

# Hausdorff Dimension of a Class of Weierstrass Functions

Huojun Ruan\* and Na Zhang

School of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

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**Abstract.** It was proved by Shen that the graph of the classical Weierstrass function  $\sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x)$  has Hausdorff dimension  $2 + \log \lambda / \log b$ , for every integer  $b \geq 2$  and every  $\lambda \in (1/b, 1)$  [Hausdorff dimension of the graph of the classical Weierstrass functions, *Math. Z.*, 289 (2018), 223–266]. In this paper, we prove that the dimension formula holds for every integer  $b \geq 3$  and every  $\lambda \in (1/b, 1)$  if we replace the function  $\cos$  by  $\sin$  in the definition of Weierstrass function. A class of more general functions are also discussed.

**Key Words:** Hausdorff dimension, Weierstrass function, SRB measure.

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## 1 Introduction

Weierstrass functions are classical fractal functions. The non-differentiability of these functions were studied by Weierstrass and Hardy [2]. Recently, Shen [7] proved that the graph of the classical Weierstrass function  $\sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x)$  has Hausdorff dimension  $2 + \log \lambda / \log b$ , for every integer  $b \geq 2$  and every  $\lambda \in (1/b, 1)$ , which solved a long-standing conjecture. Some relevant results can be found in [1, 3–5, 8]. Naturally, we want to study the Hausdorff dimension of the graph of Weierstrass functions with the following form:

$$W_{\lambda,b,\theta}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x + \theta), \quad x \in \mathbb{R},$$

where  $b \geq 2$  is an integer,  $\lambda \in (1/b, 1)$  and  $\theta \in \mathbb{R}$ .

Denote  $D_{\lambda,b} = 2 + \log \lambda / \log b$ . Denote by  $\dim_{\text{H}} \Gamma W_{\lambda,b,\theta}$  the Hausdorff dimension of the graph of  $W_{\lambda,b,\theta}$ . Our main result is:

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\*Corresponding author. *Email addresses:* ruanhj@zju.edu.cn (H. J. Ruan), 21535049@zju.edu.cn (N. Zhang)

**Theorem 1.1.** *If  $\theta = -\pi/2$ , then  $\dim_{\mathbb{H}} \Gamma W_{\lambda,b,\theta} = D_{\lambda,b}$  for every integer  $b \geq 3$  and every  $\lambda \in (1/b, 1)$ . If the integer  $b \geq 7$ , then the dimension formula holds for every  $\lambda \in (1/b, 1)$  and every  $\theta \in \mathbb{R}$ .*

The paper is organized as follows. In next section, we present necessary notations and properties introduced by Shen [7] and Tsujii [8]. In Sections 3 and 4, we prove the main result.

## 2 Preliminaries

In this section, we present necessary notations and properties introduced in [7,8]. Denote  $\gamma = 1/(\lambda b)$ ,  $\phi_{\theta}(x) = \cos(2\pi x + \theta)$ , and  $\psi_{\theta}(x) = \phi'_{\theta}(x)$ . Let  $\mathcal{A} = \{0, 1, \dots, b - 1\}$ . Given  $x \in \mathbb{R}$  and  $\mathbf{u} = \{u_n\}_{n=1}^{\infty} \in \mathcal{A}^{\mathbb{Z}^+}$ , we define

$$S_{\theta}(x, \mathbf{u}) = \sum_{n=1}^{\infty} \gamma^{n-1} \psi_{\theta}(x(\mathbf{u}|_n)),$$

where  $\mathbf{u}|_n = (u_1, \dots, u_n)$  and

$$x(\mathbf{u}|_n) = \frac{x}{b^n} + \frac{u_1}{b^n} + \frac{u_2}{b^{n-1}} + \dots + \frac{u_n}{b}.$$

For simplicity, we will use  $S(x, \mathbf{u})$  to denote  $S_{\theta}(x, \mathbf{u})$  if no confusion occurs.

Given  $\varepsilon, \delta > 0$ . Two words  $\mathbf{i}, \mathbf{j} \in \mathcal{A}^{\mathbb{Z}^+}$  are called  $(\varepsilon, \delta)$ -tangent at a point  $x_0 \in \mathbb{R}$  if

$$|S(x_0, \mathbf{i}) - S(x_0, \mathbf{j})| \leq \varepsilon \quad \text{and} \quad |S'(x_0, \mathbf{i}) - S'(x_0, \mathbf{j})| \leq \delta.$$

Let  $E(q, x_0; \varepsilon, \delta)$  denote the set of pairs  $(\mathbf{k}, \mathbf{l}) \in \mathcal{A}^q \times \mathcal{A}^q$  for which there exist  $\mathbf{u}, \mathbf{v} \in \mathcal{A}^{\mathbb{Z}^+}$  such that  $\mathbf{k}\mathbf{u}$  and  $\mathbf{l}\mathbf{v}$  are  $(\varepsilon, \delta)$ -tangent at  $x_0$ . Let

$$e(q, x_0; \varepsilon, \delta) = \max_{\mathbf{k} \in \mathcal{A}^q} \#\{\mathbf{l} \in \mathcal{A}^q : (\mathbf{k}, \mathbf{l}) \in E(q, x_0; \varepsilon, \delta)\},$$

$$E(q, x_0) = \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} E(q, x_0; \varepsilon, \delta),$$

$$e(q, x_0) = \max_{\mathbf{k} \in \mathcal{A}^q} \#\{\mathbf{l} \in \mathcal{A}^q : (\mathbf{k}, \mathbf{l}) \in E(q, x_0)\}.$$

For  $J \subset \mathbb{R}$ , define

$$E(q, J; \varepsilon, \delta) = \bigcup_{x_0 \in J} E(q, x_0; \varepsilon, \delta),$$

$$E(q, J) = \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} E(q, J; \varepsilon, \delta),$$

$$e(q, J) = \max_{\mathbf{k} \in \mathcal{A}^q} \#\{\mathbf{l} \in \mathcal{A}^q : (\mathbf{k}, \mathbf{l}) \in E(q, J)\}.$$

Tsujii’s notation  $e(q)$  is defined as

$$e(q) = \lim_{p \rightarrow \infty} \max_{k=0}^{b^p-1} e\left(q, \left[\frac{k}{b^p}, \frac{k+1}{b^p}\right]\right).$$

It is well-known that  $e(q) = \max_{x \in [0,1]} e(q, x)$ . For details, please see [7].

Now we define another useful function  $\sigma(q)$  introduced by Shen [7]. A measurable function  $\omega : [0, 1) \rightarrow [0, \infty)$  is called a weight function if  $\|\omega\|_\infty < \infty$  and  $\|1/\omega\|_\infty < \infty$ . A testing function of order  $q$  is a measurable function  $V : [0, 1) \times \mathcal{A}^q \times \mathcal{A}^q \rightarrow [0, \infty)$ . A testing function of order  $q$  is called admissible if there exist  $\varepsilon > 0$  and  $\delta > 0$  such that the following hold: For any  $x \in [0, 1)$ , if  $(\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)$ , then

$$V(x, \mathbf{u}, \mathbf{v})V(x, \mathbf{v}, \mathbf{u}) \geq 1.$$

Given a weight function  $\omega$  and an admissible testing function  $V$  of order  $q$ , we define a new measurable function  $\Sigma_{V,\omega}^q : [0, 1) \rightarrow \mathbb{R}$  as follows: for each  $x \in [0, 1)$ , let

$$\Sigma_{V,\omega}^q(x) = \sup \left\{ \frac{\omega(x)}{\omega(x(\mathbf{u}))} \sum_{\mathbf{v} \in \mathcal{A}^q} V(x, \mathbf{u}, \mathbf{v}) : \mathbf{u} \in \mathcal{A}^q \right\}.$$

Define

$$\sigma(q) = \inf \|\Sigma_{V,\omega}^q\|_\infty,$$

where the infimum is taken over all weight functions  $\omega$  and admissible testing functions  $V$  of order  $q$ .

