Analysis in Theory and Applications
Anal. Theory Appl., Vol. **36**, No. 4 (2020), pp. 468-481

DOI: 10.4208/ata.OA-SU4

A Note on $Card(X)$

Weituo Dai, Meng Wang* and Limin Sun

School of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China

Received 17 September 2020; Accepted (in revised version) 21 September 2020

Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. The main interests here are to study the relationship between $card(X)$ and $card(\mathcal{P}(X))$ and the connection between the separability of a space X and cardinality of some function space on it. We will convert the calculation of $card(\mathcal{P}(X))$ to the calculation of $card(\mathcal{F}(X \to \mathbb{Q}))$. The main tool we used here is Zorn Lemma.

Key Words: Cardinality, separability of space, Zorn Lemma.

AMS Subject Classifications: 03E10

1 Introduction

Let X be a set. If X is a finite set, we call the number of elements of X the cardinality of X, and denote it by $card(X)$. For two infinite sets X and Y, we can use this notion to compare the "number" of two sets X and Y. The following expressions are well-known:

(i) $card(X) \le card(Y)$ if there exists an injective map $\phi : X \to Y$;

(ii) $card(X) \ge card(Y)$ if there exists a surjective map $\phi : X \to Y$;

(iii) $card(X) = card(Y)$ if there exists a bijective map $\phi : X \to Y$.

Let X and Y be two sets. We recall the following theorems in [1–3].

Theorem 1.1. *$card(X) = card(Y)$ if and only if $card(X) \le card(Y)$ and $card(X) \ge card(Y)$ both hold.*

Theorem 1.2. *Either $card(X) < card(Y)$ or $card(Y) < card(X)$ or $card(X) = card(Y)$.*

Theorem 1.3. *$card(X) < card(\mathcal{P}(X))$.*

*Corresponding author. *Email address:* `mathdreamcn@zju.edu.cn` (M. Wang)

In this paper, we use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ to denote the set of positive integers, integers, rational numbers, real numbers and complex numbers respectively. The number filed F mentioned here is a subfield of $\mathbb{C}$, thus $\mathbb{Q}$ is the minimal number field and $F \supset \mathbb{Q}$. Given two sets X and Y, we denote

$$\mathcal{F}(X \to Y) = \{\text{map } f : X \to Y\}. \tag{1.1}$$

Especially, there is a natural algebra structure on $\mathcal{F}(X \to F)$ if F is a field. As usual, we use (X,ρ) to denote a metric space with a metric map $\rho : X \times X \to [0,+\infty)$, which satisfies

(i) $\rho(x_1,x_2) = 0$ if and only if $x_1 = x_2$;

(ii) $\rho(x_1,x_2) = \rho(x_2,x_1)$;

(iii) $\rho(x_1,x_3) \le \rho(x_1,x_2) + \rho(x_2,x_3)$, where x_1,x_2,x_3 are arbitrary points of X.

We use $(X,\mathcal{M},\mu)$ to denote a measure space, where $\mathcal{M}$ is a σ-algebra on X, and μ is a measure, i.e., $\mu : \mathcal{M} \to [0,+\infty]$ is a map, satisfying

(i) $\mu(\phi) = 0$;

(ii) $\mu(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{+\infty} \mu(E_j)$, where $E_j \in \mathcal{M}$ and $E_{j_1} \cap E_{j_2} = \emptyset$, $(j_1 \neq j_2)$.

We denote $card(\mathbb{N}) = c_0$, which is the minimal cardinality of all infinite sets. Denote $card(\mathbb{R}) = c$, which is called "cardinality of the continuum".

Let X and Y be two sets and $\alpha = card(X)$, $\beta = card(Y)$. We have the following definitions,

Definition 1.1. If $X \cap Y = \emptyset$, we define $\alpha + \beta = card(X \cup Y)$.

Definition 1.2. Define $\alpha \cdot \beta = card(X \times Y)$.

Definition 1.3. Define $\beta^\alpha = card(\mathcal{F}(X \to Y))$.

We verify that these three definitions are well-defined. Suppose two sets X_1 and Y_1 satisfy $card(X_1) = card(X)$, $card(Y_1) = card(Y)$ and $X_1 \cap Y_1 = \emptyset$ (in Definition 1.1). Then, we have bijective maps $\phi : X \to X_1$ and $\psi : Y \to Y_1$. We construct three maps ω, θ, η as follows:

$$\omega : X \cup Y \to X_1 \cup Y_1, \quad \omega(z) = \begin{cases} \phi(x), & \text{if } z = x \in X, \\ \psi(y), & \text{if } z = y \in Y, \end{cases} \tag{1.2a}$$

$$\theta : X \times Y \to X_1 \times Y_1 : \ \theta(x,y) = (\phi(x),\psi(y)), \tag{1.2b}$$

where $x \in X, y \in Y$.

$$\eta : \mathcal{F}(X \to Y) \to \mathcal{F}(X_1 \to Y_1) : \ \eta(f) = \psi \circ f \circ \phi^{-1}, \tag{1.3}$$

where $f \in \mathcal{F}(X \to Y)$, "$\circ$" represents the composition of maps. It is easy to verify that ω,θ,η are bijective. Thus these definitions are well-defined.

Remark 1.1. (1) Note that in Definition 1.1, if $\beta=\alpha$, we have $\alpha+\alpha=card(X\times\{0,1\})$, where $\alpha=card(X)$. (2) In some literature, $2^\alpha=card(\mathcal{P}(X))$ where $\alpha=card(X)$. We will see that this coincides with the Definition 1.3, which will be explained in the Theorem 1.8.

We state the theorems about the cardinal computation as follows and leave the discussion about cardinality and the separability of a space X in Section 3.

