

Eigenvalues of a Differential Operator and Zeros of the Riemann ζ -Function

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. The eigenvalues of a differential operator on a Hilbert-Pólya space are determined. It is shown that these eigenvalues are exactly the nontrivial zeros of the Riemann ζ -function. Moreover, their corresponding multiplicities are the same.

Key Words: Hilbert-Pólya space, zeros of zeta function, differential operator, eigenvalue.

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1 Introduction and preliminaries

In studying the Riemann hypothesis it is important to establish a connection between spectrum of operators and zeros of the Riemann zeta function. This is our continued effort for finding such an interplay.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space on \mathbb{R} . Functions in $\mathcal{S}(\mathbb{R})$ are smooth and rapid decreasing when $|x| \rightarrow \infty$. That is,

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) \mid \sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)| < \infty \text{ for } m, n \in \mathbb{N} \right\},$$

where $\mathbb{N} = \{0, 1, \dots\}$. Denote by \mathcal{H}_\cap the subspace of all even functions f in $\mathcal{S}(\mathbb{R})$ with $f(0) = \mathfrak{F}f(0) = 0$. Here

$$\mathfrak{F}f(x) = \int_{-\infty}^{\infty} f(y) e^{-2\pi ixy} dy$$

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is the Fourier transform of f . In this paper, we may identify functions in \mathcal{H}_\cap with their restrictions on $(0, \infty)$.

Denote by \mathcal{H}_- the set of all smooth functions on $(0, \infty)$ that decrease rapidly when $x \rightarrow 0+$ and ∞ . Specifically,

$$\mathcal{H}_- = \left\{ f \in C^\infty(0, \infty) \mid \sup_{x>0} |x^m f^{(n)}(x)| < \infty \text{ for } |m|, n \in \mathbb{N} \right\}.$$

For $f \in \mathcal{H}_\cap$ we denote by Z the action given by

$$Zf(x) = \sum_{n=1}^{\infty} f(nx), \quad x > 0.$$

By Poisson's summation formula, $Z\mathcal{H}_\cap \subseteq \mathcal{H}_-$. The quotient space $\mathcal{H}_-/Z\mathcal{H}_\cap$, denoted by \mathcal{H} , is regarded as a Hilbert-Pólya space, although such a quotient does not carry a Hilbert space structure.

The fundamental differential operator D on \mathcal{H}_- is given by

$$Df(x) = -xf'(x), \quad f \in \mathcal{H}_-.$$

Notice that $D\mathcal{H}_- \subseteq \mathcal{H}_-$ and $D(Z\mathcal{H}_\cap) \subseteq Z\mathcal{H}_\cap$. The operator D thus induces a differential operator on \mathcal{H} , which we denote by D_- . We follow the notation introduced in [5]. For basics on the Riemann zeta function $\zeta(s)$, we refer to [6].

The Hilbert-Pólya conjecture states that the non-trivial zeros of the Riemann ζ -function correspond (in certain canonical way) to the eigenvalues of some operator, and Riemann Hypothesis is equivalent to the self-adjointness of the operator.

In the direction of the Hilbert-Pólya conjecture, a spectral interpretation for zeros on the critical line $\text{Re}(s) = \frac{1}{2}$ was given by A. Connes [1]. He constructed a closed, densely defined, unbounded differential operator D_χ and a Hilbert-Pólya space \mathcal{H}_χ . His operator D_χ has discrete spectrum, which is the set of imaginary parts of critical zeros of the L -function with Grössencharacter χ [1, Theorem 1, pp. 40].

In [2], L. Ge studied the action of the differential operator D on a Hilbert-Pólya space, which is slightly different from \mathcal{H} , and obtained that the point spectrum of D coincides with the nontrivial zeros of $\zeta(s)$.

Meyer proved in [5, Corollary 4.2, pp. 8] that the eigenvalues of the transpose D_-^t of D_- (acting on the space of continuous linear functionals on \mathcal{H}) are exactly the non-trivial zeros of $\zeta(s)$, and the algebraic multiplicity of an eigenvalue is equal to the vanishing order of the corresponding zero of $\zeta(s)$.

Li [4] proved by an explicit construction that the nontrivial zeros of $\zeta(s)$ are the eigenvalues of D_- acting on \mathcal{H} (instead of D_-^t in [5]), which we state as follows (Theorem 1.1 and Theorem 1.4 in [4]).

