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Lyapunov Center Theorem of Infinite Dimensional Hamiltonian Systems

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. In this paper we reformulate a Lyapunov center theorem of infinite dimensional Hamiltonian systems arising from PDEs. The proof is based on a modified KAM iteration for periodic case.

Key Words: Hamiltonian systems, KAM iteration, small divisors, lower dimensional tori.

AMS Subject Classifications: 37K55, 35B10, 35J10, 35Q40

1 Introduction

The KAM theory of infinite dimensional Hamiltonian systems arising from PDEs has been studied widely. Earlier works in this area are due to Wayne, Craig, Bourgain, Kuksin, Pöschel, Eliasson and etc. [4, 6–8, 11, 13, 15]. More recently, the infinite dimensional KAM theory has been developed and there are many infinite dimensional KAM-type theorems. For some related results, see [1–3, 5, 9, 10, 12, 14, 18] and the references therein. We also refer to [17] for a survey on both finite and infinite dimensional KAM theory.

However, most of previous works mainly focus on quasi-periodic case, since periodic case can be regarded as a special quasi-periodic case. But Craig and Wayne constructed periodic solutions of the nonlinear wave equations and nonlinear Schrödinger equations with periodic boundary value by the Nash-Moser methods in weaker small divisor conditions, see [6, 7]. Their results give an infinite dimensional version of the Lyaponuv center theorem.

In this paper, we want to extend the finite dimensional Lyapunov center theorem to infinite dimensional Hamiltonian systems. The result can be regarded as a complement of infinite dimensional KAM theorems. Our results can also be applied to some partial differential equations, but here we do not pursue this problem.

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2 Main results

Consider a nearly integrable infinite dimensional Hamiltonian $H = N + P$. The normal form

$$N = \omega I + \frac{1}{2}\sum_{j=1}^{\infty}\Omega_j(\omega)(u_j^2 + v_j^2), \quad (\theta, I, u, v) \in \Gamma^{a,p} = \mathbb{T} \times \mathbb{R} \times \ell^{a,p} \times \ell^{a,p},$$

where $\mathbb{T}$ is the usual 1-torus, $\mathbb{R}$ is the 1-dimensional real space and the Hilbert space

$$\ell^{a,p} = \left\{w = (w_1, w_2, \cdots) \middle| \|w\|_{a,p} < +\infty\right\}$$

with

$$\|w\|_{a,p}^2 = \sum_{j=1}^{\infty}|w_j|^2 j^{2p} e^{2aj}, \quad a \geq 0, \quad p \geq 0.$$

The frequency $\omega \in O = (\beta_1, \beta_2)$ is regarded as parameter and the normal frequencies $\Omega_1, \Omega_2, \cdots$ are Lipschitz-continuous in the parameter ω.

The small perturbation $P = P(\theta, I, u, v)$ is analytic in (θ, I, u, v) and Lipschitz-continuous in ω, here and below, the dependence of P in the parameter ω is usually implied and not written explicitly only for simplicity if there is no any confusion.

The Hamiltonian system of H is

$$\dot{\theta} = H_I = \omega + P_I, \qquad \dot{I} = -H_\theta = -P_\theta, \tag{2.1a}$$
$$\dot{u} = H_v = \Omega v + P_v, \qquad \dot{v} = -H_u = -\Omega u - P_u, \tag{2.1b}$$

where $\Omega = (\Omega_1, \Omega_2, \cdots)$ and $\Omega u = (\Omega_1 u_1, \Omega_2 u_2, \cdots)$.

If $P = 0$, the system (2.1) is integrable and admits a family of invariant tori

$$\mathcal{T}_\omega = \mathbb{T} \times \{0\} \times \{0\} \times \{0\} \subset \Gamma^{a,p}, \quad \forall \omega \in O,$$

on which there are periodic trajectories

$$\theta = \omega t + \theta_0, \quad \forall \theta_0 \in \mathbb{T}, \quad I = 0, \quad u = 0, \quad v = 0.$$

If $P \neq 0$, the system (2.1) is not integrable in general. If the space $\ell^{a,p}$ is finite dimensional, the Lyaponuv center theorem says, if $k\omega \pm \Omega_j \neq 0, \forall k \in \mathbb{Z}, \forall j \geq 1$, and the perturbation P is sufficiently small, then the Hamiltonian system (2.1) has many 1-dimensional invariant tori with the frequency ω, which are embeddings of $\mathcal{T}_\omega$ in $\Gamma^{a,p}$.

In this paper we will prove a similar result in infinite dimensional case. To state our main results, we first introduce some notations.

Let $f(\theta)$ be a 2π-periodic function and its Fourier series expansion is

$$f(\theta) = \sum_{k \in \mathbb{Z}} f_k e^{\mathrm{i}k\theta}.$$

If $f(\theta)$ is analytic on the trip domain $D(s)=\{\theta||Im\theta|\leq s\}$, denote the super norm of f by

$$|f|_s=\sup_{\theta\in D(s)}|f(\theta)|.$$

Define a little stronger norm by

$$\|f\|_s=\sum_{k\in\mathbb{Z}}|f_k|e^{|k|s}.$$

If f also depends on $\omega\in O$, define the Lipschitz norm of f by

$$|f|_s^L=\sup_{\theta\in D(s)}|f(\theta)|^L,$$

where

$$|f(\theta)|^L=\sup_{\omega_1\neq\omega_2}|L_{\omega_1,\omega_2}f(\theta)|,\quad L_{\omega_1,\omega_2}f(\theta)=\frac{f(\theta;\omega_2)-f(\theta;\omega_1)}{\omega_2-\omega_1},$$

where we write $f(\theta)=f(\theta;\omega)$ for simplicity. Define by

$$\|f\|_s^L=\sum_{k\in\mathbb{Z}}|f_k|^Le^{|k|s}.$$

Sometimes we denote $|f|_s$ and $|f|_s^L$ by $|f|_s^\lambda$ for simplicity. Similarly, we have $\|f\|_s^\lambda$. For some properties of the norms $|\cdot|$ and $||\cdot||$, see Lemma A.1 in Appendix of this paper.

Denote by

$$D(s,r)=\{(\theta,I,u,v)||Im\,\theta|\leq s,\ |I|\leq r,\ \|u\|_{a,p}\leq r,\ \|v\|_{a,p}\leq r\}.$$

Suppose P is analytic in (θ,I,u,v) on $D(s,r)$ and Lipschitz-continuous in ω on O. Let

$$P=\sum_{l,q,\bar{q}}P_{lq\bar{q}}(\theta)I^lu^qv^{\bar{q}}.$$

Define

$$\|P\|_{D(s,r)}^\lambda=\sup_{|I|\leq r,\ \|u\|_{a,p}\leq r,\ \|v\|_{a,p}\leq r}\sum_{l,q,\bar{q}}\|P_{lq\bar{q}}\|_s^\lambda I^lu^qv^{\bar{q}}.$$

If $w(\theta,I,u,v)$ is an analytic map from $D(s,r)$ to $\ell^{a,\bar{p}}$ and Lipschitz-continuous in ω, define

$$\|w\|_{a,\bar{p},D(s,r)}^\lambda=\left(\sum_{j=1}^\infty(\|w_j\|_{D(s,r)}^\lambda)^2j^{2\bar{p}}e^{2aj}\right)^{1/2}.$$

Denote the Hamiltonian vector field of P by $X_P=(P_I,\ -P_\theta,\ P_v,\ -P_u)$. Define

$$|||X_P|||_{D(s,r)}^\lambda=\|P_I\|_{D(s,r)}^\lambda+\|P_\theta\|_{D(s,r)}^\lambda+\|P_u\|_{a,\bar{p},D(s,r)}^\lambda+\|P_v\|_{a,\bar{p},D(s,r)}^\lambda.$$

Assumption 2.1 (Asymtotics of Normal Frequencies). *Suppose*

$$|\Omega_j| \geq bj^d \quad \text{and} \quad |\Omega_j|^L \leq \bar{b}j^{\bar{d}}, \quad \forall j \geq 1,$$

where $b > 0, \bar{b} > 0, d > 1$ and $\bar{d} < d$ are constants. Moreover, there exist positive integers $\bar{N}$, $j_0 > 0$ and a small positive number $\bar{\alpha}$, such that for $\forall j \geq j_0$, $|\Omega_{j+\bar{N}} - \Omega_j| \geq \bar{\alpha}$ hold.

Assumption 2.2 (Regularity of Perturbation). *Let $\bar{p} \geq p$. Suppose P is analytic in (θ, I, u, v) on $D(s,r)$ and Lipschitz- continuous in ω on O and X_P is an analytic mapping from $D(s,r)$ to $\Gamma^{a,\bar{p}}$ and Lipschitz-continuous in ω on O.*

Theorem 2.1 (Infinite Dimensional Lyapunov Center Theorem). *Let*

$$|||X_P|||^{\lambda}_{D(s,r)} = \epsilon.$$

If Assumptions 2.1 and 2.2 hold, then there exist a positive integer $J > 0$ and a sufficiently small constant $\gamma > 0$ independent of ϵ, such that for sufficiently small $0 < \alpha \leq 1$, if

$$|k\omega \pm \Omega_j| \geq \alpha > 0, \quad \forall \omega \in O, \quad \forall j \leq J, \quad \forall k \in \mathbb{Z},$$

and $\epsilon \leq \gamma\alpha^2$, then there exists a nonempty Cantor-like subset $O_\alpha \subset O$, a Lipschitz continuous family of tori embeddings $\Phi: \mathbb{T} \times O_\alpha \to \Gamma^{a,\bar{p}}$ and a Lipschitz-continuous mapping $\omega_*: O_\alpha \to \mathbb{R}$, such that for $\omega \in O_\alpha$, the map Φ restricted to $\mathbb{T} \times \{\omega\}$ is a real analytic embedding of a rotational torus with frequency $\omega_*(\omega)$ for the Hamiltonian H at ω. Thus, for $\omega \in O_\alpha$, $\Phi(\omega_* t + \theta_0; \omega)$ is a periodic trajectory of the system (2.1). Moreover,

$$|||\Phi - \Phi_0|||_{D(s/2,r/2)} \leq c\epsilon/\alpha,$$

where Φ_0 is the trivial embedding

$$\begin{aligned}&\mathbb{T} \times O_\alpha \to \mathbb{T} \times \{0\} \times \{0\} \times \{0\} \subset \Gamma^{a,\bar{p}},\\ &|\omega^*(\omega) - \omega| \leq c\epsilon, \quad \omega \in O_\alpha,\end{aligned}$$

and

$$meas(O \setminus O_\alpha) \leq c\alpha,$$

where c is independent of ϵ, α.

The measure estimates of $O \setminus O_\alpha$ is based on the following theorem.

Theorem 2.2. *If Assumption 2.1 holds and $\omega_k : \omega \in O \to \mathbb{R}$ satisfying*

$$\inf_{\omega \in O} |\omega_k| \geq \beta, \quad |\omega_k|^L \geq \frac{1}{2}, \quad \forall k \in \mathbb{Z},$$

then there exists $J > 0$, such that

$$meas\left(\cup_{j \geq J,\, k \in \mathbb{Z}} \mathcal{R}_{kj}(\alpha)\right) \leq c\alpha,$$

where $\mathcal{R}_{kj}(\alpha) = \{\omega \,||k\omega_k \pm \Omega_j| \leq \alpha\}$ and c is a constant independent of α.

Remark 2.1. It is well known that small divisor conditions are necessary for KAM theorems. However, in periodic case the small divisor condition is very special. In finite dimensional case, the small divisor condition can hold for the parameter on a interval, so the invariant tori exist continuously depending on the frequency ω. But this is not true for infinite dimensional Hamiltonian systems. Since there are infinitively many normal frequencies, the small divisor conditions can only hold on a Cantor-like subset even in the periodic case, which causes much difficulty in the estimates of Lipschitz-norms with respect to the parameter ω. Moreover, the smoothness of the parameter is in the sense of Whitney [9]. Furthermore, the Lipschitz-norms with respect to the parameter ω can result in some loss of regularity about the angle variable θ, so the traditional implicit function theorem is not valid for our problem, the proof is still based on KAM iteration.

Remark 2.2. The asymtotics of normal frequencies that $\Omega_j \geq bj^d$ with $d > 1$ is very important for small divisor conditions $|k\omega \pm \Omega_j| \geq \alpha$. In fact, if $0 < d \leq 1$, the above small divisor conditions may not hold.

3 Proof of main theorems

At first we make the complex conjugate coordinates change

$$z = (u - \mathrm{i}v)/\sqrt{2}, \quad \bar{z} = (u + \mathrm{i}v)/\sqrt{2},$$

then the associated symplectic form is $d\theta \wedge dI + \mathrm{i}dz \wedge d\bar{z}$. In the new coordinates,

$$N = \omega I + \langle \Omega z, \bar{z} \rangle \quad \text{and} \quad P = \sum_{l,q,\bar{q}} P_{lq\bar{q}}(\theta) I^l z^q \bar{z}^{\bar{q}},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the Hilbert space $\ell^2 = \ell^{0,0}$. Below, without confusion we use the same notations $''\ell^{a,p}''$ and $''\Gamma^{a,\bar{p}}''$ to denote their complexification respectively. Thus, $D(s,r)$ is a complex neighbourhood of $T \times \{0\} \times \{0\} \times \{0\}$ in $\Gamma^{a,p}$.