Let  $\mathbb{P}$  be the Bernoulli measure on  $\mathcal{A}^{\mathbb{Z}^+}$  with uniform probabilities  $\{1/b, 1/b, \dots, 1/b\}$ . For each  $x \in \mathbb{R}$ , define a Borel probability measure  $m_x$  on  $\mathbb{R}$  by

$$m_x(A) = \mathbb{P}(\{\mathbf{v} : S(x, \mathbf{v}) \in A\}), \quad A \subset \mathbb{R}.$$

Then  $m_x$ ’s are the conditional measures along vertical fibers of the unique SRB measure  $\nu$  of the skew product map  $T : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ ,

$$T(x, y) = (bx \pmod 1, \gamma y + \psi_\theta(x)).$$

That is, the SRB measure  $\nu$  can be defined by

$$\nu(B) = \int_0^1 m_x(B_x) dx$$

for each Borel set  $B \subset \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ , where  $B_x = \{y \in \mathbb{R} : (x, y) \in B\}$ .

We will use the following two theorems to prove our main result.

**Theorem 2.1** ([7]). *If there exists  $q \in \mathbb{Z}^+$ , such that  $\sigma(q) < (\gamma b)^q$ , then the SRB measure  $\nu$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$  with square integrable density. In particular, for Lebesgue a.e.  $x \in [0, 1)$ ,  $m_x$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$  and with square integrable density. As a result,  $\dim_{\text{H}} \Gamma W_{\lambda,b,\theta} = D_{\lambda,b}$ .*

**Theorem 2.2** ([7]).  $\sigma(q) \leq e(q)$ .

We remark that Theorem 2.1 strengthens a similar result by Tsujii [8], and the dimension formula  $\dim_{\mathbb{H}} \Gamma W_{\lambda,b,\theta} = D_{\lambda,b}$  follows from Ledrappier’s theorem [6]. For details, please see [7].

The following result can be derived from the definitions of  $E(q, x)$  and  $E(q, J; \varepsilon, \delta)$ . The proof for general case is same as the special case  $\theta = 0$ , which is presented in [7]. Thus we omit the details.

**Lemma 2.1** ([7]). *Let  $x_0 \in \mathbb{R}$ , and  $\mathbf{k}, \mathbf{l} \in \mathcal{A}^q$ . Then*

- (1)  $(\mathbf{k}, \mathbf{l}) \in E(q, x_0)$  if and only if there exist  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathcal{A}^{\mathbb{Z}^+}$  such that  $F(x) = S(x, \mathbf{ku}) - S(x, \mathbf{lv})$  has a multiple zero at  $x_0$ , that is,  $F(x_0) = F'(x_0) = 0$ .
- (2) If  $(\mathbf{k}, \mathbf{l}) \notin E(q, x_0)$ , then there is a neighborhood  $U$  of  $x_0$  and  $\varepsilon, \delta > 0$  such that  $(\mathbf{k}, \mathbf{l}) \notin E(q, U; \varepsilon, \delta)$ .
- (3) For any compact  $K \subset \mathbb{R}$ , if  $(\mathbf{k}, \mathbf{l}) \notin E(q, K)$ , then there exist  $\varepsilon, \delta > 0$ , such that  $(\mathbf{k}, \mathbf{l}) \notin E(q, K; \varepsilon, \delta)$ .
- (4) For any  $\varepsilon > \varepsilon' > 0, \delta > \delta' > 0$ , there exists  $\eta > 0$ , such that if  $|x - x_0| < \eta$ ,  $(\mathbf{k}, \mathbf{l}) \notin E(q, x_0; \varepsilon, \delta)$ , then  $(\mathbf{k}, \mathbf{l}) \notin E(q, x; \varepsilon', \delta')$ .

The following three lemmas are very useful in the proof of the results in [7]. They still hold in our case.

**Lemma 2.2** ([7]). *Assume that for all  $x \in [0, 1)$ ,  $E(q, x) \neq \mathcal{A}^q \times \mathcal{A}^q$ . Then*

$$\sigma(q) \leq b^q - 2 + 2/\alpha,$$

where  $\alpha = \alpha(b, q) > 1$  satisfies  $2 - \alpha = (b^q - 2)\alpha(\alpha - 1)$ .

**Lemma 2.3** ([7]). *Let  $q \in \mathbb{Z}^+$ . Suppose that there are constants  $\varepsilon > 0$  and  $\delta > 0$  and  $K \subset [0, 1)$  with the following properties:*

- (1) For  $x \in K$ ,  $e(q, x; \varepsilon, \delta) = 1$  and for  $x \in [0, 1) \setminus K$ ,  $e(q, x; \varepsilon, \delta) \leq 2$ ;
- (2) If  $(\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)$  for some  $x \in [0, 1) \setminus K$  and  $\mathbf{u} \neq \mathbf{v}$ , then both  $x(\mathbf{u})$  and  $x(\mathbf{v})$  belong to  $K$ .

Then  $\sigma(q) \leq \sqrt{2}$ .

**Lemma 2.4** ([7]). *Let  $q \in \mathbb{Z}^+$ . Suppose that there are constants  $\varepsilon > 0$  and  $\delta > 0$  and  $K \subset [0, 1)$  with the following properties:*

- (1) For  $x \in K$ ,  $e(q, x; \varepsilon, \delta) = 1$  and for  $x \in [0, 1) \setminus K$ ,  $e(q, x; \varepsilon, \delta) \leq 2$ ;
- (2) If  $(\mathbf{u}, \mathbf{v}) \in E(q, x; \varepsilon, \delta)$  for some  $x \in [0, 1) \setminus K$  and  $\mathbf{u} \neq \mathbf{v}$ , then either  $x(\mathbf{u}) \in K$  or  $x(\mathbf{v}) \in K$ .

Then  $\sigma(q) \leq (\sqrt{5} + 1)/2$ .

### 3 The case when $b$ is large

If  $e(1) = 1$ , then form  $\gamma b = 1/\lambda > 1$ , we have  $e(1) < \gamma b$ . Thus, we always assume that  $e(1) \geq 2$ . From  $e(1) = \max_{x \in [0,1]} e(1, x)$ , there exists  $x^* \in [0, 1)$ , such that  $e(1, x^*) = e(1)$ . We will fix  $x^*$  in the sequel of the paper.

