small) ε, whose values may change in different calculations.

If we let $\rho = \rho_\varepsilon =: |u|$, then ρ and φ satisfy

$$\operatorname{div}(\rho^2\nabla\varphi) = 0 \qquad \text{in} \quad U, \tag{3.36a}$$

$$-\Delta\rho = \frac{1}{\varepsilon^2}\rho(1-\rho^2) - \rho|\nabla\varphi|^2 \qquad \text{in} \quad U, \tag{3.36b}$$

$$\left|\frac{\partial\varphi}{\partial\nu}\right| \le \frac{C}{\varepsilon^s} \qquad \text{on} \quad \Gamma, \tag{3.36c}$$

$$\left|\frac{\partial\rho}{\partial\nu}\right| \le \frac{C}{\varepsilon^s} \qquad \text{on} \quad \Gamma, \tag{3.36d}$$

for (3.36c) and (3.36d), we use (3.33a).

Step 1. Proof of (3.34a). Choose a cutoff function $\zeta \in C^\infty(\mathbb{R}^2; [0,1])$ such that

$$\zeta = 1 \quad \text{in} \quad V = V_\varepsilon =: B_{3r/4}(y), \quad \zeta = 0 \quad \text{on} \quad \partial U \setminus \Gamma \quad \text{and} \quad |\nabla\zeta| \le \frac{C}{\varepsilon^s}. \tag{3.37}$$

Multiplying (3.36a) by $\zeta^2\varphi$, we obtain (via (3.33a), (3.33b), (3.36c) and (3.37))

$$\begin{aligned}\int_U \zeta^2\rho^2|\nabla\varphi|^2 &= \int_\Gamma \zeta^2\rho^2\varphi\frac{\partial\varphi}{\partial\nu} - 2\int_U \zeta\rho^2\nabla\varphi\cdot\nabla\zeta \le C' - 2\int_U \zeta\rho^2\nabla\varphi\cdot\nabla\zeta \\ &\le C' + \frac{1}{2}\int_U \zeta^2\rho^2|\nabla\varphi|^2 + 2\int_U \rho^2|\nabla\zeta|^2 \le C'' + \frac{1}{2}\int_U \zeta^2\rho^2|\nabla\varphi|^2,\end{aligned}$$

and therefore

$$\int_V \rho^2|\nabla\varphi|^2 \le \int_U \zeta^2\rho^2|\nabla\varphi|^2 \le C. \tag{3.38}$$

Applying (3.33a) once more, we obtain (3.34a).

Step 2. Proof of (3.34b) and (3.34c). Consider some $R = R_\varepsilon \in (r, 3r/4)$ such that

$$\int_{C(y,R)\cap\Omega} |\nabla\varphi|^2 \le \frac{C}{r} \le \frac{C'}{R} \tag{3.39}$$

the existence of such R follows from (3.38). We consider the conjugate ψ of φ, defined in U, up to a constant, by

$$\partial_y\psi = \rho^2\partial_x\varphi, \quad \partial_x\psi = -\rho^2\partial_y\varphi, \tag{3.40}$$

the existence, for small ε, of ψ, is a consequence of (3.36a) and of the fact that U is simply connected. In view of (3.36c), (3.39) and (3.40), ψ satisfies

$$\operatorname{div}\left(\frac{1}{\rho^2}\nabla\psi\right) = 0 \qquad \text{in} \quad U, \tag{3.41a}$$

$$\left|\frac{\partial\psi}{\partial\tau}\right| \le \frac{C}{\varepsilon^s} \qquad \text{on} \quad \Gamma, \tag{3.41b}$$

$$\int_{C(y,R)\cap\Omega} |\nabla\psi|^2 \le \frac{C'}{R}. \tag{3.41c}$$

Set

$$W = W_\varepsilon =: \omega_R(y) = B_R(y)\cap\Omega. \tag{3.42}$$

We consider a bi-Lipschitz homeomorphism $\Phi = \Phi_\varepsilon : \overline{W} \to \overline{\mathbb{D}}$ such that

$$\frac{C'}{R}|x-y| \le |\Phi(x)-\Phi(y)| \le \frac{C}{R}|x-y|, \quad \forall x, y \in \overline{W}, \quad \text{with} \quad 0 < C' \le C < \infty. \tag{3.43}$$

Set $\xi(z) =: \psi\circ\Phi^{-1}(z)$, $\forall z \in \mathbb{D}$. Using (3.41a)–(3.43), we find that ξ satisfies

$$\operatorname{div}(f\nabla\xi) = 0 \quad \text{in} \quad \mathbb{D} \quad \text{for some} \quad f = f_\varepsilon \quad \text{such that} \tag{3.44a}$$

$$C \le f \le C', \tag{3.44b}$$

$$\int_{\mathbb{S}^1} |\xi'|^2 \le C''. \tag{3.44c}$$

By (3.44a)–(3.44c) and Lemma 3.4, we find that for some $2 < p \le 4$ we have

$$\|\nabla\xi\|_{L^p(\mathbb{D})} \le C. \tag{3.45}$$

Set $\mu =: \varphi\circ\Phi^{-1}$. From (3.45), (3.40) and (3.33a), we find that

$$\|\nabla\mu\|_{L^p(\mathbb{D})} \le C. \tag{3.46}$$

By the Morrey embedding, we also have

$$|\mu|_{C^{0,1-2/p}(\mathbb{D})} \le C. \tag{3.47}$$

Combining (3.46) and (3.47) with (3.43), we obtain

$$\|\nabla\varphi\|_{L^p(W)} \le C\varepsilon^{s(2/p-1)}, \tag{3.48a}$$

$$|\varphi|_{C^{0,1-2/p}(W)} \le C\varepsilon^{s(2/p-1)}, \tag{3.48b}$$

in particular, (3.34b) and (3.34c) hold.