The proof of Theorem 2.1 is based on a KAM iteration. We construct a sequence of symplectic maps $\Phi_0, \Phi_1, \cdots \Phi_{n-1}$ such that the zero-th and the first order coefficients of $H \circ \Phi_0 \circ \Phi_1 \circ \cdots \Phi_{n-1}$ with respect to I, z and $\bar{z}$ become smaller and smaller and finally vanish. In this paper, the special small divisor conditions $|k\omega \pm \Omega_j| \geq \alpha$ do not cause the loss of smoothness, so the zero-th and the first order coefficients of $H \circ \Phi_0 \circ \Phi_1 \circ \cdots \Phi_{n-1}$ with respect to $I, z, \bar{z}$ can become smaller and smaller. This is different from quasi-periodic case.

Below, in all estimates of KAM step we always use the same c to denote the constants of estimates which are independent of KAM steps.

Outline of KAM steps

Let $H=N+P$, where $N=\omega I+\langle\Omega z,\bar z\rangle$, $P=R+\bar P$, where

$$R=P_{000}(\theta)+P_{100}(\theta)I+\langle P_{010}(\theta),z\rangle+\langle P_{001}(\theta),\bar z\rangle,$$
$$\bar P=\sum_{l+|q|+|\bar q|\ge 2}P_{lq\bar q}(\theta)I^l z^q\bar z^{\bar q},$$

where $q=(q_1,q_2,\cdots)\in\mathbb{N}^\infty$ and $|q|=|q_1|+|q_2|+\cdots$ and $\bar q$ has the same meaning as q. Let

$$|||X_{\bar P}|||^\lambda_{D(s,r)}\le\epsilon_1^\lambda,\quad |||X_R|||^\lambda_{D(s,r)}\le\epsilon_2^\lambda,\quad \epsilon_1\le\epsilon_1^L,\quad \epsilon_2\le\epsilon_2^L.$$

Note that $X_P=(P_I,-P_\theta,\mathrm{i}P_{\bar z},-\mathrm{i}P_z)$. Thus we have

$$\|P'_{000}\|^\lambda_s\le\epsilon_2^\lambda,\quad \|P_{100}\|^\lambda_s\le\epsilon_2^\lambda,\quad \|P'_{100}\|^\lambda_s\le\epsilon_2^\lambda/r, \tag{3.1a}$$
$$\|P_{010}\|^\lambda_{a,\bar p,D(s)}\le\epsilon_2^\lambda,\quad \|P_{001}\|^\lambda_{a,\bar p,D(s)}\le\epsilon_2^\lambda, \tag{3.1b}$$
$$\|P'_{010}\|^\lambda_{-a,-\bar p,D(s)}\le\epsilon_2^\lambda/r,\quad \|P'_{001}\|^\lambda_{-a,-\bar p,D(s)}\le\epsilon_2^\lambda/r, \tag{3.1c}$$

where

$$P'_{lq\bar q}=\frac{d}{d\theta}P_{lq\bar q}.$$

Let

$$P_{010}(\theta)=\sum_{k\in\mathbb{Z}}P^k_{010}e^{ik\theta},\quad \tilde P_{010}(\theta)=\sum_{|k|\le K}P^k_{010}e^{ik\theta},\quad P^K_{010}(\theta)=P_{010}(\theta)-\tilde P_{010}(\theta).$$

Similarly, we have $\tilde P_{001}(\theta)$ and $P^K_{001}(\theta)$. Let

$$\tilde R=P_{000}(\theta)+P_{100}(\theta)I+\langle\tilde P_{010}(\theta),z\rangle+\langle\tilde P_{001}(\theta),\bar z\rangle.$$

3.1 Solving linear homological equation

The linear homological equation is as follows:

$$\{N,F\}+\tilde R=\hat N, \tag{3.2}$$

where $\{\cdot,\cdot\}$ is the Poisson product, $\hat N$ and F are functions to be solved.

Let $\hat N=[P_{000}]+[P_{100}]I$, where

$$[P_{lq\bar q}]=\frac{1}{2\pi}\int_0^{2\pi}P_{lq\bar q}(\theta)d\theta,$$
$$F=F_{000}(\theta)+F_{100}(\theta)I+\langle F_{010}(\theta),z\rangle+\langle F_{001}(\theta),\bar z\rangle.$$

By (3.2) we have

$$F'_{000}(\theta)=\frac{1}{\omega}(P_{000}(\theta)-[P_{000}]),\qquad F'_{100}(\theta)=\frac{1}{\omega}(P_{100}(\theta)-[P_{100}]),\tag{3.3a}$$

$$\omega F'_{010}(\theta)+\mathrm{i}\,\Omega F_{010}(\theta)=\tilde{P}_{010}(\theta),\qquad \omega F'_{001}(\theta)-\mathrm{i}\Omega F_{001}(\theta)=\tilde{P}_{001}(\theta).\tag{3.3b}$$

By (3.1) and (3.3a), it follows easily

$$\|F'_{000}\|_s\le\frac{1}{\beta_1}\epsilon_2,\quad \|F'_{000}\|_s^L\le\Big(\frac{1}{\beta_1}+\frac{1}{\beta_1^2}\Big)\epsilon_2^L.$$

In the same way we have

$$\begin{aligned}&\|F''_{000}\|_s\le\frac{1}{\omega}\|P'_{000}\|_s\le\frac{1}{\beta_1}\epsilon_2,\\ &\|F''_{000}\|_s^L\le\frac{1}{\beta_1}\|P'_{000}\|_s^L+\frac{1}{\beta_1^2}\|P'_{000}\|_s\le\Big(\frac{1}{\beta_1}+\frac{1}{\beta_1^2}\Big)\epsilon_2^L.\end{aligned}$$

Thus we have

$$\|F'_{000}\|_s^\lambda\le c\epsilon_2^\lambda,\quad \|F''_{000}\|_s^\lambda\le c\epsilon_2^\lambda,\tag{3.4}$$

where $c=\frac{1}{\beta_1}+\frac{1}{\beta_1^2}$. In the same way it follows that

$$\|F'_{100}\|_s^\lambda\le c\epsilon_2^\lambda,\quad \|F''_{100}\|_s^\lambda\le c\epsilon_2^\lambda/r.\tag{3.5}$$

The estimates of F_{010} and F_{001} are more complicated. Let

$$F_{010}=(F^1_{010},F^2_{010},\cdots),\quad F^j_{010}=\sum_{k\in Z}F^{jk}_{010}e^{ik\theta},\quad j\ge 1.$$

If

$$|k\omega\pm\Omega_j|\ge\alpha,\quad \forall|k|\le K,\quad \forall j\ge 1,\tag{3.6}$$

from (3.3b) that

$$F^{jk}_{010}=\frac{P^{jk}_{010}}{i(k\omega+\Omega_j)}\quad\text{and}\quad |F^{jk}_{010}|\le\frac{1}{\alpha}|P^{jk}_{010}|,\quad |k|\le K,\quad j\ge 1.$$

Thus we obtain

$$\|F_{010}\|_{a,\bar{p},D(s)}\le\frac{1}{\alpha}\|P_{010}\|_{a,\bar{p},D(s)}\le\frac{\epsilon_2}{\alpha}.\tag{3.7}$$

For the Lipschitz-norms we have

$$|F_{010}^{jk}|^L \le \frac{1}{\alpha}|P_{010}^{jk}|^L + \frac{|k| + |\Omega_j|^L}{|k\omega + \Omega_j|^2}|P_{010}^{jk}|.$$

If $|\Omega_j| \ge 2\beta_2|k|$, then $|k\omega + \Omega_j| \ge |\Omega_j|$. By Assumption 2.1, it follows

$$\frac{|k| + |\Omega_j|^L}{|k\omega + \Omega_j|^2} \le 4\frac{|k| + \frac{\bar{b}}{b}|\Omega_j|}{|\Omega_j|^2} \le c,$$
$$|F_{010}^{jk}|^L \le \frac{1}{\alpha}|P_{010}^{jk}|^L + c|P_{010}^{jk}|,$$

where $c = \frac{2}{\beta_2 b} + \frac{\bar{b}}{b^2}$. If $|\Omega_j| \le 2\beta_2|k|$, then

$$\frac{|k| + |\Omega_j|^L}{|k\omega + \Omega_j|^2} \le \frac{1}{\alpha^2}\Big(|k| + \frac{\bar{b}}{b}|\Omega_j|\Big) \le \frac{c}{\alpha^2}|k|$$

and so

$$|F_{010}^{jk}|^L \le \frac{1}{\alpha}|P_{010}^{jk}|^L + \frac{c}{\alpha^2}|k||P_{010}^{jk}|.$$

Using Lemma A.1 and (3.1b) we have

$$\|F_{010}\|^L_{a,\bar{p},D(s-\rho)} \le \frac{1}{\alpha}\|P_{010}\|^L_{a,\bar{p},D(s-\rho)} + \frac{c}{\alpha^2}\|P'_{010}\|_{a,\bar{p},D(s-\rho)} \le \frac{\epsilon_2^L}{\alpha} + \frac{c\epsilon_2}{\alpha^2\rho^2}. \tag{3.8}$$

In the same way as above, by (3.1c) we have

$$\|F'_{010}\|_{-a,-\bar{p},D(s)} \le \frac{1}{\alpha}\|P'_{010}\|_{-a,-\bar{p},D(s)} \le \frac{\epsilon_2}{\alpha r}, \tag{3.9a}$$
$$||F'_{010}||^L_{-a,-\bar{p},D(s-\rho)} \le \frac{1}{\alpha}||P'_{010}||^L_{-a,-\bar{p},D(s-\rho)} + \frac{c}{\alpha^2}||P''_{010}||_{-a,-\bar{p},D(s-\rho)} \le \frac{\epsilon_2^L}{\alpha r} + \frac{c\epsilon_2}{\alpha^2\rho^2 r}. \tag{3.9b}$$

Similarly, for F_{001} we have

$$||F_{001}||_{a,\bar{p},D(s)} \le \frac{\epsilon_2}{\alpha}, \tag{3.10a}$$
$$||F_{001}||^L_{a,\bar{p},D(s-\rho)} \le \frac{\epsilon_2^L}{\alpha} + \frac{c\epsilon_2}{\alpha^2\rho^2}, \tag{3.10b}$$
$$||F'_{001}||_{-a,-\bar{p},D(s)} \le \frac{\epsilon_2}{\alpha r}, \tag{3.10c}$$
$$||F'_{001}||^L_{-a,-\bar{p},D(s-\rho)} \le \frac{\epsilon_2^L}{\alpha r} + \frac{c\epsilon_2}{\alpha^2\rho^2 r}. \tag{3.10d}$$

3.2 Coordinates transformations

The symplectic map $\Phi = X_F^1$ is the time-1 map of flow $X_F^t = (\theta^t, I^t, z^t, \bar{z}^t)$ of the following differential equations

$$\begin{cases} \dot{\theta} = F_I = F_{100}(\theta), \\ \dot{I} = -F_\theta = -F'_{000}(\theta) - F'_{100}(\theta) I - \langle F'_{010}(\theta), z\rangle - \langle F'_{001}(\theta), \bar{z}\rangle, \\ \dot{z} = \mathrm{i} F_{\bar{z}} = \mathrm{i} F_{001}(\theta), \\ \dot{\bar{z}} = -\mathrm{i} F_z = -\mathrm{i} F_{010}(\theta). \end{cases} \tag{3.11}$$

By integrating the first equation we have

$$\theta^t - \theta_+ = \int_0^t F_{100}(\theta^\tau) d\tau,$$

where $\theta^t|_{t=0} = \theta_+$.