From definition, there exists  $k \in \mathcal{A}$ , such that  $\#\{\ell \in \mathcal{A} : (k, \ell) \in E(1, x^*)\} = e(1)$ . Let  $\ell^1, \ell^2, \dots, \ell^{e(1)}$  be all elements in  $\mathcal{A}$  such that  $(k, \ell^i) \in E(1, x^*)$ , and

$$\sin(2\pi x^1 + \theta) \leq \sin(2\pi x^2 + \theta) \leq \dots \leq \sin(2\pi x^{e(1)} + \theta),$$

where  $x^i = (x^* + \ell^i)/b, i = 1, \dots, e(1)$ .

Similarly as Lemma 3.2 and Lemma 3.3 in [7], we have the following two lemmas. Since the proof are same as that of in [7], we omit the details again.

**Lemma 3.1.** *If  $(k, \ell) \in E(1, x^*)$ , then*

$$\left| \sin\left(\frac{2\pi(x^* + k)}{b} + \theta\right) - \sin\left(\frac{2\pi(x^* + \ell)}{b} + \theta\right) \right| \leq \frac{2\gamma}{1 - \gamma}, \tag{3.1a}$$

$$\left| \cos\left(\frac{2\pi(x^* + k)}{b} + \theta\right) - \cos\left(\frac{2\pi(x^* + \ell)}{b} + \theta\right) \right| \leq \frac{2\gamma}{b - \gamma}, \tag{3.1b}$$

$$4 \sin^2 \frac{\pi(k - \ell)}{b} \leq \left(\frac{2\gamma}{1 - \gamma}\right)^2 + \left(\frac{2\gamma}{b - \gamma}\right)^2. \tag{3.1c}$$

**Lemma 3.2.** *Under the above circumstances, and with the assumption that  $1 \leq i < j \leq e(1)$ , the followings hold:*

1. *If  $\ell^i = k$  or  $\ell^j = k$ , then  $\sin(2\pi x^j + \theta) - \sin(2\pi x^i + \theta) \geq \frac{2\beta_0(b, \gamma)}{b}$ ,*
2.  *$\sin(2\pi x^j + \theta) - \sin(2\pi x^i + \theta) \geq \frac{2\beta_1(b, \gamma)}{b}$ ,*
3. *If  $\ell^i - \ell^j \not\equiv \pm 1 \pmod b$ , then  $\sin(2\pi x^j + \theta) - \sin(2\pi x^i + \theta) \geq \frac{2\beta_2(b, \gamma)}{b}$ ,*

where

$$\beta_0(b, \gamma) = \sqrt{\max\left\{0, \left(b \sin \frac{\pi}{b}\right)^2 - \frac{\gamma^2 b^2}{(b - \gamma)^2}\right\}},$$

$$\beta_1(b, \gamma) = \sqrt{\max\left\{0, \left(b \sin \frac{\pi}{b}\right)^2 - \frac{4\gamma^2 b^2}{(b - \gamma)^2}\right\}},$$

$$\beta_2(b, \gamma) = \sqrt{\max\left\{0, \left(b \sin \frac{2\pi}{b}\right)^2 - \frac{4\gamma^2 b^2}{(b - \gamma)^2}\right\}}.$$

Using these two lemmas and lemmas in Section 2, we can prove the following theorem, which implies that Theorem 1.1 holds if  $b \geq 7$ .

**Theorem 3.1.** 1. If  $b \geq 7$ , then  $e(1) < \gamma b$ .

2. If  $b = 4, 5, 6$ , then either  $e(1) = 2$  or  $e(1) < \gamma b$ .

3. If  $b = 3$ , then either  $e(1) \leq 2$  or  $\sigma(1) < \gamma b$ .

*Proof.* Using the exactly same method as in [7], we can obtain the following result: if  $b \geq 4$ , then either  $e(1) = 2$  or  $e(1) < \gamma b$ ; if  $b = 3$ , then either  $e(1) \leq 2$  or  $\sigma(1) < \gamma b$ . Thus, we only need to prove the theorem holds if  $b \geq 7$  and  $e(1) = 2$ . If  $\gamma b > 2$ , then  $\gamma b > e(1)$ . Thus it suffices to show it is impossible that  $e(1) = 2$  and  $\gamma b \leq 2$ .

We will prove this by contradiction. Assume that  $e(1) = 2$  and  $\gamma b \leq 2$ , then  $(\ell^1, \ell^2) \in E(1, x^*)$ . From Lemma 3.1 and  $\gamma \leq 2/b$ , we have

$$\begin{aligned} 4 \sin^2 \frac{\pi(\ell^2 - \ell^1)}{b} &\leq \left(\frac{2\gamma}{1-\gamma}\right)^2 + \left(\frac{2\gamma}{b-\gamma}\right)^2 \\ &\leq \left(\frac{2 \cdot (2/b)}{1-2/b}\right)^2 + \left(\frac{2 \cdot (2/b)}{b-2/b}\right)^2 = \frac{16}{(b-2)^2} + \frac{16}{(b^2-2)^2}. \end{aligned}$$

Thus

$$\sin^2 \frac{\pi}{b} \leq \frac{4}{(b-2)^2} + \frac{4}{(b^2-2)^2}. \tag{3.2}$$

Consider the function  $g(t) = g_1(t) - g_2(t)$ , where

$$g_1(t) = t^2 \sin^2(\pi/t) \quad \text{and} \quad g_2(t) = \frac{4t^2}{(t-2)^2} + \frac{4t^2}{(t^2-2)^2}.$$

It is easy to check that  $g_1$  is increasing on  $(2, +\infty)$  while  $g_2$  is decreasing on  $(2, +\infty)$ . Thus, if  $b \geq 7$ , we have  $g(b) \geq g(7) > 9 - 8 > 0$ , which implies that (3.2) does not hold for  $b \geq 7$ . □

## 4 Proof of Theorem 1.1: the case $b = 3, 4, 5, 6$

In this section, we will restrict  $\theta = -\pi/2$ . We will show the following result under this restriction: for  $b = 3, 4, 5, 6$ , if  $e(1) = 2$  then  $\sigma(1) < \gamma b$ . Combining this result with Theorem 3.1, we have either  $e(1) < \gamma b$  or  $\sigma(1) < \gamma b$  for  $b = 3, 4, 5, 6$ . Thus Theorem 1.1 holds for this case.

Using the same method as in the proof of Lemma 4.1 in [7], we have the following lemma. We omit the details.