Theorem 1.4. *If α_1,α_2,β are cardinal numbers of three nonempty sets, then $\beta^{\alpha_1}\cdot\beta^{\alpha_2}=\beta^{\alpha_1+\alpha_2}$.*

Theorem 1.5. *If α,β_1,β_2 are cardinal numbers of three nonempty sets, then $\beta_1^\alpha\cdot\beta_2^\alpha=(\beta_1\cdot\beta_2)^\alpha$.*

Theorem 1.6. *Given two cardinal numbers α, β, if at least one of them is the cardinal number of an infinite set, then $\alpha+\beta=\max(\alpha,\beta)$. Especially, $\alpha+\alpha=\alpha$ when α is the cardinal number of an infinite set.*

Theorem 1.7. *If α, β are cardinal numbers of two nonempty sets and at least one of them is infinite, then $\alpha\cdot\beta=\max(\alpha,\beta)$. Especially, $\alpha\cdot\alpha=\alpha$ when α is the cardinal number of an infinite set.*

Theorem 1.8. *Suppose two cardinal numbers α,β satisfy $\alpha\geq\beta\geq 2$, and $\alpha\geq c_0$. Then*

$$\beta^\alpha=2^\alpha=card(\mathcal{P}(X)),$$

where $\alpha=card(X)$.

Theorem 1.9. *Suppose V is an infinite-dimensional linear space over the field F with a basis E. Then, $card(V)=\max(card(E),card(F))$. Especially, $card(V)=card(E)$ when $F=\mathbb{Q}$.*

Example 1.1. Let $V=\mathbb{R}$, $F=\mathbb{Q}$. We know that

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}$$

is a transcendental number, so $\{e^k\}_{k=0}^\infty$ is a linear independent set over $\mathbb{Q}$, thus $\mathbb{R}$ is an infinite-dimensional linear space over $\mathbb{Q}$. If E is a basis of $\mathbb{R}$ over $\mathbb{Q}$, then $card(E)=card(\mathbb{R})=c$.

Theorem 1.10. *For any two infinite sets X and Y, $card(X)=card(Y)$ if and only if $card(\mathcal{P}(X))=card(\mathcal{P}(Y))$ and there exists an algebra isomorphism $\Psi:\mathcal{F}(X\to\mathbb{Q})\to\mathcal{F}(Y\to\mathbb{Q})$.*

This paper is organized as follows. In Section 2, we prove Theorems 1.4–1.10. In Section 3, we discuss the connection between separability of a space X and the cardinality of some function space on it and prove several related results. Finally, several unsolved questions are raised in the context.

2 Proof of Theorems 1.4–1.10

Proof of Theorem 1.4. Suppose X_1, X_2, Y are nonempty sets satisfying $card(X_1) = \alpha_1$, $card(X_2) = \alpha_2$ and $card(Y) = \beta$, without loss of generality, we can assume that $X_1 \cap X_2 = \emptyset$. According to Definition 1.3,

$$\begin{aligned}
&\beta^{\alpha_1} = card(\mathcal{F}(X_1 \to Y)), \quad \beta^{\alpha_2} = card(\mathcal{F}(X_2 \to Y)),\\
&\beta^{\alpha_1} \cdot \beta^{\alpha_2} = card(\mathcal{F}(X_1 \to Y) \times \mathcal{F}(X_2 \to Y))\\
&\qquad\qquad = \{(f_1, f_2) : \text{ map } f_1 : X_1 \to Y, \text{ map } f_2 : X_2 \to Y\}.
\end{aligned}$$

Since $X_1 \cap X_2 = \emptyset$, $(f_1(x_1), f_2(x_2))$, $x_1 \in X_1$, $x_2 \in X_2$, corresponds bijectively to map $f : X_1 \cup X_2 \to Y$, where

$$f(x) = \begin{cases} f_1(x_1), & \text{if } \ x = x_1 \in X_1, \\ f_2(x_2), & \text{if } \ x = x_2 \in X_2. \end{cases} \tag{2.1}$$

Therefore, $\mathcal{F}(X_1 \to Y) \times \mathcal{F}(X_2 \to Y) = \mathcal{F}(X_1 \cup X_2 \to Y)$. By Definition 1.1 and Definition 1.3, we have

$$\beta^{\alpha_1} \cdot \beta^{\alpha_2} = card(\mathcal{F}(X_1 \cup X_2 \to Y)) = \beta^{card(X_1 \cup X_2)} = \beta^{\alpha_1 + \alpha_2}.$$

Thus, we complete the proof. □

Proof of Theorem 1.5. Suppose X, Y_1, Y_2 are nonempty sets satisfying $card(X) = \alpha$, $card(Y_1) = \beta_1$ and $card(Y_2) = \beta_2$. By Definition 1.3, $\beta_1^\alpha = card(\mathcal{F}(X \to Y_1))$, $\beta_2^\alpha = card(\mathcal{F}(X \to Y_2))$. By Definition 1.2, $\beta_1^\alpha \cdot \beta_2^\alpha = card(\mathcal{F}(X \to Y_1) \times \mathcal{F}(X \to Y_2))$. Besides $\mathcal{F}(X \to Y_1) \times \mathcal{F}(X \to Y_2) = \{\text{map } f : X \to Y_1 \times Y_2\} = \mathcal{F}(X \to Y_1 \times Y_2)$. By Definition 1.2, $card(Y_1 \times Y_2) = \beta_1 \cdot \beta_2$. It follows that

$$\beta_1^\alpha \beta_2^\alpha = card(\mathcal{F}(X \to Y_1 \times Y_2)) = (card(Y_1 \times Y_2))^{card(X)} = (\beta_1 \cdot \beta_2)^\alpha,$$

the second "=" follows by Definition 1.2. □

Proof of Theorem 1.6. We first consider the case when $\alpha + \alpha = \alpha$, where α is the cardinal number of an infinite set. The proof of this case can be seen in [3, Page 30].

To prove the general result, we divide the situation into two cases:

(i) $\beta > \alpha$ and α is the cardinal number of an infinite set. Suppose X, Y satisfy $X \cap Y = \emptyset$, $card(X) = \alpha$ and $card(Y) = \beta$. Since $\beta > \alpha$, there is a proper subset Y_1 of Y s.t. $card(Y_1) = card(X) = \alpha$ and $Y_1 \cap X = \emptyset$. Because $\alpha + \alpha = \alpha$, we obtain $card(X \cup Y_1) = card(Y_1) = \alpha$. So there is a bijective map $\phi : X \cup Y_1 \to Y_1$. We construct a map $\psi : X \cup Y = (X \cup Y_1) \cup (Y \setminus Y_1) \to Y$:

$$\psi(z) = \begin{cases} \phi(z), & \text{if } \ z \in X \cup Y_1, \\ z, & \text{if } \ z \in Y \setminus Y_1. \end{cases} \tag{2.2}$$

Then ψ is trivially bijective, thus $card(X \cup Y) = card(Y)$, which indicates $\alpha + \beta = \beta = \max(\alpha, \beta)$.