If $|F_{100}|_s \leq ||F_{100}||_s \leq c\epsilon_2 \leq \rho$, then θ^t exists for $\theta_+ \in D(s-\rho)$ and $|t| \leq 1$. Thus θ^t is a mapping from $D(s-\rho)$ to $D(s)$. For $\theta_+ \in D(s-\rho)$ and $|t| \leq 1$ we have

$$|\theta^t(\theta_+) - \theta_+| \leq |F_{100}|_s \leq c\epsilon_2. \tag{3.12}$$

By the above equation

$$|\theta^t|^L \leq \int_0^t |F_{100}(\theta)|_s^L d\tau + \int_0^t |F'_{100}(\theta)|_s |\theta^\tau|^L d\tau \leq c\epsilon_2^L + c\epsilon_2 \int_0^t |\theta^\tau|^L d\tau.$$

By Gronwall's inequality we have

$$|\theta^t|^L \leq c\epsilon_2^L, \quad \forall \theta_+ \in D(s-\rho), \quad \forall t \in [0,1]. \tag{3.13}$$

By Lemma A.2 and (3.5), if $\epsilon_2 \ll \rho^2$, we have

$$||\theta^t||_{s-3\rho}^L \leq \int_0^t ||F_{100}(\theta^\tau)||_{s-3\rho}^L d\tau \leq c\epsilon_2^L, \tag{3.14}$$

where $||\theta^t||_{s-3\rho}^L$ is the norm of $\theta(\theta_+)$ with respect to θ_+ and ω. Since

$$D_{\theta_+}\theta^t - 1 = \int_0^t F'_{100}(\theta^\tau) D_{\theta_+}\theta^\tau d\tau,$$

by Lemma A.2 and (3.5) it follows

$$||D_{\theta_+}\theta^t - 1||_{s-3\rho} \leq c\epsilon_2 \int_0^t ||D_{\theta_+}\theta^\tau||_{s-3\rho} d\tau.$$

By Gronwall's inequality we have

$$||D_{\theta_+}\theta^t - 1||_{s-3\rho} \leq c\epsilon_2, \tag{3.15}$$

where $||\cdot||_{s-3\rho}$ is the norm with respect to the new variable θ_+. For the Lipschitz norm, also by Lemma A.2 we have

$$\begin{aligned}||D_{\theta_+}\theta^t||^L_{s-3\rho}\leq&\int_0^t ||F'_{100}(\theta^\tau)||^L_{s-3\rho}||D_{\theta_+}\theta^\tau||_{s-3\rho}d\tau\\&+\int_0^t ||F'_{100}(\theta^\tau)||_{s-3\rho}||D_{\theta_+}\theta^\tau||^L_{s-3\rho}\,d\tau\\\leq& c\epsilon_2^L+c\epsilon_2\int_0^t ||D_{\theta_+}\theta^\tau||^L_{s-3\rho}d\tau.\end{aligned}$$

By Gronwall's inequality we have

$$||D_{\theta_+}\theta^t||^L_{s-3\rho}\leq c\epsilon_2^L. \tag{3.16}$$

Now we consider the transformations for z and $\bar{z}$. Since

$$z^t-z_+=\mathrm{i}\int_0^t F_{001}(\theta^\tau)d\tau,$$

by Lemma A.2 and (3.10a) we have

$$||z^t-z_+||_{a,\bar{p},D(s-3\rho)}\leq\frac{c\epsilon_2}{\alpha}. \tag{3.17}$$

In the same way as the above and using (3.10b) it follows that

$$||z^t-z_+||^L_{a,\bar{p},D(s-4\rho)}\leq\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}. \tag{3.18}$$

By (3.17), (3.18) and Lemma A.1 we have

$$||D_{\theta_+}z^t||_{a,\bar{p},D(s-4\rho)}\leq\frac{c\epsilon_2}{\alpha\rho^2}, \tag{3.19a}$$

$$||D_{\theta_+}z^t||^L_{a,\bar{p},D(s-5\rho)}\leq\frac{c\epsilon_2^L}{\alpha\rho^2}+\frac{c\epsilon_2}{\alpha^2\rho^4}. \tag{3.19b}$$

Moreover, by (3.10c), (3.10d) and Lemma A.2 it follows easily

$$||D_{\theta_+}z^t||_{-a,-\bar{p},D(s-3\rho)}\leq\frac{c\epsilon_2}{\alpha r}, \tag{3.20a}$$

$$||D_{\theta_+}z^t||^L_{-a,-\bar{p},D(s-4\rho)}\leq\frac{c\epsilon_2^L}{\alpha r}+\frac{c\epsilon_2}{\alpha^2\rho^2 r}. \tag{3.20b}$$

Similarly, we have

$$||\bar{z}^t - \bar{z}_+||_{a,\bar{p},D(s-3\rho)} \leq \frac{c\epsilon_2}{\alpha}, \tag{3.21a}$$

$$||\bar{z}^t - \bar{z}_+||^L_{a,\bar{p},D(s-4\rho)} \leq \frac{c\epsilon_2^L}{\alpha} + \frac{c\epsilon_2}{\alpha^2\rho^2}, \tag{3.21b}$$

$$||D_{\theta_+}\bar{z}^t||_{a,\bar{p},D(s-4\rho)} \leq \frac{c\epsilon_2}{\alpha\rho^2}, \tag{3.21c}$$

$$||D_{\theta_+}\bar{z}^t||^L_{a,\bar{p},D(s-5\rho)} \leq \frac{c\epsilon_2^L}{\alpha\rho^2} + \frac{c\epsilon_2}{\alpha^2\rho^4}, \tag{3.21d}$$

$$||D_{\theta_+}\bar{z}^t||_{-a,-\bar{p},D(s-3\rho)} \leq \frac{c\epsilon_2}{\alpha r}, \tag{3.21e}$$

$$||D_{\theta_+}\bar{z}^t||^L_{-a,-\bar{p},D(s-4\rho)} \leq \frac{c\epsilon_2^L}{\alpha r} + \frac{c\epsilon_2}{\alpha^2\rho^2 r}. \tag{3.21f}$$

Now we estimate the transformation of variable I, which is a little more complicated. By integrating the second equation of (3.11) we have

$$\begin{aligned} I^t - I_+ =& -\int_0^t F'_{000}(\theta^\tau)d\tau - \int_0^t F'_{100}(\theta^\tau)I^\tau d\tau \\ & - \int_0^t \langle F'_{010}(\theta^\tau), z^\tau\rangle\, d\tau - \int_0^t \langle F'_{001}(\theta^\tau), \bar{z}^\tau\rangle d\tau \\ =:& J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Let $0 < \frac{c\epsilon_2}{\alpha} < \delta < r$. By Lemma A.2 and using (3.4), (3.5), (3.9a), (3.10c), (3.17) and (3.21a) it follows that

$$||I^t - I_+||_{D(s-3\rho,r-\delta)} \leq \frac{c\epsilon_2}{\alpha} + c\epsilon_2 \int_0^t ||I^\tau - I_+||_{D(s-3\rho,r-\delta)} d\tau.$$

By Gronwall's inequality we have

$$||I^t - I_+||_{D(s-3\rho,r-\delta)} \leq \frac{c\epsilon_2}{\alpha}. \tag{3.22}$$

Now we estimate Lipschitz norm. By Lemma A.2 and using (3.4) and (3.6) we have

$$\begin{aligned} ||J_1||^L_{D(s-4\rho)} &\leq c\epsilon_2^L, \\ ||J_2||^L_{D(s-4\rho,r-\delta)} &\leq \int_0^t ||F'_{100}(\theta^\tau)||^L_{D(s-4\rho)}||I^\tau||_{D(s-4\rho,r-\delta)}\, d\tau \\ &\quad + \int_0^t ||F'_{100}(\theta^\tau)||_{D(s-4\rho)}||I^\tau||^L_{D(s-4\rho,r-\delta)}\, d\tau \\ &\leq c\epsilon_2^L + c\epsilon_2 \int_0^t ||I^\tau||^L_{D(s-4\rho,r-\delta)}. \end{aligned}$$

Since

$$||J_3||^L_{D(s-4\rho,r-\delta)} \leq \int_0^t ||F'_{010}(\theta^\tau)||^L_{-a,-\bar{p},D(s-4\rho)}||z^\tau||_{a,\bar{p},D(s-4\rho,r-\delta)}d\tau + \int_0^t ||F'_{010}(\theta^\tau)||_{-a,-\bar{p},D(s-4\rho)}||z^\tau||^L_{a,\bar{p},D(s-4\rho,r-\delta)},$$

by Lemma A.2 and using (3.9a), (3.9b), (3.17), (3.18), if $\epsilon_2 \leq \alpha r$, we have

$$||J_3||^L_{D(s-4\rho,r-\delta)} \leq \frac{c\epsilon_2^L}{\alpha} + \frac{c\epsilon_2}{\alpha^2\rho^2}.$$

Similarly we have

$$||J_4||^L_{D(s-4\rho,r-\delta)} \leq \frac{c\epsilon_2^L}{\alpha} + \frac{c\epsilon_2}{\alpha^2\rho^2}.$$

Thus,

$$||I^t||^L_{D(s-4\rho,r-\delta)} \leq \frac{c\epsilon_2^L}{\alpha} + \frac{c\epsilon_2}{\alpha^2\rho^2} + c\epsilon_2\int_0^t ||I^\tau||^L_{D(s-4\rho,r-\delta)}d\tau.$$

By Gronwall 's inequality we have

$$||I^t||^L_{D(s-4\rho,r-\delta)} \leq \frac{c\epsilon_2^L}{\alpha} + \frac{c\epsilon_2}{\alpha^2\rho^2}. \tag{3.23}$$

By (3.22) and (3.23) and using Lemma A.1 we have

$$||D_{\theta_+}I^t||_{D(s-4\rho,r-\delta)} \leq \frac{c\epsilon_2}{\alpha\rho^2}, \tag{3.24a}$$

$$||D_{\theta_+}I^t||^L_{D(s-5\rho,r-\delta)} \leq \frac{c\epsilon_2^L}{\alpha\rho^2} + \frac{c\epsilon_2}{\alpha^2\rho^4}. \tag{3.24b}$$

Now we estimate $||D_{\theta_+}I^t||_{D(s-3\rho,\frac{c\epsilon_2}{\alpha})}$. By derivating I^t with respect to θ_+ we have

$$\begin{aligned} D_{\theta_+}I^t = &-\int_0^t F''_{000}(\theta^\tau)D_{\theta_+}\theta^\tau d\tau - \int_0^t F''_{100}(\theta^\tau)D_{\theta_+}\theta^\tau I^\tau d\tau \\ &- \int_0^t F'_{100}(\theta^\tau)D_{\theta_+}I^\tau\, d\tau - \int_0^t \langle F''_{010}(\theta^\tau), z^\tau\rangle D_{\theta_+}\theta^\tau d\tau \\ &- \int_0^t \langle F'_{010}(\theta^\tau), D_{\theta_+}z^\tau\rangle d\tau - \int_0^t \langle F''_{001}(\theta^\tau), \bar{z}^\tau\rangle D_{\theta_+}\theta^\tau d\tau \\ &- \int_0^t \langle F_{001}(\theta^\tau), D_{\theta_+}\bar{z}^\tau\rangle d\tau. \end{aligned} \tag{3.25}$$

Without loss of generality we only estimate the second term and the forth term, since the other terms can be estimated similarly or even more simple. If

$$|I_+| \leq \frac{c\epsilon_2}{\alpha}, \quad ||z_+||_{a,\bar{p}} \leq \frac{c\epsilon_2}{\alpha}, \quad ||\bar{z}_+||_{a,\bar{p}} \leq \frac{c\epsilon_2}{\alpha},$$

it is easy to see that

$$|I^t| \leq \frac{2c\epsilon_2}{\alpha}, \quad ||z^t||_{a,\bar{p}} \leq \frac{2c\epsilon_2}{\alpha}, \quad ||\bar{z}^t||_{a,\bar{p}} \leq \frac{2c\epsilon_2}{\alpha}.$$

Using Lemma A.2, (3.5) and (3.15) we have

$$\Big\| \int_0^t F''_{100}(\theta^\tau) D_{\theta_+}\theta^\tau I^\tau \, d\tau \Big\|_{D(s-4\rho,\frac{c\epsilon_2}{\alpha})} \leq \frac{c\epsilon_2}{r} \cdot \frac{c\epsilon_2}{\alpha} \leq \frac{c\epsilon_2}{\alpha r}.$$

By Lemmas A.2 and A.1, and using (3.9a), (3.15), if $\epsilon_2 \leq \alpha\rho^2$, we have

$$\Big\| \int_0^t \langle F''_{010}(\theta^\tau), z^\tau \rangle D_{\theta_+}\theta^\tau d\tau \Big\|_{D(s-4\rho,\frac{c\epsilon_2}{\alpha})} \leq \frac{c\epsilon_2}{r\alpha\rho^2} \frac{c\epsilon_2}{\alpha} \leq \frac{c\epsilon_2}{\alpha r}.$$

By (3.25) and combining all the above estimates we have

$$||D_{\theta_+} I^t||_{D(s-4\rho,\frac{c\epsilon_2}{\alpha})} \leq \frac{c\epsilon_2}{\alpha r} + c\epsilon_2 \int_0^t ||D_{\theta_+} I^\tau||_{D(s-4\rho,\frac{c\epsilon_2}{\alpha})} d\tau.$$

By Gronwall's inequality it follows that

$$||D_{\theta_+} I^t||_{D(s-4\rho,\frac{c\epsilon_2}{\alpha})} \leq \frac{c\epsilon_2}{\alpha r}. \tag{3.26}$$

To estimate Lipschitz-norm of $D_{\theta_+} I^t$, we still only consider the second term and the forth term in (3.25) without loss of generality. By Lemma A.2 and combining (3.5), (3.15), (3.16), (3.22) and (3.23), it follows easily that

$$\begin{aligned}
&\Big\| \int_0^t F''_{100}(\theta^\tau) D_{\theta_+}\theta^\tau I^\tau d\tau \Big\|^L_{D(s-4\rho,\frac{c\epsilon_2}{\alpha})} \\
\leq & \int_0^t ||F''_{100}(\theta^\tau)||^L_{s-4\rho} ||D_{\theta_+}\theta^\tau||_{s-4\rho} ||I^\tau||_{D(s-4\rho,\frac{c\epsilon_2}{\alpha})} \, d\tau \\
& + \int_0^t ||F''_{100}(\theta^\tau)||_{s-4\rho} ||D_{\theta_+}\theta^\tau||^L_{s-4\rho} ||I^\tau||_{D(s-4\rho,\frac{c\epsilon_2}{\alpha})} \, d\tau \\
& + \int_0^t ||F''_{100}(\theta^\tau)||_{s-4\rho} ||D_{\theta_+}\theta^\tau||_{s-4\rho} ||I^\tau||^L_{D(s-4\rho,\frac{c\epsilon_2}{\alpha})} d\tau \\
\leq & \frac{c\epsilon_2^L}{\alpha r} + \frac{c\epsilon_2}{\alpha^2\rho^2 r}.
\end{aligned}$$