**Lemma 4.1.** Assume that  $0 \leq k < \ell < b$  satisfying  $(k, \ell) \in E(1, x^*)$ . Then for any  $\kappa \in (0, 1)$ , one of the followings holds: either

$$\left| \sin\left(\frac{2\pi(x^* + k)}{b}\right) - \sin\left(\frac{2\pi(x^* + \ell)}{b}\right) \right| \leq \frac{2\gamma\sqrt{1-\kappa^2}}{b} + \frac{2\gamma^2}{b(b-\gamma)}, \tag{4.1}$$

or

$$\left| \cos\left(\frac{2\pi(x^* + k)}{b}\right) - \cos\left(\frac{2\pi(x^* + \ell)}{b}\right) \right| \leq 2\kappa\gamma + \frac{2\gamma^2}{1 - \gamma}. \tag{4.2}$$

Notice that  $\theta = -\pi/2$ . For  $x \in \mathbb{R}$  and  $\mathbf{i} = \{i_n\}_{n=1}^\infty \in \mathcal{A}^{\mathbb{Z}^+}$ , we have

$$S(x, \mathbf{i}) = \sum_{n=1}^\infty \gamma^{n-1} \psi(x_n) = 2\pi \sum_{n=1}^\infty \gamma^{n-1} \cos\left(2\pi\left(\frac{x}{b^n} + \frac{i_1}{b^n} + \dots + \frac{i_n}{b^n}\right)\right). \tag{4.3}$$

**Lemma 4.2.** *If  $x \in \mathbb{R}$  and  $\mathbf{i} = \{i_n\}_{n=1}^\infty$ . Then the following equalities hold:*

$$S(x, \mathbf{i}) = S(1 - x, \mathbf{i}'), \tag{4.4a}$$

$$S'(x, \mathbf{i}) = -S'(1 - x, \mathbf{i}'), \tag{4.4b}$$

where  $\mathbf{i}' = \{i'_n\}_{n=1}^\infty, i'_n = b - 1 - i_n$ .

*Proof.* Notice that

$$\begin{aligned} & \cos\left(2\pi\left(\frac{1-x}{b^n} + \frac{b-1-i_1}{b^n} + \dots + \frac{b-1-i_n}{b^n}\right)\right) \\ &= \cos\left(2\pi\left(-\left(\frac{x}{b^n} + \frac{i_1}{b^n} + \dots + \frac{i_n}{b^n}\right) + \left(\frac{1}{b^n} + \frac{b-1}{b^n} + \dots + \frac{b-1}{b^n}\right)\right)\right) \\ &= \cos\left(2\pi\left(-\left(\frac{x}{b^n} + \frac{i_1}{b^n} + \dots + \frac{i_n}{b^n}\right) + 1\right)\right) \\ &= \cos\left(2\pi\left(\frac{x}{b^n} + \frac{i_1}{b^n} + \dots + \frac{i_n}{b^n}\right)\right). \end{aligned}$$

Thus (4.4a) holds. From

$$S'(x, \mathbf{i}) = \frac{-4\pi^2}{b} \sum_{n=1}^\infty \left(\frac{\gamma}{b}\right)^{n-1} \sin\left(2\pi\left(\frac{x}{b^n} + \frac{i_1}{b^n} + \dots + \frac{i_n}{b^n}\right)\right),$$

we can see that (4.4b) holds. □

From Lemma 4.2, we know that  $(k, \ell) \in E(1, x^*)$  is equivalent to  $(b - 1 - k, b - 1 - \ell) \in E(1, 1 - x^*)$ . Thus  $e(1, x^*) = e(1, 1 - x^*)$ . Hence, we may assume that  $x^* \in [0, \frac{1}{2}]$ .

### 4.1 The case $b = 6$

**Proposition 4.1.** *Assume  $b = 6$  and  $e(1) = 2$ . Then  $\sigma(1) < 6\gamma$ .*

*Proof.* It is clear that  $6\gamma > 6 \cdot (1/3) = e(1) \geq \sigma(1)$  if  $\gamma > \frac{1}{3}$ . Thus we may assume that  $\gamma \leq \frac{1}{3}$ . From  $e(1) = 2$ , there exist  $0 \leq k < \ell < 6$ , such that  $(k, \ell) \in E(1, x^*)$ .

From  $(k, \ell) \in E(1, x^*)$  and Lemma 3.1, we have

$$\left| \sin\left(\frac{2\pi(x^* + k)}{6}\right) - \sin\left(\frac{2\pi(x^* + \ell)}{6}\right) \right| \leq \frac{2\gamma}{6 - \gamma} \leq \frac{2 \cdot (1/3)}{6 - (1/3)} = \frac{2}{17}, \tag{4.5a}$$

$$4 \sin^2 \frac{\pi(\ell - k)}{6} \leq \left(\frac{2\gamma}{1 - \gamma}\right)^2 + \left(\frac{2\gamma}{6 - \gamma}\right)^2 \leq 1^2 + (2/17)^2 < 2. \tag{4.5b}$$

If  $\ell - k \not\equiv \pm 1 \pmod 6$ , then  $4 \sin^2 \frac{\pi(\ell - k)}{6} \geq 4 \sin^2 \frac{2\pi}{6} = 3$ , which contradicts with (4.5b). Thus  $\ell - k \equiv \pm 1 \pmod 6$ . Combining this with  $k < \ell$ , we can see that  $\ell - k = 1$  or  $5$ .

Let  $\kappa = 0.98$ . We will show that the inequality (4.2) does not hold. In fact, if (4.2) holds, then

$$\left| \cos\left(\frac{2\pi(x^* + k)}{6}\right) - \cos\left(\frac{2\pi(x^* + \ell)}{6}\right) \right| \leq 2 \cdot 0.98 \cdot (1/3) + \frac{2 \cdot (1/3)^2}{1 - 1/3} < 0.987.$$

Combining this with (4.5a), we have

$$1 = 4 \sin^2 \frac{\pi}{6} = 4 \sin^2 \left(\frac{\pi(\ell - k)}{6}\right) < (2/17)^2 + 0.987^2 < 0.989.$$

Contradiction! Thus the inequality (4.1) holds. Let  $y^* = \pi(2x^* + k + \ell)/6$ . Then  $y^* \in [0, 5\pi/3]$  and

$$\begin{aligned} |\cos(y^*)| &= \left| 2 \sin \frac{\pi}{6} \cos(y^*) \right| = \left| \sin\left(\frac{2\pi(x^* + k)}{6}\right) - \sin\left(\frac{2\pi(x^* + \ell)}{6}\right) \right| \\ &\leq \frac{2 \cdot (1/3) \cdot \sqrt{1 - 0.98^2}}{6} + \frac{2 \cdot (1/3)^2}{6 \cdot (6 - 1/3)} < 0.029 < \cos(49\pi/100). \end{aligned}$$

Thus  $y^* \in (49\pi/100, 51\pi/100) \cup (149\pi/100, 151\pi/100)$ .

**Case 1.**  $y^* \in (49\pi/100, 51\pi/100)$ . In this case,  $2x^* + k + \ell \in (294/100, 306/100)$ . If  $k + 1 = \ell$ , then  $x^* + k \in (97/100, 103/100)$ . Since  $x^* \in [0, \frac{1}{2}]$ , we have  $(k, \ell) = (1, 2)$  and  $x^* \in [0, 3/100)$ . If  $k + 5 = \ell$ , from  $x^* \geq 0$  and  $k \geq 0$ , we have  $2x^* + 2k + 5 \geq 5 > 306/100$ , a contradiction!

**Case 2.**  $y^* \in (149\pi/100, 151\pi/100)$ . In this case,  $2x^* + k + \ell \in (894/100, 906/100)$ . If  $k + 1 = \ell$ , then  $x^* + k \in (397/100, 403/100)$ . Since  $x^* \in [0, \frac{1}{2}]$ , we have  $(k, \ell) = (4, 5)$  and  $x^* \in [0, 3/100)$ . If  $k + 5 = \ell$ , we must have  $k = 0$  and  $\ell = 5$ . Thus, from  $x^* \in [0, \frac{1}{2}]$ , we can obtain  $2x^* + 2k + 5 \leq 6 < 894/100$ , a contradiction!