Step 3. Proof of (3.35). Let ζ be a cutoff function as above. We multiply (3.36b) by $\zeta^2(\rho-1)$ and find (using (3.38), (3.36d) and (3.37)) that

$$\begin{aligned}
&\int_U \zeta^2|\nabla\rho|^2 + \frac{1}{\varepsilon^2}\int_U \zeta^2\rho(1+\rho)(1-\rho)^2\\
=&\int_U \zeta^2\rho(1-\rho)|\nabla\varphi|^2 - 2\int_U \zeta(1-\rho)\nabla\rho\cdot\nabla\zeta - \int_\Gamma \zeta^2(1-\rho)\frac{\partial\rho}{\partial\nu}\\
\le& C - 2\int_U \zeta(1-\rho)\nabla\rho\cdot\nabla\zeta\\
\le& C + \frac{1}{2}\int_U \zeta^2|\nabla\rho|^2 + 2\int_U (1-\rho)^2|\nabla\zeta|^2\\
\le& C + \frac{1}{2}\int_U \zeta^2|\nabla\rho|^2 + \frac{C'}{\varepsilon^{2s}}\int_U (1-\rho)^2\\
\le& C'' + \frac{1}{2}\int_U \zeta^2|\nabla\rho|^2,
\end{aligned} \tag{3.49}$$

where in the last line we have used (3.33d) and the assumption $0<s<1$.

Combining this with(3.33a), we find that

$$\int_U \zeta^2|\nabla\rho|^2 + \frac{1}{\varepsilon^2}\int_U \zeta^2(1-\rho)^2 \le C. \tag{3.50}$$

In particular, with W as in (3.42), we have

$$\int_W |\nabla\rho|^2 + \frac{1}{\varepsilon^2}\int_W (1-\rho)^2 \le C. \tag{3.51}$$

If Φ is as in (3.43), then (by (3.51)) $\eta=\eta_\varepsilon =: \rho\circ\Phi^{-1}$ satisfies

$$\int_{\mathbb{D}} |\nabla\eta|^2 + \frac{1}{\varepsilon^{2(1-s)}}\int_{\mathbb{D}} (1-\eta)^2 \le C', \tag{3.52}$$

and thus

$$\eta \rightharpoonup 1 \quad \text{in} \quad H^1(\mathbb{D}) \qquad \text{as} \quad \varepsilon\to 0, \tag{3.53a}$$

$$\int_{\mathbb{S}^1} |1-\eta| \to 0 \qquad \text{as} \quad \varepsilon\to 0. \tag{3.53b}$$

Using (3.33a), we obtain from (3.53a) that

$$\int_{\mathbb{D}} |1-\eta|^q \to 0 \quad \text{as} \quad \varepsilon \to 0, \quad \forall 1 \le q < \infty. \tag{3.54}$$

Going back to ρ, (3.53b) and (3.54) imply (via (3.43))

$$\int_{\partial\Omega\cap B_R(y)} |1-\rho| = o(\varepsilon^s) \qquad \text{as} \quad \varepsilon \to 0, \tag{3.55a}$$

$$\int_W |1-\rho|^q = o(\varepsilon^{2s}) \qquad \text{as} \quad \varepsilon \to 0, \quad \forall 1 \le q < \infty. \tag{3.55b}$$

Let $\lambda \in C^\infty(\mathbb{R}^2;[0,1])$ be a cutoff function such that

$$\lambda = 1 \quad \text{in} \quad \omega, \ \ \lambda = 0 \quad \text{in} \ \ U \setminus W \quad \text{and} \quad |\nabla\lambda| \le \frac{C}{\varepsilon^s}. \tag{3.56}$$

Let $2 < p \le 4$ be as in (3.34b) and let q be the conjugate exponent of $p/2$. We multiply (3.36b) by $\lambda^2(\rho-1)$ and obtain (by (3.55a)–(3.56) and (3.34b)):