Theorem 1.1 ([4]). *Let ρ be a nontrivial zero of $\zeta(s)$. Let F_ρ be the function given by*

$$F_\rho(x) = \int_1^\infty Z\eta(tx)t^{\rho-1}dt,$$

where

$$\eta(x) = 8\pi x^2 \left(\pi x^2 - \frac{3}{2} \right) e^{-\pi x^2}.$$

Then F_ρ is an eigenfunction of D_- on \mathcal{H} associated with eigenvalue ρ . Moreover, the eigenvalue ρ has geometric multiplicity one.

A question is that whether or not every eigenvalue of D_- on \mathcal{H} is a nontrivial zero of the Riemann zeta function. In this paper, we give an affirmative answer to the question by proving that there is a one-to-one correspondence between the set of eigenvalues of D_- acting on \mathcal{H} and the set of nontrivial zeros of $\zeta(s)$. Moreover, we also show that the algebraic multiplicity of any eigenvalue of D_- is equal to the vanishing order of the corresponding zero of $\zeta(s)$. Recall that the algebraic multiplicity of an eigenvalue λ of an operator T is the dimension of the space $\bigcup_{n \geq 1} \ker(T - \lambda)^n$. Some of these results are hinted in [5]. The goal of this paper is to give a detailed proof of our main result which we state as the following theorem.

Theorem 1.2. *A complex number ρ is an eigenvalue of D_- on \mathcal{H} if and only if ρ is a nontrivial zero of $\zeta(s)$. Moreover, its algebraic multiplicity is equal to the vanishing order of $\zeta(s)$ at ρ .*

The proof of Theorem 1.2 will be presented in Section 3. In Section 2, we study the eigenvalues of D_- and will show that they have a similar distribution as the nontrivial zeros of $\zeta(s)$, which is stated as follows.

Theorem 1.3. *If λ is an eigenvalue of D_- on \mathcal{H} , then $0 < \operatorname{Re}(\lambda) < 1$.*

Now we give the detailed proof of this theorem in the following section.

2 Eigenvalues of D_-

Let f be a measurable function on $(0, \infty)$. Denote by

$$\widehat{f}(s) = \int_0^\infty f(x)x^{s-1}dx, \quad s \in \mathbb{C},$$

the Mellin transform of f , provided that the above integral exists for some s . For $f \in \mathcal{H}_-$, it is not hard to check that $\widehat{f}(s)$ is entire on \mathbb{C} and, from integration by parts,

$$\widehat{Df}(s) = - \int_0^\infty x f'(x)x^{s-1}dx = - \int_0^\infty x^s df(x) = s \int_0^\infty f(x)x^{s-1}dx = s\widehat{f}(s).$$

Let g be a function in \mathcal{H}_\cap . Since $g(x)$ decreases rapidly as $x \rightarrow \infty$ and g is continuous at 0, the integral

$$\int_0^\infty g(x)x^{s-1}dx$$

exists when $\operatorname{Re}(s) > 0$, and we have

$$\widehat{Zg}(s) = \int_0^\infty \left(\sum_{n=1}^\infty g(nx) \right) x^{s-1} dx = \sum_{n=1}^\infty \int_0^\infty g(y) \left(\frac{y}{n} \right)^{s-1} d \left(\frac{y}{n} \right) = \zeta(s) \widehat{g}(s)$$

when $\operatorname{Re}(s) > 1$. Now we have some key properties of the operators D and Z . We need to introduce another operator J .

Let f be a function in \mathcal{H}_- . Denote by J the operator given by $Jf(x) = x^{-1}f(x^{-1})$. It is easy to see that $J\mathcal{H}_- = \mathcal{H}_-$ and $\widehat{Jf}(s) = \widehat{f}(1-s)$. The operator J plays an important role in the symmetry of \mathbb{C} along the critical line $\operatorname{Re}(s) = 1/2$ and induces a unitary operator from $L^2(0,1)$ onto $L^2(1,\infty)$ (see [3]). Here we are interested in its connections with operators Z , \mathfrak{F} and the Mellin transform.

For $g \in \mathcal{H}_\cap$, we have $\mathfrak{F}g \in \mathcal{H}_\cap$. Applying the Poisson summation formula to $g \in \mathcal{H}_\cap$, since both g and $\mathfrak{F}g$ are even functions vanishing at zero, we have

$$JZg(x) = \frac{1}{x} \sum_{n=1}^\infty g\left(\frac{n}{x}\right) = \sum_{n=1}^\infty \mathfrak{F}g(nx) = Z\mathfrak{F}g(x).$$

Hence

$$\widehat{JZg}(s) = \widehat{Zg}(1-s) = \widehat{Z\mathfrak{F}g}(s). \quad (2.1)$$

So $JZ\mathcal{H}_\cap \subseteq Z\mathcal{H}_\cap$. Notice that J^2 is the identity operator, we have

$$JZ\mathcal{H}_\cap = Z\mathcal{H}_\cap.$$

With the above discussion, we are ready to prove the following lemma.