From Case 1 and Case 2, we can see that in the case that  $\gamma \leq \frac{1}{3}$ , if  $0 \leq k < \ell < 6$  satisfying  $(k, \ell) \in E(1, x^*)$ , then  $0 \leq x^* < 3/100$ , and  $(k, \ell) = (1, 2)$  or  $(k, \ell) = (4, 5)$ .

From above arguments,  $e(1, x) = 1$  if  $x \in [3/100, 1/2]$ . Using the fact that  $e(1, x) = e(1, 1 - x)$ , we also have  $e(1, x) = 1$  if  $x \in [1/2, 97/100]$ .



Let  $K = [3/100, 97/100]$ . Then  $e(1, K) = 1$  and  $e(1, [0, 1]) \leq 2$ . From Lemma 2.1(3), there exist  $\varepsilon > 0, \delta > 0$ , such that  $e(1, x; \varepsilon, \delta) = 1$  if  $x \in K$ , and  $e(1, x; \varepsilon, \delta) \leq 2$  if  $x \in [0, 1] \setminus K$ .

In the case that  $x \in [0, 1/2] \setminus K$ , if  $(k, \ell) = (1, 2)$ , then  $x(2) = (x + 2)/6 \subseteq K$ ; if  $(k, \ell) = (4, 5)$ , then  $x(4) = (x + 4)/6 \subseteq K$ . Using the symmetry (Lemma 4.2), we know that the conditions of Lemma 2.4 hold for  $q = 1$ . Thus  $\sigma(1) \leq (\sqrt{5} + 1)/2$ .

If  $\gamma > (\sqrt{5} + 1)/12$ , then  $6\gamma > \sigma(1)$ . Thus, it suffices to show it is impossible that  $\gamma \leq (\sqrt{5} + 1)/12$  and  $e(1) = 2$ . In fact, if this holds, then from Lemma 3.1,

$$4 \sin^2 \left( \frac{\pi(\ell - k)}{6} \right) \leq \left( \frac{2\gamma}{6 - \gamma} \right)^2 + \left( \frac{2\gamma}{1 - \gamma} \right)^2 \leq \left( \frac{(\sqrt{5} + 1)/6}{6 - (\sqrt{5} + 1)/12} \right)^2 + \left( \frac{(\sqrt{5} + 1)/6}{1 - (\sqrt{5} + 1)/12} \right)^2 \leq 0.56 < 4 \sin^2 \left( \frac{\pi}{6} \right),$$

which contradicts with  $k \neq \ell$ . □

### 4.2 The case $b = 5$

**Proposition 4.2.** Assume  $b = 5$  and  $e(1) = 2$ . Then  $\sigma(1) < 5\gamma$ .

*Proof.* If  $\gamma > 2/5$ , then  $5\gamma > 2 = e(1) \geq \sigma(1)$ . Thus we may assume that  $\gamma \leq \frac{2}{5}$ . From  $e(1) = 2$ , there exist  $0 \leq k < \ell < 5$  such that  $(k, \ell) \in E(1, x^*)$ . Now we will show that  $x^* \in (3/20, 7/20)$  and  $(k, \ell) = (3, 4)$ .

In fact, from  $(k, \ell) \in E(1, x^*)$  and Lemma 3.1,

$$\left| \sin \left( 2\pi \frac{x^* + k}{b} \right) - \sin \left( 2\pi \frac{x^* + \ell}{b} \right) \right| \leq \frac{2\gamma}{5 - \gamma} \leq \frac{2 \times \frac{2}{5}}{5 - \frac{2}{5}} = \frac{4}{23},$$

$$4 \sin^2 \left( \frac{\pi(\ell - k)}{5} \right) \leq \left( \frac{2\gamma}{1 - \gamma} \right)^2 + \left( \frac{2\gamma}{5 - \gamma} \right)^2 \leq \left( \frac{2 \times \frac{2}{5}}{1 - \frac{2}{5}} \right)^2 + \left( \frac{2 \times \frac{2}{5}}{5 - \frac{2}{5}} \right)^2 < 2.$$

Assume that  $\ell - k \not\equiv \pm 1 \pmod{5}$ . Then  $\ell - k \in \{2, 3\}$ . Thus  $4 \sin^2 (\pi(k - \ell)/5) \geq 4 \sin^2 (2\pi/5) > 3.618$ , a contradiction. Thus  $\ell - k \equiv \pm 1 \pmod{5}$ . Since  $\ell > k$ , we have  $\ell - k = 1$  or  $\ell - k = 4$ .

Let  $\kappa = \sqrt{2}/2$ . We will show that inequality (4.2) in Lemma 4.1 does not hold. In fact, if (4.2) holds, then

$$1.38 < 4 \sin^2 \left( \frac{\pi}{5} \right) = 4 \sin^2 \left( \frac{\pi(k - \ell)}{5} \right) = \left| \sin \left( 2\pi \frac{x^* + k}{5} \right) - \sin \left( 2\pi \frac{x^* + \ell}{5} \right) \right|^2 + \left| \cos \left( 2\pi \frac{x^* + k}{5} \right) - \cos \left( 2\pi \frac{x^* + \ell}{5} \right) \right|^2 \leq \left( \frac{4}{23} \right)^2 + \left( 2 \times \frac{\sqrt{2}}{2} \times \frac{2}{5} + \frac{2 \times (2/5)^2}{1 - \frac{2}{5}} \right)^2 < \left( \frac{4}{23} \right)^2 + (1.1)^2 < 1.3,$$

a contradiction. Thus the inequality (4.1) in Lemma 4.1 holds. Let  $y^* = \pi(2x^* + k + \ell)/5$ . We have

$$\begin{aligned} \left|2 \cos(y^*) \sin\left(\frac{\pi}{5}\right)\right| &= \left|2 \cos(y^*) \sin\left(\frac{\pi(\ell - k)}{5}\right)\right| \\ &= \left|\sin\left(2\pi\frac{x^* + k}{5}\right) - \sin\left(2\pi\frac{x^* + \ell}{5}\right)\right| \\ &\leq \frac{2 \times \frac{2}{5} \times \sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2}}{5} + \frac{2 \times \left(\frac{2}{5}\right)^2}{5 \times \left(5 - \frac{2}{5}\right)} < 0.128. \end{aligned}$$

Thus

$$|\cos(y^*)| \leq \frac{0.128}{2 \sin(\pi/5)} < 0.11 < \cos\left(\frac{23\pi}{50}\right).$$

Since  $y^* \in [0, 8\pi/5]$ , we have  $y^* \in (23\pi/50, 27\pi/50) \cup (73\pi/50, 77\pi/50)$ .

**Case 1.**  $y^* \in (23\pi/50, 27\pi/50)$ . In this case,  $2x^* + k + \ell \in (23/10, 27/10)$ . If  $\ell - k = 1$ , then  $x^* + k \in (13/20, 17/20)$ , which contradicts the fact that  $x^* \in [0, 1/2)$  and  $k$  is a nonnegative integer. If  $\ell - k = 4$ , then  $2x^* + 2k + 4 \geq 4 > 27/10$ , which also contradicts the fact that  $x^* \in [0, 1/2)$  and  $k$  is a nonnegative integer.