$$\begin{aligned}
&\int_U \lambda^2|\nabla\rho|^2 + \frac{1}{\varepsilon^2}\int_U \lambda^2\rho(1+\rho)(1-\rho)^2 \\
=&\int_U \lambda^2\rho(1-\rho)|\nabla\varphi|^2 - 2\int_U \lambda(1-\rho)\nabla\rho\cdot\nabla\lambda - \int_\Gamma \lambda^2(1-\rho)\frac{\partial\rho}{\partial\nu} \\
=&\int_U \lambda^2\rho(1-\rho)|\nabla\varphi|^2 - 2\int_U \lambda(1-\rho)\nabla\rho\cdot\nabla\lambda + o(1) \\
\le& C\left(\int_W |1-\rho|^q\right)^{1/q}\left(\int_W |\nabla\varphi|^p\right)^{2/p} - 2\int_U \lambda(1-\rho)\nabla\rho\cdot\nabla\lambda + o(1) \\
=&o(1) - 2\int_U \lambda(1-\rho)\nabla\rho\cdot\nabla\lambda \\
\le& o(1) + \frac{1}{2}\int_U \lambda^2|\nabla\rho|^2 + 2\int_U (1-\rho)^2|\nabla\lambda|^2 \\
\le& o(1) + \frac{1}{2}\int_U \lambda^2|\nabla\rho|^2 + \frac{C'}{\varepsilon^{2s}}\int_W (1-\rho)^2 \\
\le& o(1) + \frac{1}{2}\int_U \lambda^2|\nabla\rho|^2 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned} \tag{3.57}$$

Combining (3.57) with (3.33a), we find that

$$\frac{1}{\varepsilon^2}\int_\omega (1-\rho)^2 \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.58}$$

On the other hand, by (3.33c) we have

$$|\nabla\rho| \le \frac{C}{\varepsilon}. \tag{3.59}$$

We thus obtain (3.35) from (3.58) and (3.59). The proof of Lemma 3.5 is completed. □

For further use, we note the following interior version of Lemma 3.5.

Lemma 3.6. *Let $s \in (0,1)$ and $A \in (0,\infty)$. Set $r = r_\varepsilon =: A\varepsilon^s$ and consider some $y = y_\varepsilon \in \Omega$ such that $U = U_\varepsilon =: B_r(y) \subset \Omega$. Let $\omega = \omega_\varepsilon =: B_{r/2}(y)$. Assume that $u = u_\varepsilon$ is a critical point of E_ε, satisfying*

$$0 < C_1 \le |u| \le C_2 < \infty \quad \textit{in} \quad U, \tag{3.60a}$$

$$u = |u|e^{\imath\varphi}, \ \textit{where} \quad \varphi = \varphi_\varepsilon \in H^1(U) \cap C(\overline{U}) \ \textit{satisfies} \quad |\varphi| \le C_3 < \infty, \tag{3.60b}$$

$$|\nabla u| \le \frac{C_4}{\varepsilon} \quad \textit{for some} \quad C_4 < \infty, \tag{3.60c}$$

$$\frac{1}{\varepsilon^2}\int_U (1-|u|^2)^2 \le C_5(|\ln\varepsilon| + 1) \quad \textit{for some} \quad C_5 < \infty. \tag{3.60d}$$

Then, for sufficiently small ε, we have

$$\|\nabla\varphi\|_{L^2(\omega)} \le C_6 < \infty, \tag{3.61a}$$

$$\|\nabla\varphi\|_{L^p(\omega)} \le C_7\varepsilon^{s(2/p-1)} \quad \textit{for some} \quad 2 < p \le 4 \quad \textit{and} \quad C_7 < \infty, \tag{3.61b}$$

$$|\varphi|_{C^{0,1-2/p}(\omega)} \le C_8\varepsilon^{s(2/p-1)} \quad \textit{for some} \quad C_8 < \infty. \tag{3.61c}$$

The above constants C_6, C_7, C_8 depend on $C_1, \cdots, C_5$ and A, but not on small ε.

Moreover,

$$|u| \to 1 \quad \textit{uniformly in } \omega \textit{ as } \varepsilon \to 0. \tag{3.62}$$

The proof of Lemma 3.6 is similar to the one of Lemma 3.5 and is left to the reader.

3.3 First properties of the limit profiles

The results from the previous section allow us to pass to the limits at the ε^s scale, either at the boundary (Lemma 3.7), or inside Ω (Lemma 3.8).

We will perform a blow up analysis of critical points u_ε of E_ε satisfying, near a boundary point y_ε, uniform bounds on the energy and the modulus. For simplicity of the statements, we assume that

$$y_\varepsilon \to y \in \partial\Omega \quad \text{as} \quad \varepsilon \to 0, \tag{3.63a}$$

$$\text{the unit tangent vector to } \partial\Omega \text{ at } y \text{ is } (1,0), \tag{3.63b}$$

$$g(y) = 1. \tag{3.63c}$$

Lemma 3.7. *Assume* (3.63a)–(3.63c). *Consider constants $0 < A = A_\varepsilon < \infty$ such that*

$$\lim_{\varepsilon\to 0+} A_\varepsilon = \infty \quad \textit{and} \quad \lim_{\varepsilon\to 0+} A_\varepsilon\varepsilon^s = 0. \tag{3.64}$$

Let $r = r_\varepsilon$, $U = U_\varepsilon$, $\Gamma = \Gamma_\varepsilon$ and $u = u_\varepsilon$ be as in Lemma 3.5 *(with variable A). Assume that* (3.33a)–(3.33d) *hold.*

Set $\mathbb{R}^2_+ =: \{x = (x_1, x_2) \in \mathbb{R}^2;\, x_2 > 0\}$. *Consider also a family of* C^1*-diffeomorphisms* $\Phi = \Phi_\varepsilon : \overline{U_\varepsilon} \to \overline{\mathbb{R}^2_+} \cap \overline{B}_{A_\varepsilon}(0)$ *such that:*

$$\Phi(y_\varepsilon) = 0, \tag{3.65a}$$

$$\Phi(\Gamma) = [-A, A] \times \{0\} \sim [-A, A], \tag{3.65b}$$

$$D\Phi = \frac{1}{\varepsilon^s}(1 + o(1))I_2 \quad \textit{as} \quad \varepsilon \to 0.^{\|} \tag{3.65c}$$