In the same way as the above, if $\epsilon_2 \leq \alpha\rho^2$, by Lemmas A.1 and A.2 and using (3.9a), (3.9b) and (3.15)-(3.18) we have

$$\begin{aligned}
&\Big\| \int_0^t \langle F''_{010}(\theta^\tau), z^\tau\rangle D_{\theta_+}\theta^\tau d\tau \Big\|^L_{D(s-5\rho,\frac{c\epsilon_2}{\alpha})} \\
\le& \int_0^t ||F''_{010}(\theta^\tau)||^L_{-a,-\bar{p},D(s-5\rho)}||z^\tau||_{a,\bar{p},D(s-5\rho,\frac{c\epsilon_2}{\alpha})}||D_{\theta_+}\theta^\tau||_{s-5\rho}\,d\tau \\
&\quad + \int_0^t ||F''_{010}(\theta^\tau)||_{-a,-\bar{p},D(s-5\rho)}||z^\tau||^L_{a,\bar{p},D(s-5\rho,\frac{c\epsilon_2}{\alpha})}||D_{\theta_+}\theta^\tau||_{s-5\rho}\,d\tau \\
&\quad + \int_0^t ||F''_{010}(\theta^\tau)||_{-a,-\bar{p},D(s-5\rho)}||z^\tau||_{a,\bar{p},D(s-5\rho,\frac{c\epsilon_2}{\alpha})}||D_{\theta_+}\theta^\tau||^L_{s-5\rho}\,d\tau \\
\le& \frac{c\epsilon_2^L}{\alpha r} + \frac{c\epsilon_2}{\alpha^2\rho^2 r}.
\end{aligned}$$

By (3.25) and using the above estimates we have

$$||D_{\theta_+}I^t||^L_{D(s-5\rho,\frac{c\epsilon_2}{\alpha})} \le \frac{c\epsilon_2^L}{\alpha r} + \frac{c\epsilon_2}{\alpha^2\rho^2 r} + c\epsilon_2\int_0^t ||D_{\theta_+}I^\tau||^L_{D(s-5\rho,\frac{c\epsilon_2}{\alpha})}d\tau.$$

By Gronwall inequality it follows that

$$||D_{\theta_+}I^t||^L_{D(s-5\rho,\frac{c\epsilon_2}{\alpha})} \le \frac{c\epsilon_2^L}{\alpha r} + \frac{c\epsilon_2}{\alpha^2\rho^2 r}. \tag{3.27}$$

Since

$$D_{I_+}I^t - 1 = -\int_0^t F'_{100}(\theta^\tau)D_{I_+}I^\tau\,d\tau,$$

in the same way as the above and by (3.5) it follows that

$$||D_{I_+}I^t - 1||_{D(s-3\rho,r-\delta)} \le c\epsilon_2, \tag{3.28a}$$
$$||D_{I_+}I^t||^L_{D(s-3\rho,r-\delta)} \le c\epsilon_2^L. \tag{3.28b}$$

For the derivatives of I^t with respect to z_+, we have

$$D_{z_+}I^t = -\int_0^t F'_2(\theta^\tau)D_{z_+}I^\tau d\tau - \int_0^t F'_3(\theta^\tau)d\tau.$$

In the same way as the above, by (3.5) and (3.7) it follows that

$$||D_{z_+}I^t||_{a,\bar{p},D(s-4\rho)} \le c\epsilon_2\int_0^t ||D_{z_+}I^t||_{a,\bar{p},D(s-4\rho)}d\tau + \frac{c\epsilon_2}{\alpha\rho^2},$$

so,

$$||D_{z_+}I^t||_{a,\bar{p},D(s-4\rho)} \le \frac{c\epsilon_2}{\alpha\rho^2}. \tag{3.29}$$

For Lipschitz norm, by (3.5), (3.8) and Lemma A.1, we have

$$\begin{aligned}
&||D_{z_+}I^t||^L_{a,\bar{p},D(s-4\rho)}\\
\leq&\int_0^t ||F'_{100}(\theta^\tau)||^L_{D(s-4\rho)}||D_{z_+}I^\tau||_{a,\bar{p},D(s-4\rho)}d\tau\\
&\quad+\int_0^t ||F'_{100}(\theta^\tau)||_{D(s-4\rho)}||D_{z_+}I^\tau||^L_{a,\bar{p},D(s-4\rho)}d\tau+\int_0^t ||F'_3(\theta^\tau)||^L_{a,\bar{p},D(s-4\rho)}\,d\tau\\
\leq&\frac{c\epsilon_2^L}{\alpha\rho^2}+\frac{c\epsilon_2}{\alpha^2\rho^4}+c\epsilon_2\int_0^t ||D_{z_+}I^\tau||^L_{a,\bar{p},D(s-4\rho)},
\end{aligned}$$

so

$$||D_{z_+}I^t||^L_{a,\bar{p},D(s-4\rho)}\leq\frac{c\epsilon_2^L}{\alpha\rho^2}+\frac{c\epsilon_2}{\alpha^2\rho^4}. \tag{3.30}$$

Similarly we have

$$||D_{\bar{z}_+}I^t||_{a,\bar{p},D(s-4\rho)}\leq\frac{c\epsilon_2}{\alpha\rho^2}, \tag{3.31a}$$

$$||D_{\bar{z}_+}I^t||^L_{a,\bar{p},D(s-4\rho)}\leq\frac{c\epsilon_2^L}{\alpha\rho^2}+\frac{c\epsilon_2}{\alpha^2\rho^4}. \tag{3.31b}$$

Let $s_+=s-5\rho$, $r_+=r-2\delta$. Combining above estimates we have

$$|||\Phi-id|||_{D(s_+,r_+)}\leq\frac{c\epsilon_2}{\alpha},\quad |||D\Phi-Id|||_{D(s_+,r_+)}\leq\frac{c\epsilon_2}{\alpha\rho^2}. \tag{3.32}$$

3.3 Estimates of new perturbation

Now we use Lemma 3.1 to estimate the Hamiltonian vector field of the new perturbation $P_+=P^1+P^2+P^3$, where

$$\begin{aligned}
&P^1=\bar{P}\circ\Phi,\quad P^2=\int_0^1\{t\tilde{R}+(1-t)\hat{N},F\}\circ X^t_F dt,\\
&P^3=\langle P^K_{010},z\rangle\circ\Phi+\langle P^K_{001},\bar{z}\rangle\Phi.
\end{aligned}$$

To estimate the new perturbation we need the following lemma.

Lemma 3.1. *If $Q(\theta,I,z,\bar{z})$ is analytic in $(\theta,I,z,\bar{z})$ on $D(s,r)$ and Lipschitz continuous in ω on O, then*

$$||Q\circ\Phi||_{D(s-3\rho,r-\delta)}\leq\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||Q||_{D(s,r)}.$$

Proof. Let

$$Q=\sum_{l,q,\bar{q}}Q_{lq\bar{q}}(\theta)I^l z^q\bar{z}^{\bar{q}}.$$

Then

$$Q\circ\Phi=\sum_{l,q,\bar q}Q_{lq\bar q}[\theta(\theta_+)][I(\theta_+,I_+,z_+,\bar z_+)]^l[z(\theta_+,z_+)]^q[\bar z(\theta_+,\bar z_+)]^{\bar q}.$$

By Lemma A.2 we have

$$||Q_{lq\bar q}\circ\theta]||_{s-3\rho}\le\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||Q_{lq\bar q}||_s.$$

Again, by (3.17), (3.21b) and (3.22) we have

$$||z^t||_{a,\bar p,D(s-3\rho,r-\delta)}\le r,\quad ||\bar z^t||_{a,\bar p,D(s-3\rho,r-\delta)}\le r,\quad ||I^t||_{D(s-3\rho,r-\delta)}\le r.$$

So

$$\begin{aligned}&||Q\circ\Phi||_{D(s-3\rho,r-\delta)}\\ \le&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)\sum_{l,q,\bar q}||Q_{lq\bar q}(\theta)||_s||I^l||_{D(s-3\rho,r-\delta)}||z^q||_{D(s-3\rho,r-\delta)}||\bar z^{\bar q}||_{D(s-3\rho,r-\delta)}\\ \le&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)\sup_{|I|\le r,||z||_{a,\bar p}\le r,||\bar z||_{a,\bar p}\le r}\sum_{l,q,\bar q}||Q_{lq\bar q}(\theta)||_sI^lz^q\bar z^{\bar q}\\ =&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||Q||_{D(s,r)}.\end{aligned}$$

Thus, we complete the proof. □

Now we first consider P^1. Note that

$$P^1_{\theta_+}=\bar P_\theta\circ\Phi D_{\theta_+}\theta+\bar P_I\circ\Phi D_{\theta_+}I+\langle\bar P_z\circ\Phi,D_{\theta_+}z\rangle+\langle\bar P_{\bar z}\circ\Phi,D_{\theta_+}\bar z\rangle.\tag{3.33}$$

Since

$$||D_{\theta_+}z||_{-a,-\bar p}\le||D_{\theta_+}z||_{a,\bar p},\quad ||D_{\theta_+}\bar z||_{-a,-\bar p}\le||D_{\theta_+}\bar z||_{a,\bar p},$$

by Lemma 3.1 and using (3.15), (3.24a), (3.19a) and (3.21c) it follows that

$$\begin{aligned}||P^1_{\theta_+}||_{D(s-5\rho,r-\delta)}\le&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar P_\theta||_{D(s,r)}(1+c\epsilon_2)+||\bar P_I||_{D(s,r)}\frac{c\epsilon_2}{\alpha\rho^2}\\ &+||\bar P_z||_{a,\bar p,D(s,r)}\frac{c\epsilon_2}{\alpha\rho^2}+||\bar P_{\bar z}||_{a,\bar p,D(s,r)}\frac{c\epsilon_2}{\alpha\rho^2}\\ \le&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar P_\theta||_{D(s,r)}+\frac{c\epsilon_2}{\alpha\rho^2}|||X_{\bar P}|||_{D(s,r)}.\end{aligned}\tag{3.34}$$

Moreover, by (3.15), (3.26), (3.20a), (3.21e), it is easy to see that

$$\begin{aligned}||P^1_{\theta_+}||_{D(s-5\rho,\frac{c\epsilon_2}{\alpha})}\le&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar P_\theta||_{D(s,\frac{2c\epsilon_2}{\alpha})}(1+c\epsilon_2)+||\bar P_I||_{D(s,\frac{2c\epsilon_2}{\alpha})}\frac{c\epsilon_2}{\alpha r}\\ &+||\bar P_z||_{a,\bar p,D(s,\frac{2c\epsilon_2}{\alpha})}\frac{c\epsilon_2}{\alpha r}+||\bar P_{\bar z}||_{a,\bar p,D(s,\frac{c\epsilon_2}{\alpha})}\frac{c\epsilon_2}{\alpha r}.\end{aligned}$$

Since $\bar{P}$ only consists of more than 1-order terms of I, z and $\bar{z}$, so

$$||\bar{P}_\theta||_{D(s,\frac{2c\epsilon_2}{\alpha})} \leq ||\bar{P}_\theta||_{D(s,r)}\Big(\frac{2c\epsilon_2}{\alpha r}\Big)^2, \qquad ||\bar{P}_I||_{D(s,\frac{2c\epsilon_2}{\alpha})} \leq ||\bar{P}_I||_{D(s,r)}\frac{2c\epsilon_2}{\alpha r},$$
$$||\bar{P}_z||_{a.,\bar{p},D(s,\frac{2c\epsilon_2}{\alpha})} \leq ||\bar{P}_z||_{a,\bar{p},D(s,r)}\frac{2c\epsilon_2}{\alpha r}, \qquad ||\bar{P}_{\bar{z}}||_{a,\bar{p},D(s,\frac{2c\epsilon_2}{\alpha})} \leq ||\bar{P}_{\bar{z}}||_{a,\bar{p},D(s,r)}\frac{2c\epsilon_2}{\alpha r}.$$

Thus, we have

$$||P^1_{\theta_+}||_{D(s-5\rho,\frac{c\epsilon_2}{\alpha})} \leq \Big(\frac{c\epsilon_2}{\alpha r}\Big)^2|||X_{\bar{P}}|||_{D(s,r)}. \tag{3.35}$$

To estimate Lipschitz norms, we first estimate the following norms:

$$||\bar{P}_\theta\circ\Phi||^L_{D(s-5\rho,r-2\delta)}, \qquad ||\bar{P}_I\circ\Phi||^L_{D(s-5\rho,r-2\delta)},$$
$$||\bar{P}_z\circ\Phi||^L_{a,\bar{p},D(s-5\rho,r-2\delta)}, \qquad ||\bar{P}_{\bar{z}}\circ\Phi||^L_{D(s-5\rho,r-2\delta)}.$$