**Case 2.**  $y^* \in (73\pi/50, 77\pi/50)$ . In this case,  $2x^* + k + \ell \in (73/10, 77/10)$ . If  $\ell - k = 1$ , then  $x^* + k \in (63/20, 67/20)$ . Thus  $(k, \ell) = (3, 4)$  and  $x^* \in (3/20, 7/20)$ . If  $\ell - k = 4$ , then  $x^* + k \in (33/20, 37/20)$ , which also contradicts the fact that  $x^* \in [0, 1/2)$  and  $k$  is a nonnegative integer.

Thus, in the case that  $\gamma \in (0, 2/5]$ , if  $0 \leq k < \ell < 5$  satisfying  $(k, \ell) \in E(1, x^*)$ , then  $x^* \in (3/20, 7/20)$  and  $(k, \ell) = (3, 4)$ .

From above arguments,  $e(1, x) = 1$  if  $x \in [0, 3/20] \cup [7/20, 1/2]$ . Using the fact that  $e(1, x) = e(1, 1 - x)$ , we have  $e(1, x) = 1$  if  $x \in [1/2, 13/20] \cup [17/20, 1]$ .

Let  $K = [0, 3/20] \cup [7/20, 13/20] \cup [17/20, 1]$ . Then  $e(1, K) = 1$  and  $e(1, [0, 1]) \leq 2$ . From Lemma 2.1, there exist  $\varepsilon, \delta > 0$ , such that  $e(1, x; \varepsilon, \delta) = 1$  if  $x \in K$ , and  $e(1, x; \varepsilon, \delta) \leq 2$  if  $x \in [0, 1] \setminus K$ .

If  $x \in (3/20, 1/4)$ , we have  $x(3) = (x + 3)/5 \in (7/20, 13/20) \subseteq K$ . If  $x \in [1/4, 7/20)$ , we have  $x(4) = (x + 4)/5 \in [17/20, 1) \subseteq K$ . From Lemma 2.4, we have  $\sigma(1) \leq (\sqrt{5} + 1)/2$ .

If  $\gamma > (\sqrt{5} + 1)/10$ , then  $5\gamma > \sigma(1)$ . Thus, it suffices to show it is impossible that  $\gamma \leq (\sqrt{5} + 1)/10$  and  $e(1) = 2$ . In fact, if this holds, then from Lemma 3.1,

$$\begin{aligned} 4 \sin^2\left(\frac{\pi(\ell - k)}{5}\right) &\leq \left(\frac{2\gamma}{5 - \gamma}\right)^2 + \left(\frac{2\gamma}{1 - \gamma}\right)^2 \\ &\leq \left(\frac{(\sqrt{5} + 1)/5}{5 - (\sqrt{5} + 1)/10}\right)^2 + \left(\frac{(\sqrt{5} + 1)/5}{1 - (\sqrt{5} + 1)/10}\right)^2 < 0.9348 < 4 \sin^2\left(\frac{\pi}{5}\right), \end{aligned}$$

which contradicts with  $k \neq \ell$ . □

### 4.3 The case $b = 4$

**Proposition 4.3.** Assume that  $b = 4$  and  $e(1) = 2$ . Then  $\sigma(1) < 4\gamma$ .

*Proof.* If  $\gamma > \frac{1}{2}$ , then  $4\gamma > 2 = e(1) \geq \sigma(1)$ . Thus we may assume that  $\gamma \leq \frac{1}{2}$ . From  $e(1) = 2$ , there exist  $0 \leq k < \ell < 4$  such that  $(k, \ell) \in E(1, x^*)$ . Now we will show that  $x^* \in (9/25, 1/2]$  and  $(k, \ell) = (2, 3)$ .

In fact, from  $(k, \ell) \in E(1, x^*)$  and Lemma 3.1,

$$\left| \sin\left(2\pi\frac{x^*+k}{4}\right) - \sin\left(2\pi\frac{x^*+\ell}{4}\right) \right| \leq \frac{2\gamma}{4-\gamma} \leq \frac{2 \times \frac{1}{2}}{4-\frac{1}{2}} = \frac{2}{7}. \tag{4.6}$$

Let  $\kappa = \frac{1}{3}$ . We will show that the inequality (4.2) in Lemma 4.1 does not hold. In fact, if (4.2) holds, then

$$\begin{aligned} 2 &= 4 \sin^2\left(\frac{\pi}{4}\right) \leq 4 \sin^2\left(\frac{\pi(k-\ell)}{4}\right) \\ &= \left| \sin\left(2\pi\frac{x^*+k}{4}\right) - \sin\left(2\pi\frac{x^*+\ell}{4}\right) \right|^2 + \left| \cos\left(2\pi\frac{x^*+k}{4}\right) - \cos\left(2\pi\frac{x^*+\ell}{4}\right) \right|^2 \\ &\leq \left(\frac{2}{7}\right)^2 + \left(2 \times \frac{1}{3} \times \frac{1}{2} + \frac{2 \times (1/2)^2}{1-\frac{1}{2}}\right)^2 < \left(\frac{2}{7}\right)^2 + \left(\frac{4}{3}\right)^2 < 2. \end{aligned}$$

A contradiction. Thus (4.1) in Lemma 4.1 holds. Let  $y^* = \pi(2x^* + k + \ell)/4$ . We have

$$\begin{aligned} \left| 2 \cos(y^*) \sin\left(\frac{\pi}{4}\right) \right| &\leq \left| 2 \cos(y^*) \sin\left(\frac{\pi(\ell-k)}{4}\right) \right| \\ &= \left| \sin\left(2\pi\frac{x^*+k}{4}\right) - \sin\left(2\pi\frac{x^*+\ell}{4}\right) \right| \\ &\leq \frac{2 \times \frac{1}{2} \times \sqrt{1 - \left(\frac{1}{3}\right)^2}}{4} + \frac{2 \times \left(\frac{1}{2}\right)^2}{4 \times \left(4 - \frac{1}{2}\right)} = \frac{\sqrt{2}}{6} + \frac{1}{28} < 0.28. \end{aligned}$$

Thus  $|\cos(y^*)| \leq 0.28/\sqrt{2} < 0.2 < \cos(43\pi/100)$ . Since  $y^* \in [0, 3\pi/2]$ , we have  $y^* \in (43\pi/100, 57\pi/100) \cup (143\pi/100, 3\pi/2)$ .

If  $y^* \in (43\pi/100, 57\pi/100)$ , then  $2x^* + k + \ell \in (43/25, 57/25)$ . In this case, we have  $k + \ell = 2$  so that  $(k, \ell) = (0, 2)$  and  $x^* \in [0, 7/50]$ . If  $y^* \in (143\pi/100, 3\pi/2]$ , then  $2x^* + k + \ell \in (143/25, 6]$ . In this case, we have  $k + \ell = 5$  so that  $(k, \ell) = (2, 3)$  and  $x^* \in (9/25, 1/2]$ .

In the case that  $\gamma \leq \frac{1}{2}$ , we have  $(0, 2) \notin E(1, x)$  for all  $x \in [0, \frac{1}{2}]$ . In fact, assume that  $(0, 2) \in E(1, x)$ . Then there exist  $\mathbf{k} = \{k_n\}_{n=1}^\infty$  and  $\mathbf{l} = \{\ell_n\}_{n=1}^\infty$ , such that  $S(x, \mathbf{k}) - S(x, \mathbf{l}) = 0$ , where  $k_1 = 0, \ell_1 = 2$ .