(ii) $\beta > \alpha$, where α is the cardinal number of a finite set, β is the cardinal number of an infinite set. Suppose $X = \{x_1, \cdots, x_k\}$, Y is an infinite set, $\beta = card(Y)$ and $X \cap Y = \emptyset$. Select a sequence $\{y_j\}_{j=1}^{\infty}$ in Y. Thus,

$$X \cup Y = \Big(\{y_j\}_{j=1}^{\infty} \cup \{x_1, \cdots, x_k\}\Big) \cup \Big(Y \setminus \{y_j\}_{j=1}^{\infty}\Big).$$

As any infinite countable set has card c_0, there exists bijection η: $\{y_j\}_{j=1}^{\infty} \cup \{x_1, \cdots, x_k\} \to \{y_j\}_{j=1}^{\infty}$. Let $\omega : X \cup Y \to Y$ be defined as

$$\omega(z) = \begin{cases} \eta(z), & \text{if } z \in \{y_j\}_{j=1}^{\infty} \cup \{x_1, \cdots, x_k\}, \\ z, & \text{if } z \in Y \setminus \{y_j\}_{j=1}^{\infty}. \end{cases} \tag{2.3}$$

It is easy to see that ω is bijective, so $card(X \cup Y) = card(Y)$. We conclude that $\alpha + \beta = \beta = \max(\alpha, \beta)$. □

Remark 2.1. Applying the principal of induction, the special case in Theorem 1.6 $\alpha + \alpha = \alpha$ can be extended to

$$\underbrace{\alpha + \cdots + \alpha}_{k} = \alpha,$$

where $k \in \mathbb{N}$, α is the cardinal number of an infinite set.

Proof of Theorem 1.7. We first consider the case when $\alpha \cdot \alpha = \alpha$, where α is the cardinal of an infinite set. Suppose X satisfies $card(X) = \alpha$. Define

$$\mathcal{X} = \{(A, \phi) : A \subset X \text{ satisfies } card(A \times A) = card(A),\ \phi : A \to A \times A \text{ bijective}\}. \tag{2.4}$$

Since X is infinite, it has countably infinite subset $A_0 = \{x_j\}_{j=1}^{\infty} \subset X$ and we have bijection $\phi_0 : A_0 \to A_0 \times A_0$. Thus, $\mathcal{X}$ is nonempty. We can define the order relation in $\mathcal{X}$: $(A_1, \phi_1) < (A_2, \phi_2) \Leftrightarrow A_1 \subset A_2$ and $\phi_2|_{A_1} = \phi_1$. Suppose $\mathcal{X}_1$ is a totally ordered subset of $\mathcal{X}$. Let $\tilde{A} = \cup_{A \in \mathcal{X}_1} A$, then

$$\tilde{A} \times \tilde{A} = \cup_{A \in \mathcal{X}_1, B \in \mathcal{X}_1} (A \times B) = \cup_{A \in \mathcal{X}_1} (A \times A).$$

Construct the map $\tilde{\phi} : \tilde{A} \to \tilde{A} \times \tilde{A}$, where $\tilde{\phi}(x) = \phi(x)$, if $x \in A \in \mathcal{X}_1$. So $(\tilde{A}, \tilde{\phi}) \in \mathcal{X}$ and $(\tilde{A}, \tilde{\phi})$ is an upper bound for $\mathcal{X}_1$ in $\mathcal{X}$. By Zorn Lemma, $\mathcal{X}$ has maximal element (A^*, ϕ^*). Here are three cases as follows:

(i) $X \setminus A^* = \emptyset$, i.e., $X = A^*$. Then $\alpha \cdot \alpha = \alpha$.

(ii) $card(X \setminus A^*) \leq card(A^*)$. Let $B = X \setminus A^*$, then $card(B) \leq card(A^*)$, $X = A^* \cup B$ and $A^* \cap B = \emptyset$. By Theorem 1.6,

$$\alpha = card(X) = card(A^* \cup B) = \max(card(A^*), card(B)) = card(A^*).$$

So $(A^*, \phi^*) \in \mathcal{X}$, $card(A^*) = card(A^* \times A^*)$. It follows that $\alpha \cdot \alpha = \alpha$.

(iii) $card(X \setminus A^*) > card(A^*)$. This case cannot occur because of the reason that: If $card(X \setminus A^*) > card(A^*)$, then X has proper subset $B \subset X \setminus A^*$ such that $card(B) = card(A^*)$, i.e., there exists a bijection from B to A^*, so $card(A^* \times B) = card(B \times A^*) = card(B \times B) = card(A^* \times A^*)$. Since $B \cap A^* = \emptyset$, any two of the four product sets in the above equation do not intersect. Note the following equality:

$$(A^* \cup B) \times (A^* \cup B) = (A^* \times A^*) \cup \{(A^* \times B) \cup (B \times A^*) \cup (B \times B)\}. \tag{2.5}$$

Applying Theorem 1.6 and the result in Remark 2.1, we obtain following equality:

$$\begin{aligned} &card(\{(A^* \times B) \cup (B \times A^*) \cup (B \times B)\}) \\ =&card(A^* \times A^*) = card(A^*) = card(B). \end{aligned} \tag{2.6}$$

Thus, there is a bijection $\eta : B \to \{(A^* \times B) \cup (B \times A^*) \cup (B \times B)\}$. We construct the map $\omega : A^* \cup B \to (A^* \cup B) \times (A^* \cup B)$ as follows:

$$\omega(z) = \begin{cases} \phi^*(z), & \text{if } z \in A^*, \\ \eta(z), & \text{if } z \in B. \end{cases} \tag{2.7}$$

Then ω is bijective and $\omega|_{A^*} = \phi^*$. Therefore, $(A^* \cup B, \omega) \in \mathcal{X}$ and $(A^* \cup B, \omega) > (A^*, \phi^*)$. This contradicts that (A^*, ϕ^*) is a maximal element for $\mathcal{X}$, it follows that $\alpha \cdot \alpha = \alpha$.