Set

$$v(z) = v_\varepsilon(z) =: u \circ \Phi^{-1}(z), \quad \forall z \in \overline{\mathbb{R}^2_+} \cap \overline{B}_{A_\varepsilon}(0). \tag{3.66}$$

Then, possibly up to a subsequence, v_ε *converges, locally uniformly on compacts of* $\overline{\mathbb{R}^2_+}$ *and weakly in* $H^1(Y)$*, for each bounded open set* $Y \subset \mathbb{R}^2_+$*, to* $v = e^{\imath\psi}$*, where* $\psi \in C^\infty(\overline{\mathbb{R}^2_+}; \mathbb{R})$ *solves*

$$\Delta\psi = 0 \qquad \textit{in} \quad \overline{\mathbb{R}^2_+}, \tag{3.67a}$$

$$\frac{\partial\psi}{\partial\nu} = \sin\psi\,(\cos\psi - \cos\alpha) \qquad \textit{on} \quad \mathbb{R} \times \{0\} \sim \mathbb{R}. \tag{3.67b}$$

If, in addition, u_ε *is a minimizer of* E_ε *in* U *with respect with its own boundary conditions, i.e.,*

$$E_\varepsilon(u_\varepsilon) \le E_\varepsilon(w), \quad \forall w \in H^1(\Omega) \quad \textit{such that} \quad w = u_\varepsilon \quad \textit{on} \quad \Omega \setminus U, \tag{3.68}$$

then ψ *is a locally minimizing solution of* (3.67a)–(3.67b)*, i.e.,*

$$\begin{aligned}\int_{\mathbb{R}^2_+} [|\nabla(\psi + \chi)|^2 - |\nabla\psi|^2] + \int_{\mathbb{R}} [(\cos(\psi + \chi) - \cos\alpha)^2 \\ - (\cos\psi - \cos\alpha)^2] \ge 0, \quad \forall \chi \in C^\infty_c(\overline{\mathbb{R}^2_+}).\end{aligned} \tag{3.69}$$

Proof. We write $v_\varepsilon = |v_\varepsilon| e^{\imath\psi_\varepsilon} = \eta_\varepsilon e^{\imath\psi_\varepsilon}$. By (3.35), $\eta_\varepsilon \to 1$ uniformly on compacts of $\overline{\mathbb{R}^2_+}$. Set $b =: 1 - 2/p$, with p as in (3.34b)–(3.34c). By (3.34c) and (3.65c), ψ_ε is bounded in $C^{0,b}(K)$, for each compact $K \subset \overline{\mathbb{R}^2_+}$, and thus possibly up to a subsequence we have $\psi_\varepsilon \to \psi$ uniformly on compacts, for some ψ such that $\psi \in C^{0,b}(K)$, $\forall K$. Moreover, by (3.34b), (3.65c) and the uniform bound of ψ_ε in $C^{0,b}(K)$, ψ_ε is bounded in $H^1(Y)$, for each bounded open set $Y \subset \mathbb{R}^2_+$. Combining this with the uniform convergence of ψ_ε to ψ on compacts, we find that $\psi_\varepsilon \rightharpoonup \psi$ in $H^1(Y)$. Using, in addition, (3.33a), (3.35) and (3.51), we deduce that $v_\varepsilon \to v$ uniformly on compacts and weakly in $H^1(Y)$, $\forall Y$.

We next determine the equation satisfied by ψ. Let $\chi \in C^\infty_c(\overline{\mathbb{R}^2_+})$ and set $\chi_\varepsilon(x) =: \chi \circ \Phi_\varepsilon(x)$, $\forall x \in U_\varepsilon$. Since $\varphi = \varphi_\varepsilon$ satisfies (3.36a) and the boundary condition

$$\frac{\partial\varphi}{\partial\nu} = \underbrace{\frac{1}{\varepsilon^s}\left[\langle e^{\imath\varphi}, g\rangle - \frac{1}{\rho}\cos\alpha\right](g_1\sin\varphi + g_2\cos\varphi)}_{h_\varepsilon} \quad \text{on} \quad \Gamma_\varepsilon, \tag{3.70}$$

[‖] Existence of Φ follows from assumption (3.63a), (3.63b) and (3.64).

we have, for ε sufficiently small such that supp $\chi_\varepsilon \subset B_{A_\varepsilon}(0)$,

$$\int_U \rho_\varepsilon^2 \nabla\varphi_\varepsilon \cdot \nabla\chi_\varepsilon = -\int_{\Gamma_\varepsilon} h_\varepsilon \chi_\varepsilon. \tag{3.71}$$

Going back to $\mathbb{R}^2_+$ and using (3.65c), (3.35), (3.63a), (3.63c) and the convergence of ψ_ε to ψ weakly in $H^1(Y)$ and uniformly on compacts, we find that

$$\int_{\mathbb{R}^2_+} \nabla\psi \cdot \nabla\chi = -\int_{\mathbb{R}} \sin\psi(\cos\psi - \cos\alpha)\chi, \quad \forall \chi \in C_c^\infty(\overline{\mathbb{R}^2_+}), \tag{3.72}$$

i.e., ψ satisfies (3.67a)–(3.67b).

