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Hausdorff Dimension of a Class of Weierstrass Functions

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. It was proved by Shen that the graph of the classical Weierstrass function $\sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x)$ has Hausdorff dimension $2+\log\lambda/\log b$, for every integer $b \geq 2$ and every $\lambda \in (1/b,1)$ [Hausdorff dimension of the graph of the classical Weierstrass functions, Math. Z., 289 (2018), 223–266]. In this paper, we prove that the dimension formula holds for every integer $b \geq 3$ and every $\lambda \in (1/b,1)$ if we replace the function cos by sin in the definition of Weierstrass function. A class of more general functions are also discussed.

Key Words: Hausdorff dimension, Weierstrass function, SRB measure.

AMS Subject Classifications: 28A80

1 Introduction

Weierstrass functions are classical fractal functions. The non-differentiability of these functions were studied by Weierstrass and Hardy [2]. Recently, Shen [7] proved that the graph of the classical Weierstrass function $\sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x)$ has Hausdorff dimension $2+\log\lambda/\log b$, for every integer $b \geq 2$ and every $\lambda \in (1/b,1)$, which solved a long-standing conjecture. Some relevant results can be found in [1, 3–5, 8]. Naturally, we want to study the Hausdorff dimension of the graph of Weierstrass functions with the following form:

$$W_{\lambda,b,\theta}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x + \theta), \quad x \in \mathbb{R},$$

where $b \geq 2$ is an integer, $\lambda \in (1/b,1)$ and $\theta \in \mathbb{R}$.

Denote $D_{\lambda,b} = 2+\log\lambda/\log b$. Denote by $\dim_{\mathrm{H}} \Gamma W_{\lambda,b,\theta}$ the Hausdorff dimension of the graph of $W_{\lambda,b,\theta}$. Our main result is:

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Theorem 1.1. *If* $\theta = -\pi/2$, *then* $\dim_{\mathrm{H}} \Gamma W_{\lambda,b,\theta} = D_{\lambda,b}$ *for every integer* $b \geq 3$ *and every* $\lambda \in (1/b, 1)$. *If the integer* $b \geq 7$, *then the dimension formula holds for every* $\lambda \in (1/b, 1)$ *and every* $\theta \in \mathbb{R}$.

The paper is organized as follows. In next section, we present necessary notations and properties introduced by Shen [7] and Tsujii [8]. In Sections 3 and 4, we prove the main result.

2 Preliminaries

In this section, we present necessary notations and properties introduced in [7,8]. Denote $\gamma = 1/(\lambda b)$, $\phi_\theta(x) = \cos(2\pi x + \theta)$, and $\psi_\theta(x) = \phi'_\theta(x)$. Let $\mathcal{A} = \{0, 1, \cdots, b-1\}$. Given $x \in \mathbb{R}$ and $\mathbf{u} = \{u_n\}_{n=1}^{\infty} \in \mathcal{A}^{\mathbb{Z}^+}$, we define

$$S_\theta(x, \mathbf{u}) = \sum_{n=1}^{\infty} \gamma^{n-1} \psi_\theta(x(\mathbf{u}|_n)),$$

where $\mathbf{u}|_n = (u_1, \cdots, u_n)$ and

$$x(\mathbf{u}|_n) = \frac{x}{b^n} + \frac{u_1}{b^n} + \frac{u_2}{b^{n-1}} + \cdots + \frac{u_n}{b}.$$

For simplicity, we will use $S(x, \mathbf{u})$ to denote $S_\theta(x, \mathbf{u})$ if no confusion occurs.

Given $\varepsilon, \delta > 0$. Two words $\mathbf{i}, \mathbf{j} \in \mathcal{A}^{\mathbb{Z}^+}$ are called (ε, δ)-*tangent* at a point $x_0 \in \mathbb{R}$ if

$$|S(x_0, \mathbf{i}) - S(x_0, \mathbf{j})| \leq \varepsilon \quad \text{and} \quad |S'(x_0, \mathbf{i}) - S'(x_0, \mathbf{j})| \leq \delta.$$