Lemma 2.1. *Let λ be an eigenvalue of D_- on \mathcal{H} . Then $\bar{\lambda}$ and $1 - \lambda$ are both eigenvalues of D_- on \mathcal{H} .*

Proof. For any given λ , an eigenvalue of D_- on \mathcal{H} , there is a function $f \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$ and a $g \in \mathcal{H}_\cap$ such that $Df - \lambda f \in Z\mathcal{H}_\cap$ and

$$-xf'(x) = \lambda f(x) + Zg(x). \quad (2.2)$$

Taking conjugate we have

$$-x\overline{f'(x)} = \overline{\lambda f(x)} + \overline{Zg(x)}, \quad \text{i.e., } D\bar{f} = \bar{\lambda}\bar{f} + Z\bar{g}.$$

Notice that $\bar{f} \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$ and $\bar{g} \in \mathcal{H}_\cap$, we see that $\bar{\lambda}$ is an eigenvalue of D_- on \mathcal{H} .

Next we prove that $1 - \lambda$ is also an eigenvalue of D_- on \mathcal{H} by showing that Jf is an eigenfunction with eigenvalue $1 - \lambda$. Since $J\mathcal{H}_- = \mathcal{H}_-$ and $JZ\mathcal{H}_\cap = Z\mathcal{H}_\cap$, we have $Jf \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$. Applying the Mellin transform to (2.2), we have

$$s\widehat{f}(s) = \lambda\widehat{f}(s) + \zeta(s)\widehat{g}(s).$$

Replacing s by $1 - s$ and applying (2.1), we obtain

$$\begin{aligned} (1 - s)\widehat{f}(1 - s) &= \lambda\widehat{f}(1 - s) + \zeta(1 - s)\widehat{g}(1 - s) \\ &= \lambda\widehat{f}(1 - s) + \widehat{JZg}(s) = \lambda\widehat{f}(1 - s) + \widehat{Z\mathfrak{F}g}(s). \end{aligned}$$

This formula implies that

$$s\widehat{f}(1 - s) = (1 - \lambda)\widehat{f}(1 - s) - \zeta(s)\widehat{\mathfrak{F}g}(s).$$

Notice that $\widehat{Jf}(s) = \widehat{f}(1 - s)$, we thus have

$$s\widehat{Jf}(s) = (1 - \lambda)\widehat{Jf}(s) - \zeta(s)\widehat{\mathfrak{F}g}(s).$$

Since $Jf \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$ and $\mathfrak{F}g \in \mathcal{H}_\cap$, applying the Mellin inverse transform we obtain

$$-x(Jf)'(x) = (1 - \lambda)Jf(x) + Z(-\mathfrak{F}g)(x).$$

Hence $1 - \lambda$ is also an eigenvalue of D_- on \mathcal{H} . This completes the proof of the lemma. \square

We already see that the integral $\int_0^\infty g(x)x^{s-1}dx$ exists when $\text{Re}(s) > 0$, where g is a function in \mathcal{H}_\cap . In fact, we can show that the above integral exists when $\text{Re}(s) > -2$. Since g is an even Schwartz function, the derivative g' is odd which implies that $g'(0) = 0$. Let

$$c = \sup_{|x| \leq 1} |g''(x)|$$

be the maximum of g'' on $[-1, 1]$. For $x \in [-1, 1]$, we have, by Mean Value Theorem,

$$|g(x) - g(0)| = |g(x)| \leq |g'(\xi)x| = |g'(\xi) - g'(0)||x| \leq |g''(\xi')x||x| \leq cx^2,$$

where $-1 < \xi, \xi' < 1$. Hence for $s \in \mathbb{C}$ with $\text{Re}(s) > -2$, the integral $\int_0^1 g(x)x^{s-1}dx$ exists. While $\int_1^\infty g(x)x^{s-1}dx$ is well defined for any $s \in \mathbb{C}$, $\int_0^\infty g(x)x^{s-1}dx$ exists when $\text{Re}(s) > -2$.