Since

$$\begin{aligned}
&||\bar{P}_\theta\circ\Phi||^L_{D(s-5\rho,r-2\delta)}\\
\leq& sup_{\omega_1\neq\omega_2}||(L_{\omega_1,\omega_2}\bar{P}_\theta)\circ\Phi||_{D(s-5\rho,r-2\delta)}\\
&+||\bar{P}_{\theta\theta}\circ\Phi||_{D(s-5\rho,r-2\delta)}||\theta||^L_{D(s-5\rho)}+||\bar{P}_{\theta I}\circ\Phi||_{D(s-5\rho,r-2\delta)}||I||^L_{D(s-5\rho,r-2\delta)}\\
&+||\bar{P}_{\theta z}\circ\Phi||_{-a,-\bar{p},D(s-5\rho,r-2\delta)}||z||^L_{a,\bar{p},D(s-5\rho,r-2\delta)}\\
&+||\bar{P}_{\theta\bar{z}}\circ\Phi||_{-a,-\bar{p},D(s-5\rho,r-2\delta)}||\bar{z}||^L_{a,\bar{p},D(s-5\rho,r-2\delta)},
\end{aligned}$$

by Lemma 3.1 and using (3.14), (3.23), (3.18) and (3.21b) it follows that

$$\begin{aligned}
&||\bar{P}_\theta\circ\Phi||^L_{D(s-5\rho,r-2\delta)}\\
\leq&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_\theta||^L_{D(s-\rho,r-\delta)}+c\epsilon_2^L||\bar{P}_{\theta\theta}||_{D(s-\rho,r-\delta)}\\
&+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{\theta I}||_{D(s-\rho,r-\delta)}+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{\theta z}||_{-a,-\bar{p},D(s-\rho,r-\delta)}\\
&+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{\theta\bar{z}}||_{-a,-\bar{p},D(s-\rho,r-\delta)}.
\end{aligned}$$

By Lemma A.1 and Cauchy's inequality it follows that

$$\begin{aligned}
&||\bar{P}_\theta\circ\Phi||^L_{D(s-5\rho,r-2\delta)}\\
\leq&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_\theta||^L_{D(s,r)}+\Big(\frac{c\epsilon_2^L}{\alpha\delta}+\frac{c\epsilon_2^L}{\rho^2}+\frac{c\epsilon_2}{\alpha^2\rho^2\delta}\Big)||\bar{P}_\theta||_{D(s,r)}.
\end{aligned} \tag{3.36}$$

In the same way it follows that

$$\begin{aligned}
&||\bar{P}_\theta\circ\Phi||^L_{D(s-5\rho,\frac{c\epsilon_2}{\alpha})}\\
\le&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_\theta||^L_{D(s-\rho,\frac{2c\epsilon_2}{\alpha})}+c\epsilon_2^L||\bar{P}_{\theta\theta}||_{D(s-\rho,\frac{2c\epsilon_2}{\alpha})}\\
&+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{\theta I}||_{D(s-\rho,\frac{2c\epsilon}{\alpha})}\\
&+\Big(\frac{\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{\theta z}||_{-a,-\bar{p},D(s-\rho,\frac{2c\epsilon_2}{\alpha})}\\
&+\Big(\frac{\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{\theta\bar{z}}||_{-a,-\bar{p},D(s-\rho,\frac{2c\epsilon_2}{\alpha})}.
\end{aligned}$$

Since $\bar{P}$ only consists of more than 1-order terms with respect to I, z, $\bar{z}$, by Lemma A.2 and Cauchy's inequality it follows that

$$\begin{aligned}
&||\bar{P}_\theta||^L_{D(s-\rho,\frac{2c\epsilon_2}{\alpha})}\le||\bar{P}_\theta||^L_{D(s,r)}\Big(\frac{2c\epsilon_2}{\alpha r}\Big)^2,\\
&||\bar{P}_{\theta\theta}||_{D(s-\rho,\frac{2c\epsilon_2}{\alpha})}\le\frac{1}{\rho^2}\Big(\frac{2c\epsilon_2}{\alpha r}\Big)^2||\bar{P}_\theta||_{D(s,r)},\\
&||\bar{P}_{\theta I}||_{D(s-\rho,\frac{2c\epsilon_2}{\alpha})}\le\Big(\frac{8c\epsilon_2}{\alpha r^2}\Big)||\bar{P}_\theta||_{D(s,r)},\\
&||\bar{P}_{\theta z}||_{-a,-\bar{p},D(s-\rho,\frac{2c\epsilon_2}{\alpha})}\le\frac{8c\epsilon_2}{\alpha r^2}||\bar{P}_\theta||_{D(s,r)},\\
&||\bar{P}_{\theta\bar{z}}||_{-a,-\bar{p},D(s-\rho,\frac{2c\epsilon_2}{\alpha})}\le\frac{8c\epsilon_2}{\alpha r^2}||\bar{P}_\theta||_{D(s,r)}.
\end{aligned}$$

If $\epsilon_2\le\alpha\rho^2$, combining all the above estimates we have

$$||\bar{P}_\theta\circ\Phi||^L_{D(s-5\rho,\frac{c\epsilon_2}{\alpha})}\le\Big(\frac{c\epsilon_2}{\alpha r}\Big)^2||\bar{P}_\theta||^L_{D(s,r)}+\frac{c\epsilon_2}{\alpha r^2}\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_\theta||_{D(s,r)}. \tag{3.37}$$

In the same way as the estimates of $\bar{P}_\theta\circ\Phi$, by Lemma A.2 and Cauchy's inequality it follows that

$$\begin{aligned}
&||\bar{P}_I\circ\Phi||^L_{D(s-5\rho,r-2\delta)}\\
\le&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_I||^L_{D(s,r)}+\Big(\frac{c\epsilon_2^L}{\alpha\delta}+\frac{c\epsilon_2^L}{\rho^2}+\frac{c\epsilon_2}{\alpha^2\rho^2\delta}\Big)||\bar{P}_I||_{D(s,r)}.
\end{aligned}\tag{3.38}$$

Also we have

$$\begin{aligned}
&||\bar{P}_I\circ\Phi||^L_{D(s-5\rho,\frac{c\epsilon_2}{\alpha})}\\
\le&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_I||^L_{D(s-\rho,\frac{2c\epsilon_2}{\alpha})}+c\epsilon_2^L||\bar{P}_{I\theta}||_{D(s-\rho,\frac{c\epsilon_2}{\alpha})}
\end{aligned}$$

$$+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{II}||_{D(s-\rho,\frac{2c\epsilon}{\alpha})}+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{Iz}||_{-a,-\bar{p},D(s-\rho,\frac{2c\epsilon_2}{\alpha})}$$
$$+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{I\bar{z}}||_{-a,-\bar{p},D(s-\rho,\frac{2c\epsilon_2}{\alpha})}.$$

Since $\bar{P}$ only consists of more than 1-order terms with respect to I, z, $\bar{z}$, by Lemma A.2 and Cauchy's inequality it follows that

$$||\bar{P}_I||^L_{D(s,\frac{2c\epsilon_2}{\alpha})}\le||\bar{P}_I||^L_{D(s,r)}\Big(\frac{2c\epsilon_2}{\alpha r}\Big),\qquad ||\bar{P}_{\theta I}||_{D(s-\rho,\frac{2c\epsilon_2}{\alpha})}\le\frac{1}{\rho^2}\frac{2c\epsilon_2}{\alpha r}||\bar{P}_I||_{D(s,r)},$$
$$||\bar{P}_{II}||_{D(s,\frac{2c\epsilon_2}{\alpha})}\le c||\bar{P}_I||_{D(s,r)},\qquad ||\bar{P}_{Iz}||_{-a,-\bar{p},D(s,\frac{2c\epsilon_2}{\alpha})}\le c||\bar{P}_I||_{D(s,r)},$$
$$||\bar{P}_{I\bar{z}}||_{-a,-\bar{p},D(s,\frac{2c\epsilon_2}{\alpha})}\le c||\bar{P}_I||_{D(s,r)}.$$

Thus,

$$||\bar{P}_I\circ\Phi||^L_{D(s-5\rho,\frac{c\epsilon_2}{\alpha})}\le\frac{c\epsilon_2}{\alpha r}||\bar{P}_I||^L_{D(s,r)}+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha\rho^2}\Big)||\bar{P}_I||_{D(s,r)}. \tag{3.39}$$

Now we estimate Lipschitz-norm of $\bar{P}_z\circ\Phi$. By Lemma 3.1 it follows that

$$\begin{aligned}&||\bar{P}_z\circ\Phi||^L_{a,\bar{p},D(s-5\rho,r-2\delta)}\\ \le&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_z||^L_{a,\bar{p},D(s-\rho,r-\delta)}+c\epsilon_2^L||\bar{P}_{z\theta}||_{a,\bar{p},D(s-\rho,r-\delta)}\\ &+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{zI}||_{a,\bar{p},D(s-\rho,r-\delta)}+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)|||\bar{P}_{zz}|||_{a,\bar{p},D(s-\rho,r-\delta)}\\ &+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)|||\bar{P}_{z\bar{z}}|||_{a,\bar{p},D(s-\rho,r-\delta)},\end{aligned}$$

where $|||\cdot|||_{a,\bar{p}}$ is the operator norm from $\ell^{a,\bar{p}}$ to $\ell^{a,\bar{p}}$. By the generalized Cauchy's inequality of analytic function on Banach space it follows that

$$\begin{aligned}&||\bar{P}_z\circ\Phi||^L_{a,\bar{p},D(s-5\rho,r-2\delta)}\\ \le&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_z||^L_{a,\bar{p},D(s,r)}+\Big(\frac{c\epsilon_2^L}{\alpha\delta}+\frac{c\epsilon_2^L}{\rho^2}+\frac{c\epsilon_2}{\alpha^2\rho^2\delta}\Big)||\bar{P}_z||_{a,\bar{p},D(s,r)}.\end{aligned} \tag{3.40}$$

Also note that

$$\begin{aligned}&||\bar{P}_z\circ\Phi||^L_{a,\bar{p},D(s-5\rho,\frac{c\epsilon_2}{\alpha})}\\ \le&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_z||^L_{a,\bar{p},D(s-\rho,\frac{2c\epsilon_2}{\alpha})}+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{z\theta}||_{a,\bar{p},D(s-\rho,\frac{2c\epsilon_2}{\alpha})}\\ &+c\epsilon_2^L||\bar{P}_{zI}||_{a,\bar{p},D(s-\rho,\frac{2c\epsilon}{\alpha})}+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)|||\bar{P}_{zz}|||_{a,\bar{p},D(s-\rho,\frac{2c\epsilon_2}{\alpha})}\\ &+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)|||\bar{P}_{z\bar{z}}|||_{a,\bar{p},D(s-\rho,\frac{2c\epsilon_2}{\alpha})}.\end{aligned}$$

In the same way as the above, if $\epsilon_2 \le \alpha\rho^2$ and $\frac{2c\epsilon_2}{\alpha} \le \frac{1}{2}r$, we have

$$||\bar{P}_z \circ \Phi||^L_{a,\bar{p},D(s-5\rho,\frac{c\epsilon_2}{\alpha})} \le \frac{c\epsilon_2}{\alpha r}||\bar{P}_z||^L_{a,\bar{p},D(s,r)} + \Big(\frac{c\epsilon_2^L}{\alpha} + \frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_z||_{a,\bar{p},D(s,r)}. \tag{3.41}$$

Moreover, we have

$$\begin{aligned}&||\bar{P}_{\bar{z}} \circ \Phi||^L_{a,\bar{p},D(s-5\rho,r-2\delta)}\\ &\le \Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_{\bar{z}}||^L_{a,\bar{p},D(s,r)} + \Big(\frac{c\epsilon_2^L}{\alpha\delta} + \frac{c\epsilon_2}{\rho^2} + \frac{c\epsilon_2}{\alpha^2\rho^2\delta}\Big)||\bar{P}_{\bar{z}}||_{a,\bar{p},D(s,r)},\end{aligned} \tag{3.42a}$$

$$||\bar{P}_{\bar{z}} \circ \Phi||^L_{a,\bar{p},D(s-5\rho,\frac{c\epsilon_2}{\alpha})} \le \frac{c\epsilon_2}{\alpha r}||\bar{P}_{\bar{z}}||^L_{a,\bar{p},D(s,r)} + \Big(\frac{c\epsilon_2^L}{\alpha} + \frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_{\bar{z}}||_{a,\bar{p},D(s,r)}. \tag{3.42b}$$

By (3.33), (3.15), (3.16), (3.24a) (3.24b), (3.20a), (3.20b), (3.21e), (3.21f) and combining (3.36), (3.38), (3.40) and (3.42a), we have

$$\begin{aligned}||P^1_{\theta_+}||^L_{D(s-5\rho,r-2\delta)} \le& \Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_\theta||^L_{D(s,r)} + \frac{c\epsilon_2}{\alpha\rho^2}|||X_{\bar{P}}|||^L_{D(s,r)}\\ &+\Big(\frac{c\epsilon_2^L}{\alpha\delta} + \frac{c\epsilon_2^L}{\alpha\rho^2} + \frac{c\epsilon_2}{\alpha^2\rho^2\delta} + \frac{c\epsilon_2}{\alpha^2\rho^4}\Big)|||X_{\bar{P}}|||_{D(s,r)}.\end{aligned} \tag{3.43}$$

In the same way as the above, by (3.37), (3.39), (3.41) and (3.42b) we have

$$||P^1_{\theta_+}||^L_{D(s-5\rho,\frac{c\epsilon_2}{\alpha})} \le \Big(\frac{c\epsilon_2}{\alpha r}\Big)^2|||X_{\bar{P}}|||^L_{D(sr)} + \frac{c\epsilon_2}{\alpha r}\Big(\frac{c\epsilon_2^L}{\alpha} + \frac{c\epsilon_2}{\alpha^2\rho^2}\Big)|||X_{\bar{P}}|||_{D(s,r)}. \tag{3.44}$$

Now we estimate $P^1_{I_+}$. Since $P^1_{I_+} = \bar{P}_I \circ \Phi D_{I_+} I$, by Lemma 3.1 and (3.28a) we have

$$||P^1_{I_+}||_{D(s-5\rho,r-2\delta)} \le \Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_I||_{D(s,r)}. \tag{3.45}$$