Let $E(q, x_0; \varepsilon, \delta)$ denote the set of pairs $(\mathbf{k}, \mathbf{l}) \in \mathcal{A}^q \times \mathcal{A}^q$ for which there exist $\mathbf{u}, \mathbf{v} \in \mathcal{A}^{\mathbb{Z}^+}$ such that $\mathbf{ku}$ and $\mathbf{lv}$ are (ε, δ)-tangent at x_0. Let

$$\begin{aligned} e(q, x_0; \varepsilon, \delta) &= \max_{\mathbf{k} \in \mathcal{A}^{\mathbb{Z}^+}} \#\{\mathbf{l} \in \mathcal{A}^q : (\mathbf{k}, \mathbf{l}) \in E(q, x_0; \varepsilon, \delta)\}, \\ E(q, x_0) &= \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} E(q, x_0; \varepsilon, \delta), \\ e(q, x_0) &= \max_{\mathbf{k} \in \mathcal{A}^q} \#\{\mathbf{l} \in \mathcal{A}^q : (\mathbf{k}, \mathbf{l}) \in E(q, x_0)\}. \end{aligned}$$

For $J \subset \mathbb{R}$, define

$$\begin{aligned} E(q, J; \varepsilon, \delta) &= \bigcup_{x_0 \in J} E(q, x_0; \varepsilon, \delta), \\ E(q, J) &= \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} E(q, J; \varepsilon, \delta), \\ e(q, J) &= \max_{\mathbf{k} \in \mathcal{A}^q} \#\{\mathbf{l} \in \mathcal{A}^q : (\mathbf{k}, \mathbf{l}) \in E(q, J)\}. \end{aligned}$$

Tsujii's notation $e(q)$ is defined as

$$e(q)=\lim_{p\to\infty}\max_{k=0}^{b^p-1}e\Big(q,\Big[\frac{k}{b^p},\frac{k+1}{b^p}\Big]\Big).$$

It is well-known that $e(q)=\max_{x\in[0,1)}e(q,x)$. For details, please see [7].

Now we define another useful function $\sigma(q)$ introduced by Shen [7]. A measurable function $\omega:\ [0,1)\to[0,\infty)$ is called a weight function if $\|\omega\|_\infty<\infty$ and $\|1/\omega\|_\infty<\infty$. A *testing function of order* q is a measurable function $V:\ [0,1)\times\mathcal{A}^q\times\mathcal{A}^q\to[0,\infty)$. A testing function of order q is called admissible if there exist $\varepsilon>0$ and $\delta>0$ such that the following hold: For any $x\in[0,1)$, if $(\mathbf{u},\mathbf{v})\in E(q,x;\varepsilon,\delta)$, then

$$V(x,\mathbf{u},\mathbf{v})V(x,\mathbf{v},\mathbf{u})\geq 1.$$

Given a weight function ω and an admissible testing function V of order q, we define a new measurable function $\Sigma^q_{V,\omega}:\ [0,1)\to\mathbb{R}$ as follows: for each $x\in[0,1)$, let

$$\Sigma^q_{V,\omega}(x)=\sup\left\{\frac{\omega(x)}{\omega(x(\mathbf{u}))}\sum_{\mathbf{v}\in\mathcal{A}^q}V(x,\mathbf{u},\mathbf{v}):\ \mathbf{u}\in\mathcal{A}^q\right\}.$$

Define

$$\sigma(q)=\inf\|\Sigma^q_{V,\omega}\|_\infty,$$

where the infimum is taken over all weight functions ω and admissible testing functions V of order q.

Let $\mathbb{P}$ be the Bernoulli measure on $\mathcal{A}^{\mathbb{Z}^+}$ with uniform probabilities $\{1/b,1/b,\cdots,1/b\}$. For each $x\in\mathbb{R}$, define a Borel probability measure m_x on $\mathbb{R}$ by

$$m_x(A)=\mathbb{P}\big(\{\mathbf{v}:\ S(x,\mathbf{v})\in A\}\big),\quad A\subset\mathbb{R}.$$

Then m_x's are the conditional measures along vertical fibers of the unique SRB measure ν of the skew product map $T:\ \mathbb{R}/\mathbb{Z}\times\mathbb{R}\to\mathbb{R}/\mathbb{Z}\times\mathbb{R}$,

$$T(x,y)=(bx\mod 1,\gamma y+\psi_\theta(x)).$$

That is, the SRB measure ν can be defined by

$$\nu(B)=\int_0^1 m_x(B_x)dx$$

for each Borel set $B\subset\mathbb{R}/\mathbb{Z}\times\mathbb{R}$, where $B_x=\{y\in\mathbb{R}:\ (x,y)\in B\}$.

We will use the following two theorems to prove our main result.

Theorem 2.1 ([7]). *If there exists $q\in\mathbb{Z}^+$, such that $\sigma(q)<(\gamma b)^q$, then the SRB measure ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ with square integrable density. In particular, for Lebesgue a.e. $x\in[0,1)$, m_x is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and with square integrable density. As a result,* $\dim_{\rm H}\Gamma W_{\lambda,b,\theta}=D_{\lambda,b}$.