Finally, assume that u_ε satisfies (3.68). Then, for small ε, we have

$$\begin{aligned}&\int_{U_\varepsilon} \rho_\varepsilon^2 [|\nabla(\varphi_\varepsilon + \chi_\varepsilon)|^2 - |\nabla\varphi_\varepsilon|^2] \\ &\quad + \frac{1}{\varepsilon^s}\int_{\Gamma_\varepsilon} [(\rho_\varepsilon\langle e^{\imath(\varphi_\varepsilon+\chi_\varepsilon)}, g\rangle - \cos\alpha)^2 - (\rho_\varepsilon\langle e^{\imath\varphi_\varepsilon}, g\rangle - \cos\alpha)^2] \ge 0.\end{aligned} \tag{3.73}$$

Transferring (3.73) to $\mathbb{R}^2_+$ and arguing as above, we find that (3.69) holds. □

Similarly, we have the following interior blow up analysis result.

Lemma 3.8. *Consider some* $0 < A = A_\varepsilon < \infty$ *such that*

$$\lim_{\varepsilon\to 0+} A_\varepsilon = \infty \quad \text{and} \quad \lim_{\varepsilon\to 0+} A_\varepsilon \varepsilon^s = 0. \tag{3.74}$$

Let $r = r_\varepsilon$, $U = U_\varepsilon$ *and* $u = u_\varepsilon$ *be as in Lemma* 3.6 *(with variable A). Assume that* (3.60a)–(3.60d) *hold. Set*

$$v(z) = v_\varepsilon(z) =: u\left(\frac{z-y}{\varepsilon^s}\right) = u_\varepsilon\left(\frac{z-y_\varepsilon}{\varepsilon^s}\right), \quad \forall z \in \mathbb{R}^2, \quad \text{such that} \quad |z| < A = A_\varepsilon. \tag{3.75}$$

Then, possibly up to a subsequence, v_ε *converges, locally uniformly on compacts of* $\mathbb{R}^2$ *and weakly in* $H^1(Y)$*, for each bounded open set* $Y \subset \mathbb{R}^2$*, to a constant* $\xi \in \mathbb{S}^1$.

Proof. Repeating the proof of Lemma 3.7 (and using Lemma 3.6 instead of Lemma 3.5), we find that, up to a subsequence, v converges, in C_{loc} and H^1_{loc}, to a map $w \in H^1_{loc}(\mathbb{R}^2; \mathbb{S}^1)$. Write $v_\varepsilon = \rho_\varepsilon e^{\imath\varphi_\varepsilon}$. Using (3.61a) and the conformal invariance of the Dirichlet integral, we find that w is of the form $w = e^{\imath\psi}$, with $\psi \in H^1(\mathbb{R}^2)$. On the other hand, (3.36a) implies, after rescaling, applying (3.62), and passing to the weak limits, that ψ is harmonic. Now a harmonic function ψ in $\mathbb{R}^2$ with finite Dirichlet integral is constant. Thus w is constant, whence the conclusion. □

3.4 Boundary limit profiles

In the proofs of Theorems 1.2 and 1.3, we will be in position to apply Lemma 3.7, having at our disposal more information than the assumptions of Lemma 3.7. Under these better assumptions, it is possible to identify all the solutions of (3.67a), (3.67b), (3.69) which can arise as limits of v_ε's.

Lemma 3.9. *We use the same notation as in Lemma* 3.7*. Assume* (3.33a)–(3.33d) *and* (3.63a)–(3.64)*. Assume, in addition, that*

$$\text{the constant } C_3 \text{ in (3.33b) satisfies } C_3 < \pi; \tag{3.76a}$$

$$u = u_\varepsilon \text{ is a minimizer of } E_\varepsilon \text{ in } \Omega. \tag{3.76b}$$

Let $\psi \in C^\infty(\overline{\mathbb{R}^2_+};\mathbb{R})$ *be such that, possibly up to a subsequence, we have* $v_\varepsilon \to e^{\imath\psi}$ *locally uniformly on compacts of* $\overline{\mathbb{R}^2_+}$ *and weakly in* $H^1(Y)$*, for each bounded open set* $Y \subset \mathbb{R}^2_+$*. Then:*