When g is not even and $g \in \mathcal{S}(\mathbb{R})$, we do not know the behavior of $g(x)$ near $x = 0$. But we can show that \widehat{g} admits a meromorphic continuation on \mathbb{C} . In fact, from integration by parts,

$$\int_0^\infty g(x)x^{s-1}dx = \frac{x^s}{s}g(x) \Big|_0^\infty - \frac{1}{s} \int_0^\infty g'(x)x^s dx = -\frac{1}{s} \int_0^\infty g'(x)x^s dx$$

for $s \in \mathbb{C}$ with $\text{Re}(s) > 0$. Since $g \in \mathcal{S}(\mathbb{R})$, we have $g' \in \mathcal{S}(\mathbb{R})$. So the last integral exists for $\text{Re}(s) > -1$. Hence we obtain a meromorphic continuation of \widehat{g} on $\{s \in \mathbb{C} \mid \text{Re}(s) > -1\}$. Repeating this procedure, we obtain a meromorphic continuation of \widehat{g} to the complex plane \mathbb{C} . Such technique can be applied to the extension of the Mellin transform of more general functions.

To prove Theorem 1.3, we need to establish a lemma on functions in \mathcal{H}_\cap .

Lemma 2.2. Assume $g \in \mathcal{H}_\cap$. Suppose that $\widehat{g}(s_0) = 0$ for some $s_0 \in \mathbb{C}$ with $\operatorname{Re}(s_0) > -1$ and $\operatorname{Re}(s_0) \neq 1$. Define, for $x \in \mathbb{R}$,

$$f(x) = \int_1^\infty g(tx)t^{s_0-1} dt.$$

Then we have

i) $f(x) = \int_1^\infty g(tx)t^{s_0-1} dt = -\int_0^1 g(tx)t^{s_0-1} dt;$

ii) $f \in \mathcal{H}_\cap$; and

iii) $\widehat{g}(s) = \widehat{f}(s)(s - s_0).$

Proof. First we prove i). Notice that $|g(x)| \leq c|x|$ for $c = \sup_{x \in \mathbb{R}} |g'(x)| < \infty$. The integral $\int_0^\infty g(t)t^{s-1}$ exists when $\operatorname{Re}(s) > -1$. Hence we only need to show that

$$\int_0^\infty g(tx)t^{s_0-1} dt = 0$$

holds for any $x \in \mathbb{R}$.

Note that

$$\int_0^\infty g(t)t^{s_0-1} dt = \widehat{g}(s_0) = 0.$$

For any given $x > 0$, substituting t by tx in the above equation, we have

$$x^{s_0} \int_0^\infty g(tx)t^{s_0-1} dt = \int_0^\infty g(t)t^{s_0-1} dt = \widehat{g}(s_0) = 0.$$

So

$$\int_0^\infty g(tx)t^{s_0-1} dt = 0$$

for $x > 0$. For $x < 0$, since g is even, we also have

$$\int_0^\infty g(tx)t^{s_0-1} dt = \int_0^\infty g(-tx)t^{s_0-1} dt = 0.$$

For $x = 0$, we have

$$\int_0^\infty g(t \cdot 0)t^{s_0-1} dt = \int_0^\infty g(0)t^{s_0-1} dt = 0$$

as $g(0) = 0$. Hence

$$\int_0^\infty g(tx)t^{s_0-1} dt = 0$$

for $x \in \mathbb{R}$. Therefore,

$$f(x) = \int_1^\infty g(tx)t^{s_0-1}dt = - \int_0^1 g(tx)t^{s_0-1}dt.$$

Next we verify ii). It is clear that f is an even function. First we check the smoothness of f . Since g is a Schwartz function, by Lebesgue's Dominated Convergence Theorem, we see that f is smooth on $\mathbb{R} \setminus \{0\}$ and, for $x \neq 0, n \in \mathbb{N}$,

$$f^{(n)}(x) = \int_1^\infty g^{(n)}(tx)t^{n+s_0-1}dt = - \int_0^1 g^{(n)}(tx)t^{n+s_0-1}dt.$$

It remains to show that f is smooth at 0. We start with the continuity of f at 0. Since $|g(x)| \leq c|x|$ for $x \in [0, 1]$, we have

$$|f(x)| \leq c \int_0^1 txt^{\text{Re}(s_0)-1}dt \leq cx.$$

Hence

$$\lim_{x \rightarrow 0} f(x) = 0 = f(0).$$

Next we prove inductively that $f^{(n)}(0)$ exists for $n \geq 1$. Notice that, by Mean Value Theorem, the function $\frac{g(t)-g(0)}{t}$ is bounded by $\sup_{x \in \mathbb{R}} |g'(x)|$ for $t \neq 0$. Since $\text{Re}(s_0) > -1$, we have, by Lebesgue's dominated convergence theorem,

$$\begin{aligned} f'(0) &= \lim_{x \rightarrow 0} \frac{- \int_0^1 g(tx)t^{s_0-1}dt + \int_0^1 g(0)t^{s_0-1}dt}{x} \\ &= \lim_{x \rightarrow 0} - \int_0^1 \frac{g(tx) - g(0)}{tx} t^{s_0} dt = - \int_0^1 g'(0)t^{s_0} dt. \end{aligned}$$