Moreover, by (3.38), (3.28a) and (3.28b) it follows that

$$||P^1_{I_+}||^L_{D(s-5\rho,r-2\delta)} \le \Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_I||^L_{D(s,r)} + \Big(\frac{c\epsilon_2^L}{\alpha\delta} + \frac{c\epsilon_2^L}{\rho^2} + \frac{c\epsilon_2}{\alpha^2\rho^2\delta}\Big)|||X_{\bar{P}}|||_{D(s,r)}. \tag{3.46}$$

At $I_+ = 0$, $z_+ = 0$, $\bar{z}_+ = 0$, it is easy to see that

$$||P^1_{I_+}||_{D(s-5\rho,0)} \le \frac{c\epsilon_2}{\alpha r}||\bar{P}_I||_{D(s,r)}. \tag{3.47}$$

By (3.39), (3.28a) and (3.28b) it follows that

$$||P^1_{I_+}||^L_{D(s-5\rho,0)} \le \frac{c\epsilon_2}{\alpha r}||\bar{P}_I||^L_{D(s,r)} + \Big(\frac{c\epsilon_2^L}{\alpha} + \frac{c\epsilon_2}{\alpha^2\rho^2}\Big)||\bar{P}_I||_{D(s,r)}. \tag{3.48}$$

Since

$$P^1_{z_+} = \bar{P}_I \circ \Phi D_{z_+} I + \bar{P}_z \circ \Phi,$$

by Lemma 3.1 and (3.29) we have

$$||P^1_{z_+}||_{a,\bar{p},D(s-3\rho,r-\delta)} \leq \Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_z||_{a,\bar{p},D(s,r)} + \frac{c\epsilon_2}{\alpha\rho^2}||\bar{P}_I||^L_{D(s,r)}. \tag{3.49}$$

Moreover, by (3.38), (3.40), (3.29) and (3.30) it follows that

$$\begin{aligned}
&||P^1_{z_+}||^L_{a,\bar{p},D(s-5\rho,r-2\delta)} \\
\leq&\Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_z||^L_{a,\bar{p},D(s,r)} + \frac{c\epsilon_2}{\alpha\rho^2}||\bar{P}_I||^L_{D(s,r)} \\
&+\Big(\frac{c\epsilon_2^L}{\alpha\delta}+\frac{c\epsilon_2^L}{\alpha\rho^2}+\frac{c\epsilon_2}{\alpha^2\rho^2\delta}+\frac{c\epsilon_2}{\alpha^2\rho^4}\Big)(||\bar{P}_z||_{a,\bar{p},D(s,r)}+||\bar{P}_I||_{D(s,r)}).
\end{aligned} \tag{3.50}$$

At $I_+=0$, $z_+=0$, $\bar{z}_+=0$, we have

$$||P^1_{z_+}||_{a,\bar{p},D(s-5\rho,0)} \leq \frac{c\epsilon_2}{\alpha r}[||\bar{P}_I||_{D(s,r)}+||\bar{P}_z||_{a,\bar{p},D(s,r)}], \tag{3.51a}$$

$$\begin{aligned}
||P^1_{z_+}||^L_{a,\bar{p},D(s-5\rho,0)} \leq& \frac{c\epsilon_2}{\alpha r}(||\bar{P}_z||^L_{a,\bar{p},D(s,r)}+||\bar{P}_I||^L_{D(s,r)}) \\
&+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)(||\bar{P}_z||_{a,\bar{p},D(s,r)}+||\bar{P}_I||_{D(s,r)}).
\end{aligned} \tag{3.51b}$$

Similarly we have

$$||P^1_{\bar{z}_+}||_{a,\bar{p},D(s-5\rho,r-2\delta)} \leq \Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_{\bar{z}}||_{a,\bar{p},D(s,r)} + \frac{c\epsilon_2}{\alpha\rho^2}||\bar{P}_I||^L_{D(s,r)}, \tag{3.52a}$$

$$\begin{aligned}
||P^1_{\bar{z}_+}||^L_{a,\bar{p},D(s-5\rho,r-2\delta)} \leq& \Big(1+\frac{c\epsilon_2}{\rho^2}\Big)||\bar{P}_{\bar{z}}||^L_{D(s,r)} + \frac{c\epsilon_2}{\alpha\rho^2}||\bar{P}_I||^L_{D(s,r)} + \Big(\frac{c\epsilon_2^L}{\alpha\delta}+\frac{c\epsilon_2^L}{\alpha\rho^2} \\
&+\frac{c\epsilon_2}{\alpha^2\rho^2\delta}+\frac{c\epsilon_2}{\alpha^2\rho^4}\Big)(||\bar{P}_{\bar{z}}||_{a,\bar{p},D(s,r)}+||\bar{P}_I||_{D(s,r)}).
\end{aligned} \tag{3.52b}$$

At $I_+=0$, $z_+=0$, $\bar{z}_+=0$, we have

$$||P^1_{\bar{z}_+}||_{a,\bar{p},D(s,r)} \leq \frac{c\epsilon_2}{\alpha r}[||\bar{P}_I||_{D(s,r)}+||\bar{P}_{\bar{z}}||_{a,\bar{p},D(s,r)}], \tag{3.53a}$$

$$\begin{aligned}
||P^1_{\bar{z}_+}||^L_{a,\bar{p},D(s-5\rho,0)} \leq& \frac{c\epsilon_2}{\alpha r}(||\bar{P}_{\bar{z}}||^L_{a,\bar{p},D(s,r)}+||\bar{P}_I||^L_{D(s,r)}) \\
&+\Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)(||\bar{P}_{\bar{z}}||_{a,\bar{p},D(s,r)}+||\bar{P}_I||_{D(s,r)}).
\end{aligned} \tag{3.53b}$$

Thus by (3.34), (3.45), (3.49) and (3.52a) we have

$$|||X_{P^1}|||_{D(s-5\rho,r-2\delta)} \leq \Big(1+\frac{c\epsilon_2}{\alpha\rho^2}\Big)|||X_{\bar{P}}|||_{D(s,r)}. \tag{3.54}$$

By (3.43), (3.46), (3.50) and (3.52b) we have

$$|||X_{P^1}|||^L_{D(s-5\rho,r-2\delta)} \le \Big(1+\frac{c\epsilon_2}{\alpha\rho^2}\Big)|||X_{\bar P}|||^L_{D(s,r)} + \frac{c}{\alpha}\Big(\frac{1}{\delta}+\frac{1}{\rho^2}\Big)\Big(\epsilon_2^L+\frac{c\epsilon_2}{\alpha\rho^2}\Big)|||X_{\bar P}|||_{D(s,r)}. \tag{3.55}$$

Let $P^1 = R^1_+ + \bar P_+$, where

$$R^1_+ = R^+_1(\theta_+) + R^+_2(\theta_+)I_+ + \langle R^+_3(\theta_+), z_+\rangle + \langle \bar R^+_3(\theta_+), \bar z_+\rangle$$

only consists of the lower order terms of P^1 with respect to I_+, z_+, $\bar z_+$, and

$$\bar P_+ = \sum_{|l|+|q|+|\bar q|\ge 2} P^1_{lq\bar q}(\theta) I^l_+ z^q_+ \bar z^{\bar q}$$

consists of the higher order terms. By (3.35), (3.47), (3.51a) and (3.53a) we have

$$|||X_{R^1_+}|||_{D(s-5\rho,r-2\delta)} \le \frac{c\epsilon_2}{\alpha r}|||X_{\bar P}|||_{D(s,r)}. \tag{3.56}$$

By (3.44), (3.48), (3.51b) and (3.53b) we have

$$|||X_{R^1_+}|||^L_{D(s-5\rho,r-2\delta)} \le \frac{c\epsilon_2}{\alpha r}|||X_{\bar P}|||^L_{D(s,r)} + \Big(\frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}\Big)|||X_{\bar P}|||_{D(s,r)}. \tag{3.57}$$

Now we estimate P^2. By the construction of F and using (3.4), (3.5), (3.7), (3.9a) and (3.9b) it follows easily that

$$||F_\theta||_{D(s,r)} \le \frac{c\epsilon_2}{\alpha}, \quad ||F_\theta||^L_{D(s-\rho,r)} \le \frac{c\epsilon_2^L}{\alpha}+\frac{c\epsilon_2}{\alpha^2\rho^2}. \tag{3.58}$$

Let

$$\begin{aligned}\{\tilde R, F\} &= \tilde R_\theta F_I - \tilde R_I F_\theta + \langle \tilde R_z, F_{\bar z}\rangle - \langle \tilde R_{\bar z}, F_z\rangle \\ &= \tilde R_\theta F_{100}(\theta) - P_{100}(\theta)F_\theta + \langle \tilde P_{010}(\theta), \bar F_{001}(\theta)\rangle - \langle \tilde P_{001}(\theta), F_{010}(\theta)\rangle \\ &=: Q_1 - Q_2 + Q_3 - Q_4.\end{aligned}$$

We only estimate Q_2 and Q_3, since Q_1 and Q_4 can be estimated similarly. Since

$$\begin{aligned}D_{\theta_+}(Q_2\circ X^t_F) =& \big(P'_{100}(\theta)F_\theta + P_{100}(\theta)F_{\theta\theta}\big)\circ X^t_F D_{\theta_+}\theta^t + P_{100}(\theta^t)F'_{100}(\theta^t)D_{\theta_+}I^t \\ &+ P_{100}(\theta^t)\langle F'_{010}(\theta^t), D_{\theta_+}z^t\rangle + P_{100}(\theta^t)\langle F'_{001}(\theta^t), D_{\theta_+}\bar z^t\rangle,\end{aligned}$$

by (3.58), (3.5), (3.9a), (3.10c), (3.19a), (3.21c) and (3.24a) and using Lemma 3.1 and Cauchy's inequality it follows that if $\epsilon_2 \le \alpha r$, then

$$||D_{\theta_+}Q_2\circ X^t_F||_{D(s-5\rho,r-2\delta)} \le \frac{c\epsilon_2^2}{\alpha\rho^2}.$$

Since

$$D_{\theta_+} Q_3 \circ X_F^t = \langle \tilde{P}'_{010}(\theta^t), F_{001}(\theta^t) \rangle D_{\theta_+} \theta^t + \langle \tilde{P}_{010}(\theta^t), F'_{001}(\theta^t) \rangle D_{\theta_+} \theta^t,$$

by (3.1b), (3.1c), (3.10a), (3.10c) and using Lemmas 3.1 and A.2 it follows that

$$||D_{\theta_+} Q_3 \circ X_F^t||_{D(s-5\rho,r-2\delta)} \le \frac{c\epsilon_2^2}{\alpha r}.$$

Thus we have

$$||D_{\theta_+} \{\tilde{R}, F\} \circ X_F^t||_{D(s-5\rho,r-2\delta)} \le \frac{c\epsilon_2^2}{\alpha\rho^2} + \frac{c\epsilon_2^2}{\alpha r}. \tag{3.59}$$

In the same way as the above it follows that

$$\begin{aligned} ||D_{\theta_+} Q_2 \circ X_F^t||^L_{D(s-5\rho,r-2\delta)} &\le \frac{c\epsilon_2}{\alpha\rho^2}\Big(\epsilon_2^L + \frac{\epsilon_2}{\alpha\rho^2}\Big), \\ ||D_{\theta_+} Q_3 \circ X_F^t||^L_{D(s-5\rho,r-2\delta)} &\le \frac{c\epsilon_2 \epsilon_2^L}{\alpha r} + \frac{c\epsilon_2^2}{\alpha^2\rho^2 r} = \frac{c\epsilon_2}{\alpha r}\Big(\epsilon_2^L + \frac{\epsilon_2}{\alpha\rho^2}\Big). \end{aligned}$$

Thus we have

$$||D_{\theta_+} \{\tilde{R}, F\} \circ X_F^t||^L_{D(s-5\rho,r-2\delta)} \le \Big(\frac{c\epsilon_2}{\alpha r} + \frac{c\epsilon_2}{\alpha\rho^2}\Big)\Big(\epsilon_2^L + \frac{\epsilon_2}{\alpha\rho^2}\Big). \tag{3.60}$$

The other derivatives of $\{\tilde{R}, F\} \circ X_F^t$ with respect to I_+, z_+ and $\bar{z}_+$ can be estimated in the same way or even more simple. Thus, by (3.59) and (3.60) we have

$$|||X_{P^2}|||_{D(s-5\rho,r-2\delta)} \le \frac{c\epsilon_2^2}{\alpha\rho^2} + \frac{c\epsilon_2^2}{\alpha r}, \tag{3.61a}$$

$$|||X_{P^2}|||^L_{D(s-5\rho,r-2\delta)} \le \Big(\frac{c\epsilon_2}{\alpha r} + \frac{c\epsilon_2}{\alpha\rho^2}\Big)\Big(\epsilon_2^L + \frac{\epsilon_2}{\alpha\rho^2}\Big). \tag{3.61b}$$

Now we estimate P^3. Since the two terms can be estimated similarly, we only consider the first one. By Lemma A.2 we have

$$\begin{aligned} &||D_{\theta_+} \langle P^K_{010}, z\rangle \circ \Phi||_{D(s-5\rho,r-2\delta)} \\ \le & ||{P^K_{010}}'[\theta(\theta_+)]||_{-a,-\bar{p},D(s-5\rho)} ||z||_{a,\bar{p},D(s-5\rho,r-2\delta)} ||D_{\theta_+}\theta||_{D(s-5\rho)} \\ & \quad + ||P^K_{010}[\theta(\theta_+)]||_{a,\bar{p},D(s-5\rho,r-2\delta)} ||D_{\theta_+} z||_{-a,-\bar{p},D(s-5\rho,r-2\delta)} \\ \le & c e^{-\rho K}\epsilon_2 + \frac{c\epsilon_2^2}{\alpha r}. \end{aligned}$$