Now we turn to the general case. Suppose $\beta > \alpha$, X, Y satisfy $card(X) = \alpha$, $card(Y) = \beta$, where β is the cardinal number of an infinite set. Without loss of generality, assume $X \cap Y = \emptyset$. Since X is nonempty, $\exists\, x_1 \in X$, so $\alpha \cdot \beta = card(X \times Y) \geq card(\{x_1\} \times Y) = card(Y) = \beta$. On the other hand, because $\beta > \alpha$, $\exists$ proper subset Y_1 of Y and a bijection $\psi : X \to Y_1$. Let $X_1 = X \cup (Y \setminus Y_1)$, then $X_1 \supset X$ and we can construct a bijection $\theta : X_1 \to Y$, where

$$\theta(z) = \begin{cases} \psi(x), & \text{if } z = x \in X, \\ z, & \text{if } z \in Y \setminus Y_1. \end{cases}$$

Since $\beta \cdot \beta = \beta$, $\alpha \cdot \beta = card(X \times Y) \leq card(X_1 \times Y) = \beta \cdot \beta = \beta$. Thus, $\alpha \cdot \beta = \beta = \max(\alpha, \beta)$. □

Remark 2.2. Suppose α is the cardinal number of an infinite set. Applying the principal of induction, the special case in Theorem 1.7 $\alpha \cdot \alpha = \alpha$ can be extended to $\alpha^k = \alpha \cdot \alpha \cdots \alpha = \alpha$, $k \in \mathbb{N}$. Applying Theorem 1.7, the result in Remark 2.1 can be extended to $c_0\alpha = \alpha + \alpha + \cdots + \alpha + \cdots = \max(c_0, \alpha) = \alpha$.

Question 2.1. Whether $\alpha^k = \alpha$ can be extended to $\alpha^{c_0} = \alpha$ for any $\alpha > c_0$? If not, for which kind of α, $\alpha^{c_0} = \alpha$ holds?

When $\alpha = c_0$, ${c_0}^{c_0} = c > c_0$; if $\alpha = c$, then $c^{c_0} = c$, the proof of which can be found in Theorem 3.2 in the Section 3. For a more common case, suppose $\gamma_0 = c$, $\gamma_k = 2^{\gamma_{k-1}}$, $k \in \mathbb{N}$. If $\alpha \in \{\gamma_0, \gamma_1, \cdots, \gamma_n, \cdots\}$, then $\alpha^{c_0} = \alpha$ holds. The more general case remains to be answered.

Before giving the proof of Theorem 1.8, we give some explanations. Theorem 1.8 means that given an infinite set X, then for any set Y satisfying $card(X) \geq card(Y) \geq 2$, we have $card(\mathcal{P}(X)) = card(\mathcal{F}(X \to Y))$. Especially, let $Y = \{0,1\}$, then $\mathcal{F}(X \to \{0,1\}) = \{\text{characteristic map } \chi_E : \ E \in \mathcal{P}(X)\}$. Therefore, $card(\mathcal{P}(X)) = card(\mathcal{F}(X \to Y))$, when computing the cardinal number of $\mathcal{P}(X)$, the choice of Y is flexible to a certain extend.

Proof of Theorem 1.8. Since $card(Y) \geq 2$, without loss of generality, assume $\{0,1\} \subset Y$. Then $card(\mathcal{F}(X \to Y)) \geq card(\{\chi_E : E \in \mathcal{P}(X)\}) = card(\mathcal{P}(X))$. On the other hand, each $f \in \mathcal{F}(X \to Y)$ has a graph $\{(x, f(x)) : x \in X\} \in \mathcal{P}(X \times Y)$. Thus $card(\mathcal{F}(X \to Y)) \leq card(\mathcal{P}(X \times Y)) = 2^{card(X \times Y)}$; according to Theorem 1.7 and $card(X) \geq card(Y) \geq 2$, $card(X \times Y) = card(X)$; so $card(\mathcal{F}(X \to Y)) \leq 2^{card(X)} = card(\mathcal{P}(X))$. In a word, by Theorem 1.1 we get $card(\mathcal{F}(X \to Y)) = card(\mathcal{P}(X))$. □

Proof of Theorem 1.9. Suppose V is an infinite-dimensional linear space over the field F with a basis E. Any element $x \in V$ can be written as $x = \sum_{j=1}^{n} \lambda_j e_j$ in a unique way, where $\lambda_j \in F$, $e_j \in E$, $e_{j_1} \neq e_{j_2}$, $(j_1 \neq j_2)$. Let $\alpha = card(E)$, since V is infinite-dimensional, $\alpha \geq c_0$. Let $X = \{(e_1, e_2, \cdots, e_n; \lambda_1, \cdots, \lambda_n) : e_j \in E, \lambda_j \in F, n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ fixed, the number of the selections of choosing n different vectors from E is $\alpha^n = \alpha$ (by Theorem 1.7). After choosing n different vectors, each e_j multiplies $\lambda_j \in F$, $\beta^n = \beta$ possibilities in total, where $\beta = card(F)$. Thus, the cardinal number of the set consisting of all the linear compositions of n vectors from E is $\alpha \cdot \beta$. It follows that $card(V) \leq card(X) = (\alpha \cdot \beta) + (\alpha \cdot \beta) + \cdots + (\alpha \cdot \beta) + \cdots = c_0(\alpha \cdot \beta) = \alpha \cdot \beta = \max(\alpha, \beta)$ (applying the result of Remark 2.2, note that $\alpha \geq c_0$).

On the other hand, because $V \supset E$ and $V \supset \{\lambda e_1 : \lambda \in F\}$, it follows that $card(V) \geq card(E)$ and $card(V) \geq card(F)$. As a result, $card(V) = \max(card(E), card(F))$. □

Proof the Theorem 1.10. "$\Rightarrow$" suppose X, Y are two infinite sets satisfying $card(X) = card(Y)$. Then, there exists a bijection $\phi : X \to Y$. ϕ induces the map $\Phi : \mathcal{P}(X) \to \mathcal{P}(Y) : \Phi(E) = \phi(E) = \{\phi(x), x \in E\} \in \mathcal{P}(Y)$, where $E \in \mathcal{P}(x)$. Clearly, Φ is bijective, so $card(\mathcal{P}(X)) = card(\mathcal{P}(Y))$. And ϕ can also induce the map $\Psi : \mathcal{F}(X \to \mathbb{Q}) \to \mathcal{F}(Y \to \mathbb{Q}) : f \to \Psi(f) = f \circ \phi^{-1}$. It is easy to check that Ψ is an algebra isomorphism from $\mathcal{F}(X \to \mathbb{Q})$ to $\mathcal{F}(Y \to \mathbb{Q})$.