1. *Either* $\psi \equiv \alpha$*.*
2. *Or* $\psi \equiv -\alpha$*.*
3. *Or* ψ *is (up to an isometry of* $\mathbb{R}^2_+$*) the unique (up to an isometry of* $\mathbb{R}^2_+$*) locally minimizing solution of*

$$\Delta\psi = 0 \qquad \text{in } \overline{\mathbb{R}^2_+}, \tag{3.77a}$$

$$\frac{\partial\psi}{\partial\nu} = \sin\psi\,(\cos\psi - \cos\alpha) \qquad \text{on } \mathbb{R}\times\{0\}\sim\mathbb{R}, \tag{3.77b}$$

$$\lim_{x\to\pm\infty}\psi(x) = \pm\alpha. \tag{3.77c}$$

Proof. Step 1. Identification of constant profiles. Assume that ψ is constant. Since we have $|\psi| < \pi$ (by (3.76a)), we find, via (3.67b), that either $\psi \equiv \alpha$, $\psi \equiv -\alpha$, or $\psi \equiv 0$. It remains to rule out the possibility that $\psi \equiv 0$. This is achieved by arguing by contradiction. If we write (as in the proof of Lemma 3.7) $u_\varepsilon = \rho_\varepsilon e^{\imath\varphi_\varepsilon}$ and $v_\varepsilon = \eta_\varepsilon e^{\imath\psi_\varepsilon}$, then (possibly up to a subsequence) $\psi_\varepsilon \to 0$ uniformly on compacts of $\mathbb{R}$. Set $\Gamma_{\ell,\varepsilon} =: \partial\Omega \cap B_{\ell\varepsilon^s}(y_\varepsilon)$, for every constant $\ell > 0$. Going back to φ_ε and using (3.65a)–(3.65c), we have $\varphi_\varepsilon \to 0$ uniformly on $\Gamma_{\ell,\varepsilon}$, for every ℓ. Combining this with the fact that $\rho_\varepsilon \to 1$ uniformly on $\Gamma_{\ell,\varepsilon}$ (by (3.35)), we find that

$$W(u_\varepsilon, g) \to (1-\cos\alpha)^2 \quad \text{uniformly on} \quad \Gamma_{\ell,\varepsilon} \quad \text{as} \quad \varepsilon \to 0. \tag{3.78}$$

Fix now some $0 < a < 1-\cos\alpha$ and consider corresponding bad discs B^j as in Section 2.2, satisfying (2.3), of radii $\le k\varepsilon^s$ and such that the mutual distance of two distinct boundary bad discs is $\gg \varepsilon^s$. Note that

$$W(u_\varepsilon(x), g(x)) \le a^2 < (1-\cos\alpha)^2, \quad \forall\varepsilon,\ \forall x \in \partial\Omega \setminus \bigcup_j B^j. \tag{3.79}$$

For fixed $\ell \geq 2k$ and small ε, $\Gamma_{\ell,\varepsilon}$ is not contained in the union of the boundary bad discs, and thus, by (3.79), (3.78) cannot hold. This proves that any constant profile is either α, or $-\alpha$.

Step 2. Asymptotic behavior of non constant profiles. Let ψ be a non constant limiting map. Then ψ is locally minimizing (Lemma 3.7) and, by assumption, non constant, so that

$$\text{either}\quad \frac{\partial\psi}{\partial x_1}(x_1,x_2)>0,\quad \forall(x_1,x_2)\in\overline{\mathbb{R}^2_+}\quad\text{or}\quad \frac{\partial\psi}{\partial x_1}(x_1,x_2)<0,\quad \forall(x_1,x_2)\in\overline{\mathbb{R}^2_+}; \tag{3.80}$$

see Cabré & Solà-Morales [6, Theorem 1.5].

Combining (3.80) with the bound $|\psi|\leq C_3<\pi$, we find that

$$\text{there exists } \gamma_{\pm\infty}=: \lim_{x_1\to\pm\infty}\psi(x_1,0)\in(-\pi,\pi),\quad \text{and}\quad \gamma_{-\infty}\neq\gamma_\infty. \tag{3.81}$$

Possibly by considering $-\psi$ instead of ψ, we may assume that

$$\gamma_{-\infty}<\gamma_\infty, \tag{3.82}$$

and then, in order to conclude, it suffices to prove that

$$\gamma_{-\infty}=-\alpha\quad\text{and}\quad \gamma_\infty=\alpha. \tag{3.83}$$

Indeed, granted (3.83), ψ is a locally minimizing solution of (3.77a)–(3.77b) satisfying (3.77c), and thus ψ is unique up to a translation in x_1 [6, Theorem 1.2, Theorem 1.5]. Since we also allow replacing ψ by $-\psi$ (exploiting the reflection symmetry in x_1, [6, Theorem 1.2]), we find that ψ is unique up to an isometry of $\mathbb{R}^2_+$.

In turn, (3.83) is proved as follows. Using $|\psi|<\pi$, (3.67a)–(3.67b) and standard elliptic estimates, we find that

$$\psi \text{ is bounded in } C^{0,\beta}(K) \text{ for every } 0<\beta<1 \text{ and compact } K\subset\overline{\mathbb{R}^2_+}. \tag{3.84}$$

Set now

$$\psi_h(x_1,x_2)=:\psi(x_1+h,x_2),\quad \forall(x_1,x_2)\in\overline{\mathbb{R}^2_+},\quad \forall h\in\mathbb{R}.$$

Clearly, each ψ_h is a locally minimizing solution of (3.67a)–(3.67b). By (3.84) and (3.81), there exist sequences $h_j^+\to\infty$, $h_j^-\to-\infty$ and locally minimizing solutions $\psi_{\pm\infty}$ of (3.67a)–(3.67b), satisfying

$$\psi_{h_j^\pm}\to\psi_{\pm\infty} \text{ locally uniformly on compacts of } \overline{\mathbb{R}^2_+}, \tag{3.85a}$$

$$\psi_{\pm\infty}(x_1,0)=\gamma_{\pm\infty},\quad \forall x_1\in\mathbb{R}. \tag{3.85b}$$