Since

$$D_{z_+} \langle P^K_{010}, z\rangle \circ \Phi = P^K_{010}[\theta(\theta_+)],$$

by Lemma A.2 it follows that

$$||D_{z_+}\langle P^K_{010}, z\rangle \circ \Phi||_{a,\bar{p},D(s-5\rho,r-2\delta)} \leq ce^{-\rho K}\epsilon_2.$$

Thus,

$$|||X_{P^3}|||_{D(s-5\rho,r-2\delta)} \leq ce^{-\rho K}\epsilon_2 + \frac{c\epsilon_2^2}{\alpha r}. \tag{3.62}$$

Similarly, we have

$$|||X_{P^3}|||^L_{D(s-5\rho,r-2\delta)} \leq ce^{-\rho K}\epsilon_2^L + \frac{c\epsilon_2}{\alpha r}\Big(\epsilon_2^L + \frac{\epsilon_2}{\alpha\rho^2}\Big). \tag{3.63}$$

Let $P_+ = R_+ + \bar{P}_+$, where $R_+ = R_+^1 + P^2 + P^3$. Let $s_+ = s - 5\rho$ and $r_+ = r - 2\delta$. Suppose

$$\frac{\epsilon_2}{\alpha r} \leq 1, \quad \frac{2c\epsilon_2}{\alpha} \leq \delta, \quad e^{-\rho K} \leq \frac{\epsilon_1}{\alpha r}. \tag{3.64}$$

Then, by (3.54)-(3.57) and (3.61a)-(3.63) we have

$$\begin{cases} |||X_{\bar{P}_+}|||_{D(s_+,r_+)} \leq \Big(1 + \dfrac{c\epsilon_2}{\alpha\rho^2}\Big)\epsilon_1 \triangleq \epsilon_{1,+}, \\ |||X_{R_+}|||_{D(s_+,r_+)} \leq \dfrac{c\epsilon_1}{\alpha r}\epsilon_2 + \dfrac{c\epsilon_2^2}{\alpha r} + \dfrac{c\epsilon_2^2}{\alpha\rho^2} \triangleq \epsilon_{2,+}, \\ |||X_{\bar{P}_+}|||^L_{D(s_+,r_+)} \leq \Big(1 + \dfrac{c\epsilon_2}{\alpha\rho^2}\Big)\epsilon_1^L + \frac{c}{\alpha}\Big(\dfrac{1}{\delta} + \dfrac{1}{\rho^2}\Big)\Big(\epsilon_2^L + \dfrac{\epsilon_2}{\alpha\rho^2}\Big)\epsilon_1 \triangleq \epsilon_{1,+}^L, \\ |||X_{R_+}|||^L_{D(s_+,r_+)} \leq \dfrac{c\epsilon_2}{\alpha r}\epsilon_1^L + \Big(\dfrac{c\epsilon_1}{\alpha r_+} + \dfrac{c\epsilon_2}{\alpha r} + \dfrac{c\epsilon_2}{\alpha\rho^2}\Big)\Big(\epsilon_2^L + \dfrac{\epsilon_2}{\alpha\rho^2}\Big) \triangleq \epsilon_{2,+}^L. \end{cases} \tag{3.65}$$

Obviously, if $\epsilon_1 \leq \epsilon_1^L$ and $\epsilon_2 \leq \epsilon_2^L$, then $\epsilon_{1,+} \leq \epsilon_{1,+}^L$ and $\epsilon_{2,+} \leq \epsilon_{2,+}^L$. Below we consider the new frequency.

Let

$$\frac{2}{(K\rho)^2} = \frac{\epsilon_1}{\alpha r},$$

i.e.,

$$K = \frac{1}{\rho}\sqrt{\frac{2\alpha r}{\epsilon_1}}.$$

Then

$$e^{-K\rho} \leq \frac{\epsilon_1}{\alpha r}.$$

Let $\omega_+ = \omega + \hat{\omega}$, where ω satisfies

$$|\omega k + \Omega_j| \geq \alpha, \quad \forall |k| \leq K, \quad \forall j \geq 1.$$

By the above estimates we have $|\hat{\omega}|^\lambda \le \epsilon_2^\lambda$. So for $|k| \le K$ we have

$$|\omega_+ k + \Omega_j| \ge \alpha - \frac{\epsilon_2}{\rho}\sqrt{\frac{2\alpha r}{\epsilon_1}} = \alpha_+.$$

Let

$$\rho_+ = \frac{\rho}{2}, \quad K_+ = \frac{1}{\rho_+}\sqrt{\frac{2\alpha_+ r_+}{\epsilon_{1,+}}}, \quad \mathcal{R}_{kj} = \{\omega || \omega_+ k \pm \Omega_j| \le \alpha_+\}.$$

If

$$\omega \in O_+ = O - \cup_{K \le |k| \le K_+, j \ge 1} \mathcal{R}_{kj},$$

then

$$|\omega_+ k \pm \Omega_j| \ge \alpha_+, \quad \forall |k| \le K_+, \quad \forall j \ge 1.$$

KAM iteration

Now we go through the KAM iteration. Let

$$\begin{aligned} s_0 &= s, \qquad & s_{m+1} &= s_m - 5\rho_m, \qquad & \rho_m &= \frac{1}{10}\frac{s}{2^{m+1}}, \\ r_0 &= r, \qquad & r_{m+1} &= r_m - 2\delta_m, \qquad & \delta_m &= \frac{1}{2}\frac{r}{4^{m+1}}. \end{aligned}$$

Thus, we have $s_m \ge \frac{s}{2}$ and $r_m \ge \frac{r}{2}$. Let

$$K_m = \frac{1}{\rho_m}\sqrt{\frac{2\alpha_m r_m}{\epsilon_{1,m}}} \quad \text{and} \quad \alpha_{m+1} = \alpha_m - \frac{\epsilon_{2,m}}{\rho_m}\sqrt{\frac{2\alpha_m r_m}{\epsilon_{1,m}}},$$

where $\alpha_0 = \alpha$. Let

$$\Phi^m = \Phi_0 \circ \Phi_1 \cdots \circ \Phi_{m-1}, \quad H^m = H \circ \Phi^m = N^m + P^m,$$

with

$$N^m = \omega_m I + \langle \Omega z, \bar{z} \rangle, \quad P^m = R^m + \bar{P}^m,$$

where

$$\begin{aligned} R^m &= P^m_{000}(\theta) + P^m_{100}(\theta) I + \langle P^m_{010}(\theta), z \rangle + \langle P^m_{001}(\theta), \bar{z} \rangle, \\ \bar{P}^m &= \sum_{|l|+|q|+|\bar{q}| \ge 2} P^m_{lq\bar{q}}(\theta) I^l z^q \bar{z}^{\bar{q}}. \end{aligned}$$

At the initial step, let $\epsilon_{1,0}=\epsilon_{1,0}^L=\epsilon_{2,0}=\epsilon_{2,0}^L=\epsilon$. Then let

$$\epsilon_{1,m+1}=\Big(1+\frac{c\epsilon_{2,m}}{\alpha_m\rho_m^2}\Big)\epsilon_{1,m}, \tag{3.66a}$$

$$\epsilon_{2,m+1}=\frac{c\epsilon_{1,m}}{\alpha_m r_m}\epsilon_{2,m}+\frac{c\epsilon_{2,m}^2}{\alpha_m r_m}+\frac{c\epsilon_{2,m}^2}{\alpha_m\rho_m^2}, \tag{3.66b}$$

$$\epsilon_{1,m+1}^L=\Big(1+\frac{c\epsilon_{2,m}}{\alpha_m\rho_m^2}\Big)\epsilon_{1,m}^L+\frac{c\epsilon_{1,m}}{\alpha_m}\Big(\frac{1}{\delta_m}+\frac{1}{\rho_m^2}\Big)\Big(\epsilon_{2,m}^L+\frac{\epsilon_{2,m}}{\alpha_m\rho_m^2}\Big), \tag{3.66c}$$

$$\epsilon_{2,m+1}^L=\frac{c\epsilon_{2,m}}{\alpha_m r_m}\epsilon_{1,m}^L+\Big(\frac{c\epsilon_{1,m}}{\alpha_m r_{m+1}}+\frac{c\epsilon_{2,m}}{\alpha_m r_m}+\frac{c\epsilon_{2,m}}{\alpha_m\rho_m^2}\Big)\Big(\epsilon_{2,m}^L+\frac{\epsilon_{2,m}}{\alpha_m\rho_m^2}\Big). \tag{3.66d}$$

Then, by the previous estimates we have

$$|||X_{\bar{P}^m}|||_{D(s_m,r_m)}^{\lambda}\le\epsilon_{1,m}^{\lambda},\quad |||X_{R^m}|||_{D(s_m,r_m)}^{\lambda}\le\epsilon_{2,m}^{\lambda}. \tag{3.67}$$

Convergence of the KAM iteration

At first note that $0<\alpha\le 1$, $r_m\ge\frac{r}{2}$ and $s_m\ge\frac{s}{2}$. Suppose that $\alpha_m\ge\frac{1}{2}\alpha$. Then, by (3.66) we have

$$\epsilon_{1,m+1}\le\Big(1+\frac{c\epsilon_{2,m}}{\alpha\rho_m^2}\Big)\epsilon_{1,m}, \tag{3.68a}$$

$$\epsilon_{2,m+1}\le\frac{c\epsilon_{1,m}}{\alpha}\epsilon_{2,m}+\frac{c\epsilon_{2,m}^2}{\alpha\rho_m^2}, \tag{3.68b}$$

$$\epsilon_{1,m+1}^L\le\Big(1+\frac{c\epsilon_{2,m}}{\alpha\rho_m^2}\Big)\epsilon_{1,m}^L+\frac{c\epsilon_{1,m}}{\alpha}\Big(\frac{1}{\delta_m}+\frac{1}{\rho_m^2}\Big)\Big(\epsilon_{2,m}^L+\frac{\epsilon_{2,m}}{\alpha\rho_m^2}\Big), \tag{3.68c}$$

$$\epsilon_{2,m+1}^L\le\frac{c\epsilon_{2,m}}{\alpha}\epsilon_{1,m}^L+\Big(\frac{c\epsilon_{1,m}}{\alpha}+\frac{c\epsilon_{2,m}}{\alpha\rho_m^2}\Big)\Big(\epsilon_{2,m}^L+\frac{\epsilon_{2,m}}{\alpha\rho_m^2}\Big). \tag{3.68d}$$

Note that the constants c in the above inequalities are independent of KAM steps.

Now suppose $\epsilon\le\epsilon_{1,m}\le 2\epsilon$, $\epsilon\le\epsilon_{1,m}^L\le 2\epsilon$. Below, by induction we will prove that if $\frac{c\epsilon}{\alpha\rho_0^2}$, $\frac{c\epsilon}{\alpha^2\rho_0^4}$ and $\frac{c\epsilon}{\alpha^2\delta_0\rho_0^2}$ are sufficiently small, for all $m\ge 1$, we have

$$\begin{aligned}
&\epsilon_{1,m}\le\prod_{j=1}^{m}\Big(1+\Big(\frac{24c\epsilon}{\alpha\rho_0^2}\Big)^j\Big)\epsilon\le 2\epsilon,\\
&\epsilon_{1,m}^L\le\prod_{j=1}^{m}\Big(1+\Big(\frac{168c\epsilon}{\alpha^2\rho_0^4}\Big)^j+\Big(\frac{64c\epsilon}{\alpha^2\delta_0\rho_0^2}\Big)^j\Big)\epsilon\le 2\epsilon,\\
&\epsilon_{2,m}\le\Big(\frac{6c\epsilon}{\alpha\rho_0^2}\Big)^m\epsilon,\quad \epsilon_{2,m}^L\le\Big(\frac{8c\epsilon}{\alpha\rho_0^2}\Big)^{m+1}.
\end{aligned}$$

For $m=1$, we have

$$\epsilon_{1,1} \le \Big(1+\frac{c\epsilon_{2,0}}{\alpha\rho_0^2}\Big)\epsilon_{1,0} \le \Big(1+\frac{c\epsilon}{\alpha\rho_0^2}\Big)\epsilon,$$

$$\epsilon_{2,1} \le \frac{c\epsilon_{1,0}}{\alpha}\epsilon_{2,0} + \frac{c\epsilon_{2,0}^2}{\alpha\rho_0^2} \le \frac{2c\epsilon}{\alpha\rho_0^2}\epsilon.$$

Moreover, if $\epsilon_{1,m} \le 2\epsilon$ and $\frac{6c\epsilon}{\alpha\rho_0^2} \le \frac{1}{4}$, by (3.68), it follows that

$$\epsilon_{2,m} \le \frac{2c\epsilon}{\alpha}\epsilon_{2,m-1} + \frac{c\epsilon_{2,m-1}^2}{\alpha\rho_{m-1}^2} \le \frac{6c\epsilon}{\alpha\rho_0^2}\epsilon_{2,m-1}.$$

Then we have

$$\epsilon_{1,m} \le \Big(1+\Big(\frac{24c\epsilon}{\alpha\rho_0^2}\Big)^m\Big)\epsilon_{1,m-1}.$$