" $\Leftarrow$ " In Introduction, we have pointed out that $\mathcal{F}(X \to \mathbb{Q})$ is an algebra, now denote it by $\mathcal{A}$. Consider a special ideal family of $\mathcal{A}$, denote it by S-type: An ideal $\mathcal{I}$ of $\mathcal{A}$ belongs to S-type $\Leftrightarrow$ $\mathcal{I}$ satisfies the following conditions:

- (i) $\mathcal{I} \neq \emptyset$, $\mathcal{I} \neq \mathcal{A}$;

- (ii) $\mathcal{I}$ is a principal ideal, i.e., $\mathcal{I} = \{f_0 \cdot g : f_0 \in \mathcal{I}$ is a fixed element, $\forall g \in \mathcal{A}\}$;
- (iii) In the family of principal ideals, $\mathcal{I}$ is a maximal element.

It is not hard to prove that all S-type ideals are $\{\mathcal{I}_x : x \in X\}$, where

$$\mathcal{I}_x = \{\text{function } f : X \to \mathbb{Q};\ f(x) = 0\}.$$

Similarly, all S-type ideals of $\mathcal{F}(Y \to \mathbb{Q})$ are $\{\mathcal{I}_y : y \in Y\}$, where

$$\mathcal{I}_y = \{\text{function } h : Y \to \mathbb{Q};\ h(y) = 0\}.$$

Since $\Psi : \mathcal{F}(X \to \mathbb{Q}) \to \mathcal{F}(Y \to \mathbb{Q})$ is algebra isomorphism, $\{\Psi(I_x) : x \in X\} \subset \{\mathcal{I}_y : y \in Y\}$. So $card(X) \leq card(Y)$; and because $\{\Psi^{-1}(\mathcal{I}_y) : y \in Y\} \subset \{\mathcal{I}_x : x \in X\}$, it follows that $card(Y) \leq card(X)$. In a word, we obtain the result

$$card(X) = card(Y)$$

by Theorem 1.1. □

Remark 2.3. It seems that the requirement in Theorem 1.10 is isomorphism "$\Psi : \mathcal{F}(X \to \mathbb{Q}) \to \mathcal{F}(Y \to \mathbb{Q})$" is not important. We concern about whether this requirement can be removed. We make following hypothesis.

Hypothesis 2.1. Suppose X, Y are two sets. Then $card(X) = card(Y) \Leftrightarrow card(\mathcal{P}(X)) = card(\mathcal{P}(Y))$.

This hypothesis holds in the following three cases: (1) X and Y are finite sets. (2) the cardinal numbers of X and Y are from a special sequence $\{\alpha_0, \alpha_1, \cdots, \alpha_n, \cdots\}$, where $\alpha_0 = card(S)$, $\alpha_k = 2^{\alpha_{k-1}}$, $k \in \mathbb{N}$, and S is an arbitrary infinitely set. (3) If we accept the "continuum hypothesis", then the hypothesis holds when $\max(card(X), card(Y)) \leq c$.

3 Cardinality and separability of the space

In this section, we discuss the relationship between the cardinality and the separability of the space. A metric space is called separable if it has a countable dense subset.

Theorem 3.1. *If (X, ρ) is a separable metric space, then $card(X) \leq c$.*

Proof. Suppose $\{x_n\}_{n=1}^{\infty}$ is a countable dense subset of X. Then, for each $x \in X$, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x, i.e., $\rho(x_{n_k}, x) \to 0$, $k \to \infty$. Such sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is not unique, but if $x_1 \neq x_2$, it holds that $\{x_{n_k}^{(1)}\} \neq \{x_{n_k}^{(2)}\}$, where $x_{n_k}^{(1)} \to x_1$, $x_{n_k}^{(2)} \to x_2$. Let

$$\mathcal{X} = \left\{\{x_{n_k}\}_{k=1}^{\infty} : \{n_k\} \text{ is a subsequence of } \mathbb{N}\right\}.$$

DOI https://doi.org/10.4208/ata.OA-SU4 | Generated on 2024-11-19 20:44:51

Each $\{x_{n_k}\}$ corresponds to a real number $t=\sum_{n=1}^{\infty}\frac{\epsilon_n}{2^n}\in[0,1]$, where

$$\epsilon_n=\begin{cases}1, & n=n_k,\\ 0, & n\neq n_k.\end{cases}$$

Thus, $card(X)\leq card(\mathcal{X})=card([0,1])=c$. □

The separability of (X,ρ) does not contribute to the separability of $C(X,\rho)$. For example, Let $X=\mathbb{R}^n$ and ρ is the Euclidean metric. Then

$$\begin{aligned}C_c(\mathbb{R}^n)&=\{f\in C(\mathbb{R}^n): f \text{ has compact support}\},\\ C_0(\mathbb{R}^n)&=\{f\in C(\mathbb{R}^n): f \text{ vanishes at infinity}\}.\end{aligned}$$

It is well-known that $C_c(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$ are separable. However, $C(\mathbb{R}^n)$ is not separable. The computation of $card(C(X,\rho))$ needs further argument.

Theorem 3.2. *Suppose (X,ρ) is a separable metric space, $C(X,\rho)$ is the set of all continuous functions with respect to ρ. Then, $card(C(X,\rho))=c$.*

Proof. For each $r>0$, construct a function $f_r\in C(X,\rho)$ as follow,

$$f_r(x)=\begin{cases}r-\rho(x,x_0), & \text{if } \rho(x,x_0)\leq r,\\ 0, & \text{if } \rho(x,x_0)>r,\end{cases}\tag{3.1}$$

where x_0 is a fixed point in X. So $card(C(X,\rho))\geq card((0,+\infty))=c$. On the other hand, since (X,ρ) is separable, it has a countable dense subset $\{x_n\}_{n=1}^{\infty}\subset X$. Each $f\in C(X,\rho)$, since f is continuous, f is uniquely determined by $\{f(x_n)\}_{n=1}^{+\infty}$. Let

$$\mathcal{X}=\left\{\{\lambda_n\}_{n=1}^{\infty}:\lambda_n\in\mathbb{C},\ n\in\mathbb{N}\right\},$$

then $card(C(X,\rho))\leq card(\mathcal{X})=c^{c_0}$.