Assume that

$$f^{(k)}(0) = - \int_0^1 g^{(k)}(0)t^{k+s_0-1}dt \tag{2.3}$$

for some $k \geq 1$. Notice that the function $\frac{g^{(k)}(t)-g^{(k)}(0)}{t}$ is bounded by $\sup_{x \in \mathbb{R}} |g^{(k+1)}(x)|$ for $t \neq 0$. For $n = k + 1$, we apply the above equation (2.3), use Lebesgue's Dominated Convergence Theorem, and then obtain

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{x \rightarrow 0} \frac{- \int_0^1 g^{(k)}(tx)t^{k+s_0-1}dt + \int_0^1 g^{(k)}(0)t^{k+s_0-1}dt}{x} \\ &= \lim_{x \rightarrow 0} - \int_0^1 \frac{g^{(k)}(tx) - g^{(k)}(0)}{tx} t^{k+s_0} dt = - \int_0^1 g^{(k+1)}(0)t^{k+s_0} dt. \end{aligned}$$

Thus, (2.3) holds for all $k \in \mathbb{N}$. Hence f is smooth on \mathbb{R} .

Now we prove that $f(x)$ decays rapidly as $|x| \rightarrow \infty$, it suffices to check that $x^m f^{(n)}(x) \rightarrow 0$ when $|x| \rightarrow \infty$ for any $m, n \in \mathbb{N}$. Since g is a Schwartz function, there is a constant $C > 0$ such that

$$\begin{aligned} |x^m f^{(n)}(x)| &= \left| x^m \int_1^\infty g^{(n)}(tx) t^{n+s_0-1} dt \right| \\ &\leq C \left| \int_1^\infty x^m (tx)^{-m-1-\operatorname{Re}(s_0)} t^{n+s_0-1} dt \right| \\ &\leq \frac{C}{|x|} \int_1^\infty t^{-2} dt = \frac{C}{|x|} \rightarrow 0, \quad |x| \rightarrow \infty. \end{aligned}$$

Hence $f \in \mathcal{S}(\mathbb{R})$.

In the above, we have seen that $f(0) = 0$. It remains to show that $\mathfrak{F}f(0) = 0$. If $-1 < \operatorname{Re}(s_0) < 1$, then there is a δ such that $0 < \delta < 1$ and $\operatorname{Re}(s_0) < \delta$. Thus we have

$$\begin{aligned} |\mathfrak{F}f(0)| &= \left| \int_{-\infty}^\infty f(x) dx \right| = \left| \int_{-\infty}^\infty \int_1^\infty g(tx) t^{s_0-1} dt dx \right| \\ &\leq \int_{|x|<1} \int_1^\infty |g(tx)| t^{\operatorname{Re}(s_0)-1} dt dx + \int_{|x|\geq 1} \int_1^\infty |g(tx)| t^{\operatorname{Re}(s_0)-1} dt dx \\ &\leq \int_{|x|<1} \int_1^\infty C_1 (tx)^{-\delta} t^{\operatorname{Re}(s_0)-1} dx dt + \int_{|x|\geq 1} \int_1^\infty C_2 (tx)^{-2} t^{\operatorname{Re}(s_0)-1} dx dt \end{aligned}$$

for some constants $C_1, C_2 > 0$. So the double integral $\int_{-\infty}^\infty \int_1^\infty g(tx) t^{s_0-1} dt dx$ is absolutely integrable. By Fubini's Theorem,

$$\mathfrak{F}f(0) = \int_1^\infty t^{s_0-1} \int_{-\infty}^\infty g(tx) dx dt = \int_1^\infty t^{s_0-2} \int_{-\infty}^\infty g(x) dx dt = 0.$$