Obviously, it follows that

$$\epsilon_{1,1}^L \le \Big(1+\frac{3c\epsilon}{\alpha^2\rho_0^4}+\frac{2c\epsilon}{\alpha^2\delta_0\rho_0^2}\Big)\epsilon, \quad \epsilon_{2,1}^L \le 6c\Big(\frac{\epsilon}{\alpha\rho_0^2}\Big)^2.$$

By the induction assumption, we have

$$\begin{aligned}
\epsilon_{1,m}^L &\le \Big(1+\frac{c\epsilon_{2,m-1}}{\alpha\rho_{m-1}^2}\Big)\epsilon_{1,m-1}^L + \frac{c\epsilon_{1,m-1}}{\alpha}\Big(\frac{1}{\delta_{m-1}}+\frac{1}{\rho_{m-1}^2}\Big)\Big(\epsilon_{2,m-1}^L+\frac{\epsilon_{2,m-1}}{\alpha\rho_{m-1}^2}\Big)\\
&\le \prod_{j=1}^m \Big(1+\Big(\frac{168c\epsilon}{\alpha^2\rho_0^4}\Big)^j+\Big(\frac{64c\epsilon}{\alpha^2\delta_0\rho_0^2}\Big)^j\Big)\epsilon,\\
\epsilon_{2,m}^L &\le \frac{c\epsilon_{2,m-1}}{\alpha}2\epsilon + \Big(\frac{2c\epsilon}{\alpha}+\frac{c\epsilon_{2,m-1}}{\alpha\rho_{m-1}^2}\Big)\Big(\epsilon_{2,m-1}^L+\frac{\epsilon_{2,m-1}}{\alpha\rho_{m-1}^2}\Big) \le \Big(\frac{8c\epsilon}{\alpha\rho_0^2}\Big)^{m+1}.
\end{aligned}$$

It is easy to see that if $\frac{c\epsilon}{\alpha\rho_0^2}$, $\frac{c\epsilon}{\alpha^2\rho_0^4}$ and $\frac{c\epsilon}{\alpha^2\delta_0\rho_0^2}$ are sufficiently small, we have

$$\epsilon \le \epsilon_{1,m} \le 2\epsilon, \quad \epsilon \le \epsilon_{1,m}^L \le 2\epsilon.$$

Now we verify all the conditions required in the estimates of KAM steps. At first we prove $\alpha_m \ge \alpha/2$. Since $\alpha_m \le \alpha$, we have

$$\sum_{m=0}^{\infty}\sqrt{\frac{2\alpha_m r_m}{\epsilon_{1,m}}}\frac{\epsilon_{2,m}}{\rho_m} \le \sqrt{\frac{2\alpha r}{\epsilon}}\frac{\epsilon}{\rho_0}\sum_{m=0}^{\infty}\Big(\frac{12c\epsilon}{\alpha\rho_0^2}\Big)^m \le \sqrt{\frac{800\alpha r\epsilon}{s^2}} \le \frac{10\sqrt{2\alpha r\gamma}}{s^2}\alpha.$$

We choose γ sufficiently small such that $\frac{10\sqrt{2\alpha r\gamma}}{s^2} \le \frac{1}{4}$. Thus we have

$$\alpha_m \ge \alpha - \sum_{j=0}^{m-1}\sqrt{\frac{2\alpha r}{\epsilon_{1,j}}}\frac{\epsilon_{2,j}}{\rho_j} \ge \alpha - \frac{10\sqrt{2\alpha r\gamma}}{s^2}\alpha \ge \frac{\alpha}{2}.$$

Now we verify all conditions of (3.64). By the estimates of the KAM steps, we can choose the sufficiently small γ such that the following inequalities hold:

$$\epsilon_{2,m} \leq \alpha r_m, \quad c\epsilon_{2,m} \leq \alpha \delta_m, \quad \epsilon_{2,m} \leq \alpha_m \rho_m^2, \quad e^{-K_m \rho_m} \leq \frac{\epsilon_{1,m}}{\alpha_m r_m}.$$

Since

$$|\omega_{m+1} - \omega_m|^\lambda \leq \epsilon_{2,m}^\lambda,$$

it follows that $|\omega_* - \omega| \leq c\epsilon$ and

$$|\omega_m - \omega|^L \leq \epsilon_{2,0}^L + \sum_{j=1}^m \epsilon_{2,j}^L \leq \epsilon + \sum_{j=1}^m \Big(\frac{8c\epsilon}{\alpha \rho_0^2}\Big)^{j+1} \leq 2\epsilon \leq \frac{1}{2}$$

and so $|\omega_m|^L \geq \frac{1}{2}$. By the KAM steps we have

$$|k\omega_m \pm \Omega_j| \geq \alpha_m \geq \alpha/2, \quad \forall |k| \leq K_m, \quad \forall j \geq 1, \quad \forall \omega \in O_m.$$

Let

$$\mathcal{R}_{kj}(\alpha) = \{\omega \,||k\omega_k \pm \Omega_j| \leq \alpha\},$$

where $\omega_k = \omega_m$ if $K_m \leq |k| \leq K_{m+1}$. Let

$$\bar{O}_m = \cup_{K_m \leq |k| \leq K_{m+1}, j \geq 1} \mathcal{R}_{kj}(\alpha_m).$$

If $\omega \in O_{m+1} = O_m \setminus \bar{O}_m$, we have

$$|k\omega_{m+1} \pm \Omega_j| \geq \alpha_{m+1}, \quad \forall |k| \leq K_{m+1}, \quad \forall j \geq 1.$$

Let $O_\alpha = \cap_{m=0}^\infty O_m$. By Theorem 2.2 we have

$$\text{meas}(O \setminus O_\alpha) \leq \text{meas}(\cup_{m \geq 0} \bar{O}_m) \leq c\alpha.$$

Now we consider the convergence of $\{\Phi^m\}$. By the KAM steps we have

$$\Phi_m = (\theta_m, I_m, z_m, \bar{z}_m): \ D(s_{m+1}, r_{m+1}) \to D(s_m, r_m),$$

is a symplectic map on the space $\Gamma^{a,\bar{p}}$. Moreover, (3.32) implies the following estimates

$$|||\Phi_m - id|||_{D(s_{m+1}, r_{m+1})} \leq \frac{c\epsilon_{2,m}}{\alpha_m}, \quad |||D\Phi_m - Id|||_{D(s_{m+1}, r_{m+1})} \leq \frac{c\epsilon_{2,m}}{\alpha_m \rho_m^2}. \tag{3.69}$$

Note that Φ_m is affine about (I, u, v), by (3.69) it follows easily the convergence of $\{\Phi^m\}$; here we omit the details.

Let

$$\Phi^m \to \Phi_*, \quad H_m \to H \circ \Phi_* = H_* = N_* + P_*.$$

Then
$$N_* = \omega_* I + \langle \Omega z, \bar{z} \rangle, \quad P_*(\theta, I, z, \bar{z}) = O(|(I, z, \bar{z})|^2).$$
Moreover,
$$|||\Phi_* - id|||_{D(s/2,r/2)} \leq \frac{c\epsilon}{\alpha}.$$
Then, the Φ in Theorem 2.1 is given by $\Phi(\theta;\omega) = \Phi_*(\theta,0,0,0;\omega)$.

Proof of Theorem 2.2. Without loss of generality, we only consider the case $k \geq 0$. Suppose $0 < \alpha \leq \frac{1}{4}\beta$. Let $f_{kj}(\omega) = k\omega_k(\omega) - \Omega_j(\omega)$ and
$$\mathcal{R}_{kj}(\alpha) = \{\omega || f_{kj}(\omega)| \leq \alpha\}.$$
If $\omega \in \mathcal{R}_{kj}$, then for $k' \neq k$ we have
$$|f_{k'j}(\omega)| \geq |k - k'||\omega_k| - \alpha \geq \frac{1}{2}\beta \geq \alpha.$$
If $|j - j'| > \bar{N}$ we have
$$|f_{kj'}(\omega)| \geq \bar{\alpha} - \alpha \geq \frac{1}{2}\bar{\alpha}.$$
There exist $c_1 > 0$, $c_2 > 0$ such that if $|f_{kj}(\omega)| \leq \alpha$, then
$$c_1 j^d \leq |k| \leq c_2 j^d.$$
Since
$$|f_{kj}|^L \geq k\Big(\frac{1}{2} - \frac{bj^{\bar{d}}}{k}\Big) \geq k\Big(\frac{1}{2} - \frac{b}{c_1} j^{\bar{d}-d}\Big),$$
so there exists $J > 0$ such that if $j \geq J$, we have
$$|f_{kj}|^L \geq \frac{1}{3}k \geq \frac{c_1}{3} j^d.$$
Thus, since $d > 1$, we have
$$\text{meas}(\cup_{j \geq J, k \in \mathbb{Z}} \mathcal{R}_{kj}(\alpha)) \leq \sum_{j \geq J, k \in \mathbb{Z}} \text{meas}(\mathcal{R}_{kj}(\alpha)) \leq 2\bar{N} \sum_{j=1}^{\infty} \frac{3}{c_1} \frac{2\alpha}{j^d} \leq c\alpha.$$
Thus, we complete the proof. $\square$

Appendix

In this section we give some lemmas which are useful in KAM steps.

Lemma A.1. *If $f(\theta)$ is a 2π-periodic analytic function on $D(s)$, then*
$$|f|_s \leq ||f||_s, \quad ||f||_{s-\rho} \leq \frac{c}{\rho}|f|_s, \quad ||f'(\theta)||_{s-\rho} \leq \frac{c}{\rho^2}|f|_s.$$

Proof. The proof is easy and we omit the details. □

Lemma A.2. *Let $\Theta : \theta_+ \to \theta(\theta_+)$ is a real analytic map from $D(s-\rho)$ to $D(s)$, which depends on the parameter ω and satisfies that for $\theta_+ \in D(s-\rho)$*

$$|\theta - \theta_+| \leq \epsilon \quad \text{and} \quad |\theta|^L \leq \epsilon^L,$$

where $\epsilon < \rho \leq 1$. Let $f(\theta)$ be a 2π-periodic function depending on a parameter ω. If f is analytic in θ on $D(s)$ and Lipschitz continuous in ω on O, then

(1) $||f \circ \Theta||_{s-3\rho} \leq (1 + \frac{3\epsilon}{\rho^2})||f||_s$,

(2) $||f \circ \Theta||^L_{s-3\rho} \leq (1 + \frac{3\epsilon}{\rho^2})||f||^L_s + \frac{5\epsilon^L}{\rho^2}||f||_s$, *where $\epsilon \leq \frac{1}{4}\rho$ and $\rho \leq \frac{1}{3}s$.*

Proof. Let

$$\bar{f}(\theta_+) = f \circ \Theta(\theta_+) = f[\theta(\theta_+)] \quad \text{and} \quad \tilde{f}(\theta_+) = \bar{f}(\theta_+) - f(\theta_+).$$

Then $\tilde{f}$ is analytic in θ_+ on $D(s-\rho)$ and

$$\tilde{f}(\theta_+) = \int_0^1 f'[\theta_+ + t(\theta - \theta_+)](\theta - \theta_+)dt. \tag{A.70}$$

Thus,

$$|\tilde{f}(\theta_+)|_{s-\rho} \leq \epsilon |f'(\theta)|_s.$$

For the Fourier coefficients of $\bar{f}$, f and $\tilde{f}$, we have $\bar{f}_k = f_k + \tilde{f}_k$ and

$$\begin{aligned} ||\bar{f}||_{s-2\rho} \leq & \sum_{k\in Z} |f_k| e^{|k|(s-2\rho)} + \sum_{k \in Z} |\tilde{f}_k| e^{|k|(s-2\rho)} \\ \leq & ||f||_{s-2\rho} + |\tilde{f}|_{s-\rho} \sum_{k\in Z} e^{-|k|\rho} \leq ||f||_{s-2\rho} + \frac{3\epsilon}{\rho}|f'|_s. \end{aligned}$$

By Cauchy's estimates we have proved the result (1).

By Eq. (A.70), for $\theta_+ \in D(s-\rho)$ we have

$$\begin{aligned} |\tilde{f}(\theta_+)|^L \leq & \int_0^1 |f'[\theta_+ + t(\theta-\theta_+)]|^L |\theta - \theta_+| dt + \int_0^1 |f'[\theta_+ + t(\theta - \theta_+)]||\theta - \theta_+|^L dt \\ \leq & \epsilon \int_0^1 |f'[\theta_+ + t(\theta - \theta_+)]|^L dt + \epsilon^L |f'(\theta)|_s. \end{aligned}$$

Since

$$|f'[\theta_+ + t(\theta-\theta_+)]|^L \leq |f'(\theta)|^L_s + |f''(\theta)|_s |\theta - \theta_+|^L \leq |f'(\theta)|^L_s + \epsilon^L |f''(\theta)|_s,$$

so it follows that

$$|\tilde{f}(\theta_+)|^L \leq \epsilon^L |f'(\theta)|_s + \epsilon[|f'(\theta)|^L_s + \epsilon^L |f''(\theta)|_s].$$

In the same way as the above it follows that

$$||\bar{f}||^L_{s-2\rho} \leq ||f||^L_{s-2\rho} + \frac{3}{\rho}\{\epsilon^L |f'|_s + \epsilon[|f'|^L_s + \epsilon^L |f''|_s]\}.$$

By Cauchy's estimates we have the result (2). □

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