To complete the proof of Theorem 3.2, we have to prove $c^{c_0}=c$. Let

$$l^2=\left\{\{\lambda_n\}_{n=1}^{\infty}:\lambda_n\in\mathbb{C},\ n\in\mathbb{N} \text{ and } \sum_{n=1}^{\infty}|\lambda_n|^2<+\infty\right\}$$

be a separable Hilbert space, by Theorem 3.1, $card(l^2)=c$. l^2 consists of Hilbert cubes $\mathcal{H}=\{\{\lambda_n\}_{n=1}^{\infty}:\lambda_n\in\mathbb{C} \text{ and } |\lambda_n|\in I_n=[-\frac{1}{2^n},\frac{1}{2^n}],n\in\mathbb{N}\}=I_1\times I_2\times\cdots\times I_n\times\cdots$, where $card(I_n)=c$. It follows that $c=card(l^2)\geq card(\mathcal{H})=c^{c_0}$. And trivially $c^{c_0}\geq c$. Therefore, $c^{c_0}=c$. □

Remark 3.1. For the equation $c^{c_0}=c$, we might have such simpler explanation: since $c=2^{c_0}$, $c^{c_0}=(2^{c_0})^{c_0}=2^{c_0\times c_0}=2^{c_0}=c$. However, does $(2^{c_0})^{c_0}=2^{c_0\times c_0}$ holds? Note that 2^{α} is not the usual exponential function! In view of this and Theorem 1.4, Theorem 1.5, we pose the following question:

Question 3.1. Given arbitrary three cardinal numbers $\alpha_1, \alpha_2, \beta$ of nonempty sets, whether the equation $(\beta^{\alpha_1})^{\alpha_2} = \beta^{\alpha_1 \times \alpha_2}$ holds?

Consider the space L^p, which consists of functions defined on a measure space $(X, \mathcal{M}, \mu)$. μ is called a complete measure $\Leftrightarrow$ if $E \in \mathcal{M}$ and $\mu(E) = 0$, then $\mu(F) = 0$ for any subset $F \in \mathcal{M}$ of E. There exists a equivalence relation in $\mathcal{M}$ "$\sim$": $E_1 \sim E_2 \Leftrightarrow \mu((E_1 \setminus E_2) \cup (E_2 \setminus E_1)) = 0$. Given this equivalence relation, $\mathcal{M}$ becomes $[\mathcal{M}]$. All μ-measurable functions on X is denoted by $\mathcal{M}(X)$, define the equivalence relation "$\sim$": $f_1 \sim f_2 \Leftrightarrow f_1 - f_2 = 0$, μ-a.e.. The quotient space $\mathcal{M}(X)/\sim$ is denoted by $[\mathcal{M}(X)]$. For $0 < p < +\infty$, let

$$L^p(X, \mu) = \Big\{ f \in [\mathcal{M}(X)] : \|f\|_p = \Big(\int_X |f|^p d\mu \Big)^{1/p} < +\infty \Big\}.$$

Define the metric ρ_p as follow: for any $f_1, f_2 \in L^p$,

$$\rho_p(f_1, f_2) = \begin{cases} \|f_1 - f_2\|_p^p, & \text{if } \ 0 < p < 1, \\ \|f_1 - f_2\|_p, & \text{if } \ 1 \le p < +\infty. \end{cases} \tag{3.2}$$

Then $L^p(X, \mu)$ becomes a linear metric space, $0 < p < +\infty$. $L^\infty(X, \mu)$ is the set of μ-measurable essentially bounded functions. $L^\infty(X, \mu)$ can be a metric space with the metric induced by the essential supremum $\|f\|_\infty$.

Theorem 3.3. *Suppose $(X, \mathcal{M}, \mu)$ is a measure space with the complete measure μ. Identifying the elements in $\mathcal{M}$ differ by a set of measure zero, we get $[\mathcal{M}]$. If $card([\mathcal{M}]) \ge c_0$, then $card(L^p(X, \mu)) \le (card([\mathcal{M}]))^{c_0}$, where $0 < p < +\infty$.*

Proof. Let

$$\varphi_{\mathbb{Q}}(X, \mu) = \left\{ \sum_{j=1}^{n} (r_j + i r_j') \chi_{E_j} : r_j, r_j' \in \mathbb{Q}, E_j \in [\mathcal{M}], n \in \mathbb{N} \right\}.$$

It is clear that $\varphi_{\mathbb{Q}}(X, \mu)$ is dense in $L^p(X, \mu)$, $0 < p < +\infty$ (see [1, pp. 200]). Thus, for each $f \in L^p(X, \mu)$, there is a sequence of functions $\{f_n\}_{n=1}^\infty \subset \varphi_{\mathbb{Q}}(X, \mu)$ satisfying $\|f_n - f\|_p \to 0$, such $\{f_n\}_{n=1}^\infty$ is not unique. But if $f \ne g$, it must hold that $\{f_n\} \ne \{g_n\}$, where $\|g_n - g\|_p \to 0$. It follows that $card(L^p(X, \mu)) \le (card(\varphi_{\mathbb{Q}}(X, \mu)))^{c_0}$.

And since $\varphi_{\mathbb{Q}}(X, \mu)$ is a linear space over $\mathbb{Q} + i\mathbb{Q}$, $\{\chi_E : E \in [\mathcal{M}]\} = \mathcal{E}$ is its basis, trivially $card(\mathcal{E}) = card([\mathcal{M}])$, by Theorem 1.7,

$$card(\varphi_{\mathbb{Q}}(X, \mu)) = \max(card(\mathcal{E}), card(\mathbb{Q} + i\mathbb{Q})) = \max(card([\mathcal{M}]), card(\mathbb{Q} + i\mathbb{Q})).$$

And because $card([\mathcal{M}]) \ge c_0 = card(\mathbb{Q}) = card(\mathbb{Q} \times \{0, 1\}) = card(\mathbb{Q} + i\mathbb{Q})$, it follows that $card(\varphi_{\mathbb{Q}}(X, \mu)) = card([\mathcal{M}])$. Therefore, we get the result that $card(L^p(X, \mu)) \le (card([\mathcal{M}]))^{c_0}$, where $0 < p < +\infty$. $\square$

Remark 3.2. For general measure space $(X,\mathcal{M},\mu)$, whether $\phi_{\mathbb{Q}}(X,\mu)$ is dense in $L^\infty(X,\mu)$ is not sure. We must make an assumption of μ. For example, if $\mu(X)<+\infty$, then $\phi_{\mathbb{Q}}(X,\mu)$ is dense in $L^\infty(X,\mu)$, so the result of Theorem 3.3 also holds for the case when $p=+\infty$.