If $\operatorname{Re}(s_0) > 1$, then we can choose δ' such that $\delta' > 1$ and $\operatorname{Re}(s_0) - 1 - \delta' > -1$. Then we have

$$\begin{aligned} |\mathfrak{F}f(0)| &= \left| \int_{-\infty}^\infty f(x) dx \right| = \left| \int_{-\infty}^\infty \int_0^1 g(tx) t^{s_0-1} dt dx \right| \\ &\leq \int_{|x|<1} \int_0^1 |g(tx)| t^{\operatorname{Re}(s_0)-1} dt dx + \int_{|x|\geq 1} \int_0^1 |g(tx)| t^{\operatorname{Re}(s_0)-1} dt dx \\ &\leq \int_{|x|<1} \int_0^1 C_3 t^{\operatorname{Re}(s_0)-1} dx dt + \int_{|x|\geq 1} \int_0^1 C_4 (tx)^{-\delta'} t^{\operatorname{Re}(s_0)-1} dx dt \end{aligned}$$

for some constants $C_3, C_4 > 0$. Again, by Fubini's Theorem,

$$\mathfrak{F}f(0) = - \int_0^1 t^{s_0-1} \int_{-\infty}^\infty g(tx) dx dt = - \int_0^1 t^{s_0-2} \int_{-\infty}^\infty g(x) dx dt = 0.$$

Therefore $f \in \mathcal{H}_\cap$.

Finally, we prove statement iii). By partial integration we have

$$-xf'(x) = \int_1^\infty -xg'(tx)t^{s_0}dt = -g(tx)t^{s_0} \Big|_1^\infty + s_0 \int_1^\infty g(tx)t^{s_0-1}dt = g(x) + s_0f(x).$$

Applying the Mellin transform, we obtain that $s\widehat{f}(s) = \widehat{g}(s) + s_0\widehat{f}(s)$. Hence $\widehat{g}(s) = \widehat{f}(s)(s - s_0)$. This completes the proof of the lemma. \square

Now we are ready to prove Theorem 1.3.

Proof. By Lemma 2.1, it suffices to show that D_- has no eigenvalue λ on \mathcal{H} with $\text{Re}(\lambda) > 1$ or $\text{Re}(\lambda) = 0$.

Assume on the contrary that such λ exists, i.e., λ is an eigenvalue of D_- on \mathcal{H} satisfying $\text{Re}(\lambda) > 1$ or $\text{Re}(\lambda) = 0$. Then there is a function $f \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$ and a $g \in \mathcal{H}_\cap$ such that $-xf'(x) = \lambda f(x) + Zg(x)$. Applying the Mellin transform, we have $s\widehat{f}(s) = \lambda\widehat{f}(s) + \zeta(s)\widehat{g}(s)$. Hence

$$\frac{\widehat{f}(s)}{\zeta(s)} = \frac{\widehat{g}(s)}{s - \lambda}.$$

By the assumption that λ satisfies $\text{Re}(\lambda) = 0$ or $\text{Re}(\lambda) > 1$, we have $\zeta(\lambda) \neq 0$. So $\widehat{f}(s)/\zeta(s)$ is analytic at λ . Hence we have $\widehat{g}(\lambda) = 0$. By Lemma 2.2 iii), there is a function $g_1 \in \mathcal{H}_\cap$ such that $\widehat{g}(s) = \widehat{g}_1(s)(s - s_0)$. Thus $\widehat{f}(s) = \zeta(s)\widehat{g}_1(s)$, which implies that $f = Zg_1 \in Z\mathcal{H}_\cap$. This contradicts the fact that $f \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$. Hence D_- has no eigenvalue λ with $\text{Re}(\lambda) = 0$ or $\text{Re}(\lambda) > 1$. This completes the proof of Theorem 1.3. \square

We may summarize the argument in the proof of Theorem 1.3 and obtain the following corollary. We will apply this corollary several times in the next section.

Corollary 2.1. *Let λ be a complex number with $\text{Re}(\lambda) > -1$ and $\text{Re}(\lambda) \neq 1$. Assume that $f \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$ and $g \in \mathcal{H}_\cap$ satisfy $-xf'(x) = \lambda f(x) + Zg(x)$ for $x > 0$. Then $\widehat{g}(\lambda) \neq 0$ and $\widehat{f}(s)/\zeta(s)$ has a simple pole at λ , which implies that λ is a nontrivial zero of the Riemann zeta function.*

3 Proof of Theorem 1.2

We first prove a technical lemma.