Theorem 2.2 ([7]). $\sigma(q) \leq e(q)$.

We remark that Theorem 2.1 strengths a similar result by Tsujii [8], and the dimension formula $\dim_{\rm H} \Gamma W_{\lambda,b,\theta} = D_{\lambda,b}$ follows from Ledrappier's theorem [6]. For details, please see [7].

The following result can be derived from the definitions of $E(q,x)$ and $E(q,J;\varepsilon,\delta)$. The proof for general case is same as the special case $\theta=0$, which is presented in [7]. Thus we omit the details.

Lemma 2.1 ([7]). *Let* $x_0 \in \mathbb{R}$, *and* $\mathbf{k},\mathbf{l} \in \mathcal{A}^q$. *Then*

- *(1)* $(\mathbf{k},\mathbf{l}) \in E(q,x_0)$ *if and only if there exist* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathcal{A}^{\mathbb{Z}^+}$ *such that* $F(x) = S(x,\mathbf{ku}) - S(x,\mathbf{lv})$ *has a multiple zero at* x_0, *that is,* $F(x_0) = F'(x_0) = 0$.
- *(2) If* $(\mathbf{k},\mathbf{l}) \notin E(q,x_0)$, *then there is a neighborhood* U *of* x_0 *and* $\varepsilon,\delta > 0$ *such that* $(\mathbf{k},\mathbf{l}) \notin E(q,U;\varepsilon,\delta)$.
- *(3) For any compact* $K \subset \mathbb{R}$, *if* $(\mathbf{k},\mathbf{l}) \notin E(q,K)$, *then there exist* $\varepsilon,\delta > 0$, *such that* $(\mathbf{k},\mathbf{l}) \notin E(q,K;\varepsilon,\delta)$.
- *(4) For any* $\varepsilon > \varepsilon' > 0$, $\delta > \delta' > 0$, *there exists* $\eta > 0$, *such that if* $|x-x_0| < \eta$, $(\mathbf{k},\mathbf{l}) \notin E(q,x_0;\varepsilon,\delta)$, *then* $(\mathbf{k},\mathbf{l}) \notin E(q,x;\varepsilon',\delta')$.

The following three lemmas are very useful in the proof of the results in [7]. They still hold in our case.

Lemma 2.2 ([7]). *Assume that for all* $x \in [0,1)$, $E(q,x) \neq \mathcal{A}^q \times \mathcal{A}^q$. *Then*

$$\sigma(q) \leq b^q - 2 + 2/\alpha,$$

where $\alpha = \alpha(b,q) > 1$ *satisfies* $2-\alpha = (b^q-2)\alpha(\alpha-1)$.

Lemma 2.3 ([7]). *Let* $q \in \mathbb{Z}^+$. *Suppose that there are constants* $\varepsilon > 0$ *and* $\delta > 0$ *and* $K \subset [0,1)$ *with the following properties:*

- *(1) For* $x \in K$, $e(q,x;\varepsilon,\delta) = 1$ *and for* $x \in [0,1) \setminus K$, $e(q,x;\varepsilon,\delta) \leq 2$;
- *(2) If* $(\mathbf{u},\mathbf{v}) \in E(q,x;\varepsilon,\delta)$ *for some* $x \in [0,1) \setminus K$ *and* $u \neq v$, *then both* $x(\mathbf{u})$ *and* $x(\mathbf{v})$ *belong to* K.

Then $\sigma(q) \leq \sqrt{2}$.

Lemma 2.4 ([7]). *Let* $q \in \mathbb{Z}^+$. *Suppose that there are constants* $\varepsilon > 0$ *and* $\delta > 0$ *and* $K \subset [0,1]$ *with the following properties:*

- *(1) For* $x \in K$, $e(q,x;\varepsilon,\delta) = 1$ *and for* $x \in [0,1) \setminus K$, $e(q,x;\varepsilon,\delta) \leq 2$;
- *(2) If* $(\mathbf{u},\mathbf{v}) \in E(q,x;\varepsilon,\delta)$ *for some* $x \in [0,1) \setminus K$ *and* $\mathbf{u} \neq \mathbf{v}$, *then either* $x(\mathbf{u}) \in K$ *or* $x(\mathbf{v}) \in K$.

Then $\sigma(q) \leq (\sqrt{5}+1)/2$.