Theorem 3.4. *Suppose (X,ρ) is a separable metric space, $(X,\mathcal{M},\mu)$ is a measure space. If $C(X,\rho)$ is dense in $L^p(X,\mu)$, then $card(L^p(X,\mu))\le c$, where $0<p<+\infty$.*

Proof. (X,ρ) is separable, so $card(C(X,\rho))=c$ by Theorem 3.2. If $C(X,\rho)$ is dense in $L^p(X,\mu)$, then using a similar argument to the proof of Theorem 3.3, we can show that

$$card(L^p(X,\mu))\le(card(C(X,\rho)))^{c_0}=c^{c_0}=c.$$

Now we consider how to make $C(X,\rho)$ dense in $L^p(X,\mu)$? Use $B(x_0,r)$ to denote the open ball in X of radius r centering at x_0. $\mathcal{B}_X$ is the Borel algebra generated by open sets in X. Suppose $(X,\mathcal{M},\mu)$ is a measure space, where $\mathcal{M}\supset\mathcal{B}_X$ and μ has the following regular properties: (i) $\exists x_0\in X$ s.t. $\mu(\bar{B}(x_0,r))<+\infty$, $\forall r\in[0,\infty)$; (ii) $\forall E\in\mathcal{M}$, $\mu(E)=\inf\{\mu(V):\text{ open set }V\supset E\}=\sup\{\mu(U):\text{closed set }U\subset E\}$. □

Theorem 3.5. *(X,ρ) is a metric space and $(X,\mathcal{M},\mu)$ is a measure space. If μ is regular, then $C(X,\rho)$ is dense in $L^p(X,\mu)$, where $0<p<+\infty$.*

Proof. In the proof of Theorem 3.3, we have shown that $\phi_{\mathbb{Q}}(X,\mu)$ is dense in $L^p(X,\mu)$ $(0<p<+\infty)$. Now we consider using continuous functions to approach χ_E $(E\in[\mathcal{M}])$. For any fixed $\epsilon>0$, by the regularity of μ, there exists a closed set U_ϵ and an open V_ϵ such that $U_\epsilon\subset E\subset V_\epsilon$ and $\mu(V_\epsilon\setminus U_\epsilon)<\epsilon$. Let

$$f_\epsilon(x)=\frac{d(x,X\setminus V_\epsilon)}{d(x,X\setminus V_\epsilon)+d(x,U_\epsilon)},$$

where $d(x,W)=\inf\{\rho(x,y):y\in W\}$ is the distance between x and W. By the axiom (iii) of ρ, $d(x,W)$ is continuous function of x. It follows that $f_\epsilon\in C(X,\rho)$, $0\le f_\epsilon\le 1$ and $\chi_{U_\epsilon}\le f_\epsilon\le\chi_{V_\epsilon}$. So

$$\int_X|\chi_E-f_\epsilon|^p d\mu\le\mu(V_\epsilon\setminus U_\epsilon)<\epsilon,\quad(0<p<\infty).$$

Thus, $C(X,\rho)$ is dense in $L^p(X,\mu)$, where $0<p<\infty$. □

Combining Theorem 3.4 and Theorem 3.5, we get the following corollary.

Corollary 3.1. *Suppose $C(X,\rho)$ is a separable metric space, $(X,\mathcal{M},\mu)$ is a measure space with regular measure μ. Then, $card(L^p(X,\mu))\le c$, where $0<p<+\infty$.*

Corollary 3.2. *Suppose μ is a σ-finite measure of $(X,\mathcal{M},\mu)$ and $card([\mathcal{M}])>c$. Then, $L^p(X,\mu)$ is not separable, where $0<p\le+\infty$.*

Proof. First, we assume that μ is a finite measure, i.e., $\mu(X) < +\infty$. Then, any set $E \in [\mathcal{M}]$ corresponds to a characteristic function $\chi_E \in L^p(X,\mu)(0 < p \leq +\infty)$. Since $card([\mathcal{M}]) > c$, $card(L^p(X,\mu)) \geq card([\mathcal{M}]) > c$. Therefore, $L^p(X,\mu)$ is not separable. Otherwise, if $L^p(X,\mu)$ is separable, by Theorem 3.1, $card(L^p(X,\mu)) \leq c$. A contradiction!

When μ is a σ-finite measure, we can get the result by using the result of the finite measure case, we leave out the details here. □

Question 3.2. For general metric space $(X,\mathcal{M},\mu)$, when $card([\mathcal{M}]) = c$, whether $L^p(X,\mu)$ is separable?

We only consider the Euclidean space below. Let $X = \mathbb{R}^n$, $\mu = \mu_L$ is the Lebesgue measure. Denote the Lesbesgue measure space by $\mathcal{M}_L$. After the process of completion, we get $[\mathcal{M}_L]$. The set of all Lebesgue measurable functions is denoted by $\mathcal{M}_L(\mathbb{R}^n)$, given the equivalence relation as before, we get $[\mathcal{M}_L(\mathbb{R}^n)]$. $C(\mathbb{R}^n)$ represents the set of all continuous functions with respect to Euclidean topology on $\mathbb{R}^n$, the dual space of Schwartz space $S(\mathbb{R}^n)$ is the tempered distribution space $S'(\mathbb{R}^n)$.

Theorem 3.6. *$card(C(\mathbb{R}^n))$, $card(L^p(\mathbb{R}^n))$, $card([\mathcal{M}_L(\mathbb{R}^n)])$, $card(S'(\mathbb{R}^n)) = c$.*

Proof. (i) For $\mathbb{R}^n$ is separable, by Theorem 3.2, $card(C(\mathbb{R}^n)) = c$.