Lemma 3.1. *Let G be an entire function on \mathbb{C} . Then G is the Mellin transform of some function in \mathcal{H}_- if and only if, for every $m \in \mathbb{N}$, $s^m G(s)$ is bounded in any vertical strip with finite width.*

Proof. Let $S = \{\sigma + it \mid \alpha \leq \sigma \leq \beta, t \in \mathbb{R}\}$ be a vertical strip with finite width $\beta - \alpha$, where $-\infty < \alpha < \beta < \infty$. We first show that if $G = \widehat{g}$ for some $g \in \mathcal{H}_-$, then for every

$m \in \mathbb{N}$, $s^m G(s)$ is bounded in any vertical strip with finite width. Since $g \in \mathcal{H}_-$, for any $s \in S$, we have

$$\begin{aligned} |\widehat{g}(s)| &= \left| \int_0^\infty g(x)x^{s-1} dx \right| \\ &\leq \int_0^1 |g(x)x^{s-1}| dx + \int_1^\infty |g(x)x^{s-1}| dx \\ &\leq \int_0^1 |g(x)x^\alpha| dx + \int_1^\infty |g(x)x^{\beta+1-2}| dx \leq C_1 + C_2, \end{aligned}$$

where

$$C_1 = \sup_{0 < x < 1} |g(x)x^\alpha| \quad \text{and} \quad C_2 = \sup_{x \geq 1} |g(x)x^{\beta+1}|.$$

Hence \widehat{g} is bounded in S .

Next we show that the function $s^m \widehat{g}(s)$ is bounded in S for $m \in \mathbb{N}_{>0}$. For $s \in S$ with $|\operatorname{Im}(s)| \leq 1$, $s^m \widehat{g}(s)$ is bounded by continuity. For $s \in S$ with $|\operatorname{Im}(s)| > 1$, $\frac{s}{s+j}$ is bounded for $j = 0, 1, \dots, m-1$. Thus there is a constant $C > 0$ such that

$$|s^m| \leq C |s(s+1) \cdots (s+m-1)|.$$

By partial integration, we have

$$\begin{aligned} |s^m \widehat{g}(s)| &\leq C |s(s+1) \cdots (s+m-1)| |\widehat{g}(s)| \\ &= C \left| s(s+1) \cdots (s+m-1) \int_0^\infty g(x)x^{s-1} dx \right| \\ &= C \left| \int_0^\infty g^{(m)}(x)x^{s+m-1} dx \right|. \end{aligned}$$

The last integral in the above formula is the Mellin transform of the function $x^m g^{(m)}(x)$. Since $g^{(m)} \in \mathcal{H}_-$, we have that this integral is bounded in S . Hence $s^m \widehat{g}^{(m)}(s)$ is bounded in S . This shows that $G = \widehat{g}$ satisfies the desired property.

Conversely, suppose that G is an entire function on \mathbb{C} such that for every $m \in \mathbb{N}$, $s^m G(s)$ is bounded in any vertical strip with finite width. Let

$$h(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s)x^{-s} ds, \quad x > 0,$$

be the Mellin inverse transform of G . The above integral exists for all $c \in \mathbb{R}$ and is independent of c . It is enough to show that $h \in \mathcal{H}_-$.

Since G decreases rapidly on every vertical line, we have

$$h^{(n)}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s)x^{-s-n} (-s)(-s-1) \cdots (-s-n+1) dx$$

for $n \in \mathbb{N}$. So h is smooth on $(0, \infty)$. Also, for any $k \in \mathbb{Z}$ and $x > 0$, we have

$$\begin{aligned} \left| x^k h^{(n)}(x) \right| &= \left| \frac{1}{2\pi i} \int_{k-n-i\infty}^{k-n+i\infty} G(s)(-s)(-s-1)\cdots(-s-n+1)x^{k-n-s} ds \right| \\ &\leq \frac{1}{2\pi} \int_{k-n-i\infty}^{k-n+i\infty} |G(s)(-s)(-s-1)\cdots(-s-n+1)| ds < \infty. \end{aligned}$$

Hence $h \in \mathcal{H}_-$. This completes the proof of the lemma. □

Now we can prove our main theorem.

Proof. We first show that a complex number ρ is an eigenvalue of D_- on \mathcal{H} if and only if ρ is a nontrivial zero of the Riemann zeta function.

Let ρ be a nontrivial zero of $\zeta(s)$. We see that ρ is an eigenvalue of D_- by Li's Theorem [4, Theorem 1.1].

Conversely, let ρ be an eigenvalue of D_- . By Theorem 1.3, we have $0 < \text{Re}(\rho) < 1$. Then by Corollary 2.1, ρ is a nontrivial zero of $\zeta(s)$. This proves the first part of Theorem 1.2.

It remains to show that the algebraic multiplicity of the eigenvalue ρ is equal to the vanishing order of $\zeta(s)$ at ρ . Suppose that the function $\zeta(s)$ has vanishing order $m_0 > 0$ at ρ . We shall show that the algebraic multiplicity of D_- at ρ is also m_0 .