3 The case when b is large

If $e(1)=1$, then form $\gamma b=1/\lambda>1$, we have $e(1)<\gamma b$. Thus, we always assume that $e(1)\geq 2$. From $e(1)=\max_{x\in[0,1)} e(1,x)$, there exists $x^*\in[0,1)$, such that $e(1,x^*)=e(1)$. We will fix x^* in the sequel of the paper.

From definition, there exists $k\in\mathcal{A}$, such that $\#\{\ell\in\mathcal{A}:\ (k,\ell)\in E(1,x^*)\}=e(1)$. Let $\ell^1,\ell^2,\cdots,\ell^{e(1)}$ be all elements in $\mathcal{A}$ such that $(k,\ell^{(i)})\in E(1,x^*)$, and

$$\sin(2\pi x^1+\theta)\leq\sin(2\pi x^2+\theta)\leq\cdots\leq\sin(2\pi x^{e(1)}+\theta),$$

where $x^i=(x^*+\ell^i)/b$, $i=1,\cdots,e(1)$.

Similarly as Lemma 3.2 and Lemma 3.3 in [7], we have the following two lemmas. Since the proof are same as that of in [7], we omit the details again.

Lemma 3.1. *If $(k,\ell)\in E(1,x^*)$, then*

$$\left|\sin\left(\frac{2\pi(x^*+k)}{b}+\theta\right)-\sin\left(\frac{2\pi(x^*+\ell)}{b}+\theta\right)\right|\leq\frac{2\gamma}{1-\gamma}, \tag{3.1a}$$

$$\left|\cos\left(\frac{2\pi(x^*+k)}{b}+\theta\right)-\cos\left(\frac{2\pi(x^*+\ell)}{b}+\theta\right)\right|\leq\frac{2\gamma}{b-\gamma}, \tag{3.1b}$$

$$4\sin^2\frac{\pi(k-\ell)}{b}\leq\left(\frac{2\gamma}{1-\gamma}\right)^2+\left(\frac{2\gamma}{b-\gamma}\right)^2. \tag{3.1c}$$

Lemma 3.2. *Under the above circumstances, and with the assumption that $1\leq i<j\leq e(1)$, the followings hold:*

1. *If $\ell^i=k$ or $\ell^j=k$, then* $\sin(2\pi x^j+\theta)-\sin(2\pi x^i+\theta)\geq\frac{2\beta_0(b,\gamma)}{b}$,
2. $\sin(2\pi x^j+\theta)-\sin(2\pi x^i+\theta)\geq\frac{2\beta_1(b,\gamma)}{b}$,
3. *If $\ell^i-\ell^j\neq\pm1 \mod b$, then* $\sin(2\pi x^j+\theta)-\sin(2\pi x^i+\theta)\geq\frac{2\beta_2(b,\gamma)}{b}$,

where

$$\beta_0(b,\gamma)=\sqrt{\max\left\{0,\left(b\sin\frac{\pi}{b}\right)^2-\frac{\gamma^2b^2}{(b-\gamma)^2}\right\}},$$

$$\beta_1(b,\gamma)=\sqrt{\max\left\{0,\left(b\sin\frac{\pi}{b}\right)^2-\frac{4\gamma^2b^2}{(b-\gamma)^2}\right\}},$$

$$\beta_2(b,\gamma)=\sqrt{\max\left\{0,\left(b\sin\frac{2\pi}{b}\right)^2-\frac{4\gamma^2b^2}{(b-\gamma)^2}\right\}}.$$

Using these two lemmas and lemmas in Section 2, we can prove the following theorem, which implies that Theorem 1.1 holds if $b\geq 7$.

Theorem 3.1. *1. If $b \geq 7$, then $e(1) < \gamma b$.*

2. If $b = 4,5,6$, then either $e(1) = 2$ or $e(1) < \gamma b$.

3. If $b = 3$, then either $e(1) \leq 2$ or $\sigma(1) < \gamma b$.

Proof. Using the exactly same method as in [7], we can obtain the following result: if $b \geq 4$, then either $e(1) = 2$ or $e(1) < \gamma b$; if $b = 3$, then either $e(1) \leq 2$ or $\sigma(1) < \gamma b$. Thus, we only need to prove the theorem holds if $b \geq 7$ and $e(1) = 2$. If $\gamma b > 2$, then $\gamma b > e(1)$. Thus it suffices to show it is impossible that $e(1) = 2$ and $\gamma b \leq 2$.