Let  $x_n = (x + 4k_2 + \dots + 4^{n-1}k_n)/4^n$ ,  $y_n = (x + 2 + 4\ell_2 + \dots + 4^{n-1}\ell_n)/4^n$ . Then

$$\begin{aligned} & \left| \cos(2\pi x_1) - \cos(2\pi y_1) + \gamma(\cos(2\pi x_2) - \cos(2\pi y_2)) \right| \\ &= \left| - \sum_{n=3}^{\infty} \gamma^{n-1} (\cos(2\pi x_n) - \cos(2\pi y_n)) \right| \\ &\leq 2 \sum_{n=2}^{\infty} \gamma^n \leq \frac{2\gamma^2}{1-\gamma} \leq \frac{2 \times (\frac{1}{2})^2}{1-\frac{1}{2}} = 1. \end{aligned}$$

Notice that

$$\begin{aligned} \cos(2\pi x_2) - \cos(2\pi y_2) &= \cos\left(\frac{\pi x}{8} + \frac{k_2\pi}{2}\right) - \cos\left(\frac{\pi(x+2)}{8} + \frac{\ell_2\pi}{2}\right) \\ &= -2 \sin\left(\frac{\pi(x+1)}{8} + \frac{(k_2 + \ell_2)\pi}{4}\right) \sin\left(-\frac{\pi}{8} + \frac{(k_2 - \ell_2)\pi}{4}\right) \geq -2 \sin\frac{3\pi}{8}. \end{aligned}$$

Thus

$$\cos(2\pi x_1) - \cos(2\pi y_1) \leq 1 - \gamma(\cos(2\pi x_2) - \cos(2\pi y_2)) \leq 1 + \sin\frac{3\pi}{8} < 1.93.$$

From the inequality (4.6), we have

$$\begin{aligned} 4 &= 4 \sin^2\left(\frac{2\pi}{4}\right) = 4 \sin^2\left(\frac{\pi(\ell - k)}{4}\right) \\ &= \left| \sin(2\pi x_1) - \sin(2\pi y_1) \right|^2 + \left| \cos(2\pi x_1) - \cos(2\pi y_1) \right|^2 \\ &< \left(\frac{2}{7}\right)^2 + 1.93^2 < 3.81. \end{aligned}$$

A contradiction. Thus, in the case that  $\gamma \leq 1/2$ , if  $0 \leq k < \ell < 4$  satisfying  $(k, \ell) \in E(1, x^*)$ , then  $x^* \in (9/25, 1/2]$  and  $(k, \ell) = (2, 3)$ .

From the above arguments,  $e(1, x) = 1$  if  $x \in [0, 9/25]$ . Using the fact that  $e(1, x) = e(1, 1 - x)$ , we have  $e(1, x) = 1$  if  $x \in [16/25, 1]$ .

Let  $K = [0, 9/25] \cup [16/25, 1]$ . Then  $e(1, K) = 1$  and  $e(1, [0, 1]) \leq 2$ . From Lemma 2.1, there exist  $\varepsilon, \delta > 0$ , such that  $e(1, x; \varepsilon, \delta) = 1$  if  $x \in K$ , and  $e(1, x; \varepsilon, \delta) \leq 2$  if  $x \in [0, 1] \setminus K$ .

In the case that  $x \in (9/25, 1/2]$ , we have  $x(3) = (x + 3)/4 \in [16/25, 1] \subseteq K$ . From Lemma 2.4, we have  $\sigma(1) \leq (\sqrt{5} + 1)/2$ .

If  $\gamma > (\sqrt{5} + 1)/8$ , then  $4\gamma > \sigma(1)$ . Thus, it suffices to show it is impossible that  $\gamma \leq (\sqrt{5} + 1)/8$  and  $e(1) = 2$ . In fact, if this holds, then from Lemma 3.1,

$$\begin{aligned} 4 \sin^2\left(\frac{\pi(\ell - k)}{4}\right) &\leq \left(\frac{2\gamma}{4-\gamma}\right)^2 + \left(\frac{2\gamma}{1-\gamma}\right)^2 \\ &\leq \left(\frac{(\sqrt{5} + 1)/4}{4 - (\sqrt{5} + 1)/8}\right)^2 + \left(\frac{(\sqrt{5} + 1)/4}{1 - (\sqrt{5} + 1)/8}\right)^2 < 1.897 < 4 \sin^2\left(\frac{\pi}{4}\right), \end{aligned}$$

which contradicts with  $k \neq \ell$ . □

### 4.4 The case $b = 3$

**Proposition 4.4.** *Assume  $b = 3$  and  $e(1) = 2$ . Then  $\sigma(1) < 3\gamma$ .*

*Proof.* If  $\gamma > 2/3$ , then  $3\gamma > 2 = e(1) \geq \sigma(1)$ . Thus we may assume that  $\gamma \leq 2/3$ . From  $e(1) = 2$ , there exist  $0 \leq k < \ell < 3$  such that  $(k, \ell) \in E(1, x^*)$ .

Let  $y^* = \pi(2x^* + k + \ell)/3$ . From  $(k, \ell) \in E(1, x^*)$  and Lemma 3.1,

$$\begin{aligned} & 2 \left| \cos(y^*) \right| \left| \sin \left( \frac{\pi(\ell - k)}{3} \right) \right| \\ &= \left| \sin \left( 2\pi \frac{x^* + k}{3} \right) - \sin \left( 2\pi \frac{x^* + \ell}{3} \right) \right| \\ &\leq \frac{2\gamma}{3 - \gamma} \leq \frac{4/3}{3 - 2/3} = \frac{4}{7}. \end{aligned}$$

Thus  $|\cos(y^*)| \leq 4/(7\sqrt{3}) < \cos(39\pi/100)$ . Hence  $y^* \in (39\pi/100, 61\pi/100)$ . Since  $y^* \in [0, 4\pi/3]$ , we have  $2x^* + k + \ell \in (117/100, 183/100)$ . Thus  $(k, \ell) = (0, 1)$  and  $x^* \in (17/200, 83/200)$ .

From above arguments,  $e(1, x) = 1$  if  $x \in [0, 17/200] \cup [83/200, 1/2]$ . Using the fact that  $e(1, x) = e(1, 1 - x)$ ,  $e(1, x) = 1$  if  $x \in [1/2, 117/200] \cup [183/200, 1]$ .

Let  $K_1 = [0, 17/200] \cup [83/200, 117/200] \cup [183/200, 1]$ . From Lemma 2.1, there exist  $\varepsilon > 0, \delta > 0$ , such that  $e(1, x; \varepsilon, \delta) = 1$  if  $x \in K_1$ , and  $e(1, x; \varepsilon, \delta) \leq 2$  if  $x \in [0, 1] \setminus K_1$ .

If  $x \in (17/200, 51/200)$ , we have  $x(0) = x/3 \in (17/600, 17/200) \subseteq K_1$ .

If  $x \in [51/200, 83/200)$ , we have  $x(1) = (x + 1)/3 \in (251/600, 283/600) \subseteq K_1$ .