Let $F_1 \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$ and $g_1 \in \mathcal{H}_\cap$ such that $-xF_1'(x) = \rho F_1(x) + Zg_1$. We have, by Proposition 2.1, that the function $\widehat{F}_1(s)/\zeta(s)$ has a simple pole at ρ . So $\widehat{F}_1(s)$ has vanishing order $m_0 - 1$ at ρ .

Notice that for every $m \in \mathbb{N}$, $s^m \widehat{F}_1(s)$ is bounded in any vertical strip with finite width. Then the function $H_i, 0 \leq i < m_0$, given by

$$H_i(s) = \frac{\widehat{F}_1(s)}{(s - \rho)^i}$$

also admits the same property. By Lemma 3.1, H_i is the Mellin transform of some function F_{i+1} in \mathcal{H}_- .

To show that the algebraic multiplicity of D_- at ρ is m_0 , we let, for $n \geq 1$,

$$V_n = \{f \in \mathcal{H}_- \mid (D - \rho)^n f \in Z\mathcal{H}_\cap\}.$$

Notice that the dimension of the subspace $\cup_{n=1}^\infty V_n / Z\mathcal{H}_\cap$ in \mathcal{H} is equal to the algebraic multiplicity of the eigenvalue ρ of D_- acting on \mathcal{H} . Denote by $W_n = V_n / Z\mathcal{H}_\cap$. To prove our theorem, it is sufficient to show the following formula

$$\dim W_n = \begin{cases} n, & 1 \leq n \leq m_0, \\ m_0, & n > m_0, \end{cases} \tag{3.1}$$

holds.

Now we prove formula (3.1). First we show that $V_n = V_{n-1}$ for $n > m_0$, which implies that $W_n = W_{n-1}$ for $n \geq m_0$. Assume that there is an $f \in V_n \setminus V_{n-1}$. We show that, in this case, $n \leq m_0$. The assumption $f \in V_n \setminus V_{n-1}$ means that $(D - \rho)^n f \in Z\mathcal{H}_\cap$ and $(D - \rho)^{n-1} f \notin Z\mathcal{H}_\cap$. Denote by f_1 the function $(D - \rho)^{n-1} f$. Then there is a function $g \in \mathcal{H}_\cap$ such that $-xf_1'(x) = \rho f_1(x) + Zg(x)$ for $x > 0$. By Proposition 2.1, the function $\widehat{f}_1(s)/\zeta(s)$ has a simple pole at ρ . Since $\zeta(s)$ has vanishing order m_0 at ρ , the vanishing order of the function $\widehat{f}_1(s) = (s - \rho)^{n-1} \widehat{f}(s)$ at ρ is $m_0 - 1$. Because \widehat{f} is analytic at ρ , we must have $n \leq m_0$. Hence $W_n = W_{n-1}$ for $n > m_0$.

Next we show that $\dim W_n = n$ for $1 \leq n \leq m_0$. The case $n = 1$ follows from Li's Theorem [4, Theorem 1.4]. For $1 < n \leq m_0$, it is sufficient to show that $\dim(V_n/V_{n-1}) = 1$. Recall that $F_n \in \mathcal{H}_-$ is the Mellin inverse transform of H_{n-1} . Applying Mellin transform to $(D - \rho)^n F_n$ and $(D - \rho)^{n-1} F_n$ and noticing that $F_1 \notin Z\mathcal{H}_\cap$ but $(D - \rho)F_1 \in Z\mathcal{H}_\cap$, we conclude that $F_n \in V_n \setminus V_{n-1}$. Hence $\dim(V_n/V_{n-1}) \geq 1$.

Finally, we show that $\dim(V_n/V_{n-1}) \leq 1$. Let G_1 and G_2 be two functions in $V_n \setminus V_{n-1}$. Then $(D - \rho)^{n-1} G_1, (D - \rho)^{n-1} G_2 \in V_1 \setminus Z\mathcal{H}_\cap$. By Li's Theorem, there is a complex number μ such that

$$(D - \rho)^{n-1} G_1 - \mu(D - \rho)^{n-1} G_2 = (D - \rho)^{n-1} (G_1 - \mu G_2) \in Z\mathcal{H}_\cap.$$

So $G_1 - \mu G_2 \in V_{n-1}$. Hence $\dim(V_n/V_{n-1}) \leq 1$. Therefore

$$\dim(W_n/W_{n-1}) = \dim(V_n/V_{n-1}) = 1$$

for $1 \leq n \leq m_0$ and (3.1) holds. This completes the proof of the theorem. \square

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