We will prove this by contradiction. Assume that $e(1) = 2$ and $\gamma b \leq 2$, then $(\ell^1, \ell^2) \in E(1, x^*)$. From Lemma 3.1 and $\gamma \leq 2/b$, we have

$$4\sin^2\frac{\pi(\ell^2-\ell^1)}{b} \leq \left(\frac{2\gamma}{1-\gamma}\right)^2 + \left(\frac{2\gamma}{b-\gamma}\right)^2$$
$$\leq \left(\frac{2\cdot(2/b)}{1-2/b}\right)^2 + \left(\frac{2\cdot(2/b)}{b-2/b}\right)^2 = \frac{16}{(b-2)^2} + \frac{16}{(b^2-2)^2}.$$

Thus

$$\sin^2\frac{\pi}{b} \leq \frac{4}{(b-2)^2} + \frac{4}{(b^2-2)^2}. \tag{3.2}$$

Consider the function $g(t) = g_1(t) - g_2(t)$, where

$$g_1(t) = t^2\sin^2(\pi/t) \quad \text{and} \quad g_2(t) = \frac{4t^2}{(t-2)^2} + \frac{4t^2}{(t^2-2)^2}.$$

It is easy to check that g_1 is increasing on $(2,+\infty)$ while g_2 is decreasing on $(2,+\infty)$. Thus, if $b \geq 7$, we have $g(b) \geq g(7) > 9 - 8 > 0$, which implies that (3.2) does not hold for $b \geq 7$. □

4 Proof of Theorem 1.1: the case $b = 3,4,5,6$

In this section, we will restrict $\theta = -\pi/2$. We will show the following result under this restriction: for $b = 3,4,5,6$, if $e(1) = 2$ then $\sigma(1) < \gamma b$. Combining this result with Theorem 3.1, we have either $e(1) < \gamma b$ or $\sigma(1) < \gamma b$ for $b = 3,4,5,6$. Thus Theorem 1.1 holds for this case.

Using the same method as in the proof of Lemma 4.1 in [7], we have the following lemma. We omit the details.

Lemma 4.1. *Assume that $0 \leq k < \ell < b$ satisfying $(k,\ell) \in E(1,x^*)$. Then for any $\kappa \in (0,1)$, one of the followings holds: either*

$$\left|\sin\left(\frac{2\pi(x^*+k)}{b}\right) - \sin\left(\frac{2\pi(x^*+\ell)}{b}\right)\right| \leq \frac{2\gamma\sqrt{1-\kappa^2}}{b} + \frac{2\gamma^2}{b(b-\gamma)}, \tag{4.1}$$

or

$$\left|\cos\Big(\frac{2\pi(x^*+k)}{b}\Big)-\cos\Big(\frac{2\pi(x^*+\ell)}{b}\Big)\right|\le 2\kappa\gamma+\frac{2\gamma^2}{1-\gamma}. \tag{4.2}$$

Notice that $\theta=-\pi/2$. For $x\in\mathbb{R}$ and $\mathbf{i}=\{i_n\}_{n=1}^{\infty}\in\mathcal{A}^{\mathbb{Z}^+}$, we have

$$S(x,\mathbf{i})=\sum_{n=1}^{\infty}\gamma^{n-1}\psi(x_n)=2\pi\sum_{n=1}^{\infty}\gamma^{n-1}\cos\Big(2\pi\Big(\frac{x}{b^n}+\frac{i_1}{b^n}+\cdots+\frac{i_n}{b}\Big)\Big). \tag{4.3}$$

Lemma 4.2. *If $x\in\mathbb{R}$ and $\mathbf{i}=\{i_n\}_{n=1}^{\infty}$. Then the following equalities hold:*

$$S(x,\mathbf{i})=S(1-x,\mathbf{i}'), \tag{4.4a}$$
$$S'(x,\mathbf{i})=-S'(1-x,\mathbf{i}'), \tag{4.4b}$$

where $\mathbf{i}'=\{i'_n\}_{n=1}^{\infty}$, $i'_n=b-1-i_n$.