From Lemma 2.4, we have  $\sigma(1) \leq (\sqrt{5} + 1)/2$ . Thus, if  $\gamma > (\sqrt{5} + 1)/6$ , then  $3\gamma > \sigma(1)$ .

Now we will show: if  $\gamma \leq (\sqrt{5} + 1)/6$  and  $(k, \ell) = (0, 1)$ , then  $x^* \in (23/200, 77/200)$ . In fact, from  $(k, \ell) \in E(1, x^*)$  and Lemma 3.1,

$$\begin{aligned} & 2 \left| \sin \left( \frac{\pi(k - \ell)}{3} \right) \cos(y^*) \right| \\ &= \left| \sin \left( 2\pi \frac{x^* + k}{3} \right) - \sin \left( 2\pi \frac{x^* + \ell}{3} \right) \right| \\ &\leq \frac{2\gamma}{3 - \gamma} \leq \frac{(\sqrt{5} + 1)/3}{3 - (\sqrt{5} + 1)/6} \leq 0.4384. \end{aligned}$$

Thus  $|\cos(y^*)| \leq 0.4384/\sqrt{3} < \cos(0.41\pi)$ . Hence  $y^* \in (0.41\pi, 0.59\pi)$ . By the definition of  $y^*$ , we have  $2x^* + k + \ell \in (1.23, 1.77)$ . Combining this with  $(k, \ell) = (0, 1)$ , we have  $x^* \in (23/200, 77/200)$ .

Now we will show: if  $\gamma \leq (\sqrt{5} + 1)/6$  and  $x \in (23/200, 1/8]$ , then  $(0, 1) \notin E(1, x)$ .

In fact, assume that  $(0, 1) \in E(1, x)$ . Then from Lemma 3.1,

$$\begin{aligned} 3 &= 4 \sin^2\left(\frac{\pi}{3}\right) \\ &= \left| \cos\left(2\pi \frac{x+0}{3}\right) - \cos\left(2\pi \frac{x+1}{3}\right) \right|^2 + \left| \sin\left(2\pi \frac{x+0}{3}\right) - \sin\left(2\pi \frac{x+1}{3}\right) \right|^2 \\ &\leq \left(\sqrt{3} \sin\left(\pi \frac{2x+1}{3}\right)\right)^2 + \left(\frac{2\gamma}{3-\gamma}\right)^2 \\ &\leq \left(\sqrt{3} \sin\frac{5\pi}{12}\right)^2 + \left(\frac{(\sqrt{5}+1)/3}{3-(\sqrt{5}+1)/6}\right)^2 \leq 2.9913. \end{aligned}$$

A contradiction. If  $x \in [3/8, 77/200)$ , then  $0 \leq \sin((2x+1)\pi/3) \leq \sin(7\pi/12) = \sin(5\pi/12)$ . Thus, using the same argument, we can see that: if  $\gamma \leq (\sqrt{5}+1)/6$  and  $x \in [3/8, 77/200)$ , then  $(0, 1) \notin E(1, x)$ .

From the above arguments, in the case that  $\gamma \leq (\sqrt{5}+1)/6$ , if  $0 \leq k < \ell < 3$  satisfying  $(k, \ell) \in E(1, x^*)$ , then  $x^* \in (1/8, 3/8)$  and  $(k, \ell) = (0, 1)$ .

Let  $K_2 = [0, 1/8] \cup [3/8, 5/8] \cup [7/8, 1]$ . From Lemma 2.1, there exist  $\varepsilon, \delta > 0$ , such that  $e(1, x; \varepsilon, \delta) = 1$  if  $x \in K_2$ , and  $e(1, x; \varepsilon, \delta) \leq 2$  if  $x \in [0, 1) \setminus K_2$ .

In the case that  $x \in (1/8, 3/8)$ , we have  $x(0) = x/3 \in (1/24, 1/8) \subseteq K_2$ , and  $x(1) = (x+1)/3 \in (3/8, 11/24) \subseteq K_2$ . From Lemma 2.3,  $\sigma(1) \leq \sqrt{2}$ .

If  $\gamma > \sqrt{2}/3$ , then  $3\gamma > \sqrt{2} \geq \sigma(1)$ . Thus, it suffices to show that if  $\gamma \leq \sqrt{2}/3$ , then it is impossible that  $e(1) = 2$ . In fact, assume that there exists  $x \in (1/8, 3/8)$  satisfying  $(0, 1) \in E(1, x)$ . From Lemma 2.1, we know that there exist  $\mathbf{k} = \{k_n\}_{n=1}^\infty$  and  $\mathbf{l} = \{\ell_n\}_{n=1}^\infty$  such that  $S(x, \mathbf{k}) - S(x, \mathbf{l}) = 0$ , where  $k_1 = 0, \ell_1 = 1$ .

Let  $x_n = (x + 3k_2 + \dots + 3^{n-1}k_n)/3^n, y_n = (x + 1 + 3\ell_2 + \dots + 3^{n-1}\ell_n)/3^n$ . We have

$$\begin{aligned} & \left| \cos(2\pi x_1) - \cos(2\pi y_1) + \gamma(\cos(2\pi x_2) - \cos(2\pi y_2)) \right| \\ &= \left| - \sum_{n=3}^\infty \gamma^{n-1} (\cos(2\pi x_n) - \cos(2\pi y_n)) \right| \leq 2 \sum_{n=2}^\infty \gamma^n \leq \frac{2\gamma^2}{1-\gamma}. \end{aligned}$$

Notice that

$$\begin{aligned} & \cos(2\pi x_2) - \cos(2\pi y_2) \\ &= \cos \frac{2\pi(x+3k_2)}{9} - \cos \frac{2\pi(x+1+3\ell_2)}{9} \\ &\geq \cos\left(\frac{2\pi x}{9} + \frac{2\pi}{3}\right) - \cos\left(\frac{2\pi x}{9} + \frac{2\pi}{9}\right) \\ &= -2 \sin\left(\frac{2\pi x}{9} + \frac{4\pi}{9}\right) \sin\left(\frac{2\pi}{9}\right) \geq -2 \sin\left(\frac{2\pi}{9}\right) > -1.3. \end{aligned}$$

Thus  $\cos(2\pi x_1) - \cos(2\pi y_1) \leq 2\gamma^2/(1 - \gamma) + 1.3\gamma$ . Hence

$$\begin{aligned} 3 &= 4 \sin^2\left(\frac{\pi}{3}\right) = |\cos(2\pi x_1) - \cos(2\pi y_1)|^2 + |\sin(2\pi x_1) - \sin(2\pi y_1)|^2 \\ &\leq \left(\frac{2\gamma^2}{1 - \gamma} + 1.3\gamma\right)^2 + \left(2\left|\cos\left(\frac{\pi(2x+1)}{3}\right)\right|\sin\left(\frac{\pi}{3}\right)\right)^2 \\ &\leq \left(\frac{2 \times (\sqrt{2}/3)^2}{1 - \sqrt{2}/3} + 1.3 \times \frac{\sqrt{2}}{3}\right)^2 + \left(\sqrt{3} \cos\left(\frac{5\pi}{12}\right)\right)^2 < 2.3140. \end{aligned}$$

A contradiction. □

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