(ii) For any $r > 0$, corresponds to a function $\chi_{B(0,r)} \in L^p(\mathbb{R}^n)$, so $card(L^p(\mathbb{R}^n)) \geq card((0,+\infty)) = c$. On the other hand, since $C(R^n)$ is dense in $L^p(\mathbb{R}^n)$ (by Theorem 3.5), where $0 < p < +\infty$. It follows that $card(L^p(\mathbb{R}^n)) = c$ holds when $0 < p < +\infty$. The computation of $card(L^\infty(\mathbb{R}^n))$ is consisted in the case (iii).

(iii) Let $X_k = \{f \cdot \chi_{\{k-1\leq|x|\leq k\}} : f \in L^\infty(\mathbb{R}^n)\}$. Then $f \in L^\infty(\mathbb{R}^n)$ can be written as $f = \sum_{k=1}^\infty f_k$, where $f_k \in X_k$ has compact support. Clearly, $X_k \subset L^p(\mathbb{R}^n)(0 < p < \infty)$, so $card(X_k) \leq card(L^p(\mathbb{R}^n)) = c$. It follows that $card(L^\infty(\mathbb{R}^n)) \leq card(X_1 \times \cdots \times X_k \times \cdots) \leq c^{c_0} = c$; and since $\chi_{B(0,r)} \in L^\infty(\mathbb{R}^n)$, $card(L^\infty(\mathbb{R}^n)) \geq card((0,+\infty)) = c$. Therefore, we get the result that $card(L^\infty(\mathbb{R}^n)) = c$.

Since $L^\infty(\mathbb{R}^n) \subset [\mathcal{M}(\mathbb{R}^n)]$, $card([\mathcal{M}(\mathbb{R}^n)]) \geq card(L^\infty(\mathbb{R}^n)) = c$. On the other hand, each $g \in [\mathcal{M}(\mathbb{R}^n)]$ can be written as $g = \sum_{k=1}^\infty g_k$, where $g_k = g\chi_{\{x:k-1\leq|g(x)|<k\}} \in L^\infty(\mathbb{R}^n)$. Therefore,

$$card([\mathcal{M}(\mathbb{R}^n)]) \leq (card(L^\infty(\mathbb{R}^n)))^{c_0} = c^{c_0} = c.$$

(iv) The computation of $S'(\mathbb{R}^n)$ is rather complicated, we give an outline of the proof as below: each $u \in S'(\mathbb{R}^n)$ corresponds to a sequence of $\{u_k\}$, where $u_k \in S'(\mathbb{R}^n)_k$, $k \in \mathbb{N}$, where

$$S'(\mathbb{R}^n)_k = \{u \in S'(\mathbb{R}^n) : supp(u) \subset \bar{B}(0,k+1) = \{x \in \mathbb{R}^n : |x| \leq k+1\}\}.$$

It follows that each $S'(\mathbb{R}^n)_k$ is a subset of tempered distribution with fixed compact support. We define $u_k : u_k(f) = u(\Omega_k f)$, $\forall f \in S(\mathbb{R}^n)$,

$$\Omega_k(x) = \omega_k(|x|) = \omega_k\Big(\sqrt{x_1^2 + \cdots + x_n^2}\Big)$$

and ω_k is an one-variable function:

$$\omega_k(t)=\begin{cases}1, & \text{if } \ 0\le t\le k,\\ \omega_0(t-k), & \text{if } \ k<t<k+1,\\ 0, & \text{if } \ t\ge k+1,\end{cases}$$

$$\omega_0(t)=1-\lambda_0\int_0^t \exp(-1/s(1-s))ds,$$

where

$$\lambda_0=\Big(\int_0^1 \exp(-1/s(1-s))\,ds\Big)^{-1}.$$

Then $\omega_0\in C^\infty([0,1])$, and

$$\omega_0(0)=1,\quad \omega_0(1)=0,\quad \omega^{(k)}(0)=\omega^{(k)}(1)=0,\quad (k\in\mathbb{N}).$$

Thus Ω_k has special property: $\|\partial^\alpha\Omega_k\|_\infty\le C_\alpha$, the constant C_α is independent with k. We can show that u_k weakly converges to u, i.e., for any $f\in S(\mathbb{R}^n)$, $\lim_{k\to\infty}u_k(f)=u(f)$. It follows that $card(S'(\mathbb{R}^n))\le\prod_{k=1}^\infty card(S'(\mathbb{R}^n)_k)$.

The tempered distribution with compact support u_k has Fourier transform $\widehat{u_k}\in C^\infty(\mathbb{R}^n)$ (see [1, pp. 291–296]). So $card(S'(\mathbb{R}^n)_k)\le card(C^\infty(\mathbb{R}^n))=c$. It follows that $card(S'(\mathbb{R}^n))\le c$. On the other hand,

$$card(S'(\mathbb{R}^n))\ge card(L^\infty(\mathbb{R}^n))=c.$$

Therefore, $card(S'(\mathbb{R}^n))=c$. □

Remark 3.3. the cardinal number of the Cantor set in $\mathbb{R}$ of zero measure is c, and because Lebesgue measure is complete, the cardinal number of the set of all sets of zero measure in $\mathbb{R}$ is $=2^c$. Since the union of a non-measurable set and a measurable set is non-measurable if they do not intersect, the cardinal number of the set of all non-measurable sets is $=2^c$.

Question 3.3. Suppose $\mathcal{M}_L$ is the set of all Lesbesgue measurable sets in $\mathbb{R}^n$, then all non-measurable sets $=\mathcal{P}(\mathbb{R}^n)\setminus\mathcal{M}_L$, after adding the equivalence relation (identifying the sets differ by a set of measure zero), denote it by $[\mathcal{P}(\mathbb{R}^n)\setminus\mathcal{M}_L]$, then what is $card([\mathcal{P}(\mathbb{R}^n)\setminus\mathcal{M}_L])$? This is what we will consider next.

Acknowledgements

Meng Wang is partially supported by NSF of China under Grant No. 11771388.

References

[1] G. B. Folland, Real Analysis, Modern Techniques and Their Applications, second Edition, Wiley-Interscience Publication, 2007.
[2] D. X. Xia, Z, R. Wu, S. C. Yan, and W. C. Shu, Real Analysis and Functional Analysis, People's Education Press, 1978.
[3] W. X. Zheng, and S. W. Wang, Real Analysis and Functional Analysis, Nanjing University Press, 1978.