By (3.85b) and [6, Theorem 1.5], we have $\psi_{\pm\infty}\equiv\gamma_{\pm\infty}$. As in Step 1, we have $\gamma_{\pm\infty}\in\{-\alpha,0,\alpha\}$. In order to conclude, it suffices to rule out the possibility $\psi_{\pm\infty}\equiv 0$. Argue by contradiction and assume e.g., that $\psi_\infty\equiv 0$. Fix some small number δ and some large number ℓ. Then there exists some j such that $|\psi_{h_j^+}(x_1,0)|\leq\delta$ for $|x_1|\leq\ell$. As in Step 1, by going back to φ_ε and u_ε we obtain a contradiction. The proof of Lemma 3.9 is completed. □

4 Proofs of Theorems 1.2 and 1.3

We consider a sequence $\varepsilon = \varepsilon_n \to 0$ such that Theorem 1.1 applies and we let $\zeta \in \partial\Omega$ be a light boojum. With no loss of generality, we assume that $\zeta = 0$, $g(\zeta) = 1$ (and thus, in Lemma 2.2 item 2, we may take $\theta_0 = 0$) and the tangent at ζ is $\tau = (1,0)$.

Our proof relies on Lemma 2.2 used in conjunction with the blowup analysis in Section 3. Here we make the following observation concerning the subsequence $\varepsilon = \varepsilon_n \to 0$. In Theorem 1.1, it is necessary to extract a subseqence of $\varepsilon_n \to 0$ for which each of the bad disks converge to a particular choice of limiting defect sites $\{\zeta_j\}$. In order to obtain the conclusion of Lemma 2.2, it may be necessary to extract a further subsequence $\varepsilon_n \to 0$. However, as the conclusions of Theorem 1.2 and Lemma 2.2 do not depend on the particular subsequence, by a standard argument the conclusion of Theorem 1.2 holds for the original sequence $\varepsilon_n \to 0$. This being noted, we work from now on with $\varepsilon = \varepsilon_n$ satisfying the conclusions of Lemma 2.2.

Step 1. Preliminary analysis of u_ε near ζ. Consider some $\overline{\delta} > 0$ such that

$$\beta =: \alpha + \overline{\delta} < \pi/2. \tag{4.1}$$

Let δ, γ satisfy (3.2a) and (3.2b).

By taking $\sigma > 0$ sufficiently small, we may assume that (2.11) holds, and also that

$$g(\partial\Omega \cap \overline{B}_\sigma(0)) \subset \mathbb{S}^1_\delta. \tag{4.2}$$

Using (2.11), (4.1), Lemma 2.2 item 1, the minimality of u and the smallness of σ, we find that the assumptions of Lemma 3.3 are satisfied in $D =: B_{R_\sigma}(0) \cap \Omega$, and thus, for small ε, we have (3.18a) and (3.18b).

Step 2. Proof of Theorem 1.3 item 2. By(3.18a), (3.18b), (2.2b) and (1.5), the assumptions (3.60a)–(3.60d) of Lemma 3.6 are satisfied. We obtain Theorem 1.3 item 2 by invoking Lemma 3.8.

Step 3. Proof of Theorem 1.2. Write, in a small fixed neighborhood of ζ, $u = \rho e^{\imath\varphi}$, with $\varphi \in C^1$ and $|\varphi| < 2\alpha$ (see Step 1 and (3.18b)). Consider the essential bad disc $B_{k\varepsilon^s}(w_\varepsilon)$ (with $w_\varepsilon \in \partial\Omega$) converging to ζ; see Lemma 2.1. By (2.4) and Lemma 2.1 item 3, for sufficiently small $a > 0$ and small ε we have, with $r =: k\varepsilon^s$, $R =: 2k\varepsilon^s$,

$$\begin{aligned}
&|\varphi - t^+\alpha - 2\pi m^+| < \alpha \quad \text{on} \quad \Gamma^+_{r,R},\\
&|\varphi - t^-\alpha - 2\pi m^+| < \alpha \quad \text{on} \quad \Gamma^-_{r,R},\\
&t^\pm \in \{-1,1\}, \quad t^+ - t^- = -2, \quad m^+ \in \mathbb{Z}.
\end{aligned}$$

Since, for small ε, $|\varphi| < 2\alpha$ and $|\varphi| \le \gamma < \pi/2$ (see Step 1), this leaves only the possibility $t^\pm = \mp 1$, $m^+ = 0$. It follows that

$$\varphi < 0 \quad \text{on}\, \Gamma^+_{r,R}, \quad \text{respectively} \quad \varphi > 0 \quad \text{on} \quad \Gamma^+_{r,R}. \tag{4.3}$$

We now let $y = y_\varepsilon \in \partial\Omega \cap B_{k\varepsilon^s}(w_\varepsilon)$ be such that

$$\varphi(y) = 0. \tag{4.4}$$

Using (4.3) and (4.4) in conjunction with Lemma 3.9 and (3.80), we find that (1.10)–(1.12) hold.

Step 4. Proof of Theorem 1.3 item 2. Let ρ, φ, y_ε and w_ε be as in the previous step. Consider any $z = z_\varepsilon \in \partial\Omega$ such that $z_\varepsilon \to \zeta$ and $|z_\varepsilon - y_\varepsilon| \gg \varepsilon^s$. We know that, up to a subsequence, $(u_\varepsilon)^{z_\varepsilon,\varepsilon^s} \to e^{\imath\psi}$ for some ψ as in Lemma 3.9. Argue by contradiction, assuming that ψ is nonconstant. By Lemma 3.9, there exists some $x_0 \in \mathbb{R}$ such that $\psi(x_0, 0) = 0$, and thus (going back from ψ to u_ε and using the local uniform convergence) there exists some $\widetilde{z} = \widetilde{z}_\varepsilon \in \partial\Omega$ such that $|\widetilde{z}_\varepsilon - z_\varepsilon| \le C\varepsilon^s$ and $u_\varepsilon(\widetilde{z}_\varepsilon) \to 1$. With no loss of generality, we may assume that $\widetilde{z}_\varepsilon = z_\varepsilon$, and then $\psi(0,0) = 0$.

We have $W(u_\varepsilon(z_\varepsilon), g(z_\varepsilon)) \to (1-\cos\alpha)^2$ and thus (provided we choose a sufficiently small a) there exists some boundary bad disc containing z_ε. With no loss of generality, we may assume that the disc is centered at z_ε. Since $|z_\varepsilon - w_\varepsilon| \gg \varepsilon^s$, for small ε this bad disc is not essential (see Lemma 2.1), and in particular, with $r =: k\varepsilon^s$, $R =: 2k\varepsilon^s$, we have

$$\begin{aligned}
&|\varphi - t^+\alpha - 2\pi m^+| < \alpha \quad \text{on} \quad \Gamma^+_{r,R},\\
&|\varphi - t^+\alpha - 2\pi m^+| < \alpha \quad \text{on} \quad \Gamma^-_{r,R},\\
&t^+ \in \{-1, 1\}, \quad m^+ \in \mathbb{Z}.
\end{aligned}$$

Since $|\varphi| \le \gamma < \pi/2$, we find that $m^+ = 0$, and either $t^+ = 1$, and then $\varphi > 0$ on $\Gamma^+_{r,R} \cup \Gamma^-_{r,R}$, or $t^+ = -1$, and then $\varphi < 0$ on $\Gamma^+_{r,R} \cup \Gamma^-_{r,R}$. This implies that either $\psi(x_1, 0) \ge 0$ when $k \le |x_1| \le 2k$, or $\psi(x_1, 0) \le 0$ when $k \le |x_1| \le 2k$. However, this contradicts (3.80) and the fact that $\psi(0,0) = 0$. The proofs of Theorems 1.2 and 1.3 are completed. □

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References

[1] S. Alama, L. Bronsard, and B. Galvão Sousa, Weak anchoring for a two-dimensional liquid crystal, Nonlinear Anal., 119 (2015), 74–97.

[2] S. Alama, L. Bronsard, and D. Golovaty, Thin film liquid crystals with oblique anchoring and boojums, arXiv:1907.04757, 2019.

[3] F. Bethuel, H. Brezis, and F. Hélein, Ginzburg-Landau vortices, Progress in Nonlinear Differential Equations and Their Applications, 13. Birkhäuser Boston Inc., Boston, MA, 1994.

[4] B. V. Boyarskiĭ, Homeomorphic solutions of Beltrami systems, Dokl. Akad. Nauk SSSR (N.S.), 102 (1955), 661–664.

[5] B. V. Boyarskiĭ, Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients, Mat. Sb. N.S., 43(85) (1957), 451–503.

[6] X. Cabré and J. Solà-Morales, Layer solutions in a half-space for boundary reactions, Commun. Pure Appl. Math., 58(12) (2005), 1678–1732.

[7] D. Golovaty, J.A. Montero, and P. Sternberg, Dimension reduction for the Landau-de Gennes model in planar nematic thin films, J. Nonlinear Sci., 25(6) (2015), 1431–1451.

[8] R. Ignat and M. Kurzke, Global Jacobian and Γ-convergence in a two-dimensional Ginzburg-Landau model for boundary vortices, arXiv:1910.06039 preprint, 2019.

[9] R. L. Jerrard, Lower bounds for generalized Ginzburg-Landau functionals, SIAM J. Math. Anal., 30(4) (1999), 721–746.

[10] M. Kurzke, Boundary vortices in thin magnetic films, Calc. Var. Partial Differential Equations, 26(1) (2006), 1–28.

[11] N. G. Meyers, An L^p-estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3), 17 (1963), 189–206.

[12] R. Moser, On the energy of domain walls in ferromagnetism, Interfaces Free Bound., 11(3) (2009), 399–419.

[13] É. Sandier, Lower bounds for the energy of unit vector fields and applications, J. Funct. Anal., 152(2) (1998), 379–403.

[14] M. Struwe, On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2 dimensions, Differential Integral Equations, 7(5-6) (1994), 1613–1624.

[15] P. I. C. Teixeira, T. J. Sluckin, and D. E. Sullivan, Landau-de Gennes theory of anchoring transitions at a nematic liquid crystal-substrate interface, Liquid Crystals, 14(4) (1993), 1243–1253.