Proof. Notice that

$$\begin{aligned}
&\cos\Big(2\pi\Big(\frac{1-x}{b^n}+\frac{b-1-i_1}{b^n}+\cdots+\frac{b-1-i_n}{b}\Big)\Big)\\
=&\cos\Big(2\pi\Big(-\Big(\frac{x}{b^n}+\frac{i_1}{b^n}+\cdots+\frac{i_n}{b}\Big)+\Big(\frac{1}{b^n}+\frac{b-1}{b^n}+\cdots+\frac{b-1}{b}\Big)\Big)\Big)\\
=&\cos\Big(2\pi\Big(-\Big(\frac{x}{b^n}+\frac{i_1}{b^n}+\cdots+\frac{i_n}{b}\Big)+1\Big)\Big)\\
=&\cos\Big(2\pi\Big(\frac{x}{b^n}+\frac{i_1}{b^n}+\cdots+\frac{i_n}{b}\Big)\Big).
\end{aligned}$$

Thus (4.4a) holds. From

$$S'(x,\mathbf{i})=\frac{-4\pi^2}{b}\sum_{n=1}^{\infty}\Big(\frac{\gamma}{b}\Big)^{n-1}\sin\Big(2\pi\Big(\frac{x}{b^n}+\frac{i_1}{b^n}+\cdots+\frac{i_n}{b}\Big)\Big),$$

we can see that (4.4b) holds. □

From Lemma 4.2, we know that $(k,\ell)\in E(1,x^*)$ is equivalent to $\big(b-1-k,b-1-\ell\big)\in E(1,1-x^*)$. Thus $e(1,x^*)=e(1,1-x^*)$. Hence, we may assume that $x^*\in[0,\frac{1}{2}]$.

4.1 The case $b=6$

Proposition 4.1. *Assume $b=6$ and $e(1)=2$. Then $\sigma(1)<6\gamma$.*

Proof. It is clear that $6\gamma>6\cdot(1/3)=e(1)\ge\sigma(1)$ if $\gamma>\frac{1}{3}$. Thus we may assume that $\gamma\le\frac{1}{3}$. From $e(1)=2$, there exist $0\le k<\ell<6$, such that $(k,\ell)\in E(1,x^*)$.

From $(k,\ell)\in E(1,x^*)$ and Lemma 3.1, we have

$$\left|\sin\Big(\frac{2\pi(x^*+k)}{6}\Big)-\sin\Big(\frac{2\pi(x^*+\ell)}{6}\Big)\right|\le\frac{2\gamma}{6-\gamma}\le\frac{2\cdot(1/3)}{6-(1/3)}=\frac{2}{17}, \tag{4.5a}$$

$$4\sin^2\frac{\pi(\ell-k)}{6}\le\Big(\frac{2\gamma}{1-\gamma}\Big)^2+\Big(\frac{2\gamma}{6-\gamma}\Big)^2\le 1^2+(2/17)^2<2. \tag{4.5b}$$

If $\ell-k\neq\pm1 \mod 6$, then $4\sin^2\frac{\pi(\ell-k)}{6}\ge 4\sin^2\frac{2\pi}{6}=3$, which contradicts with (4.5b). Thus $\ell-k=\pm1 \mod 6$. Combining this with $k<\ell$, we can see that $\ell-k=1$ or 5.

Let $\kappa=0.98$. We will show that the inequality (4.2) does not hold. In fact, if (4.2) holds, then

$$\left|\cos\Big(\frac{2\pi(x^*+k)}{6}\Big)-\cos\Big(\frac{2\pi(x^*+\ell)}{6}\Big)\right|\le 2\cdot0.98\cdot(1/3)+\frac{2\cdot(1/3)^2}{1-1/3}<0.987.$$

Combining this with (4.5a), we have

$$1=4\sin^2\frac{\pi}{6}=4\sin^2\Big(\frac{\pi(\ell-k)}{6}\Big)<(2/17)^2+0.987^2<0.989.$$

Contradiction! Thus the inequality (4.1) holds. Let $y^*=\pi(2x^*+k+\ell)/6$. Then $y^*\in[0,5\pi/3]$ and

$$\begin{aligned}|\cos(y^*)|=\Big|2\sin\frac{\pi}{6}\cos(y^*)\Big|&=\left|\sin\Big(\frac{2\pi(x^*+k)}{6}\Big)-\sin\Big(\frac{2\pi(x^*+\ell)}{6}\Big)\right|\\&\le\frac{2\cdot(1/3)\cdot\sqrt{1-0.98^2}}{6}+\frac{2\cdot(1/3)^2}{6\cdot(6-1/3)}<0.029<\cos(49\pi/100).\end{aligned}$$

Thus $y^*\in\big(49\pi/100,51\pi/100\big)\cup\big(149\pi/100,151\pi/100\big)$.

Case 1. $y^*\in\big(49\pi/100,51\pi/100\big)$. In this case, $2x^*+k+\ell\in\big(294/100,306/100\big)$. If $k+1=\ell$, then $x^*+k\in\big(97/100,103/100\big)$. Since $x^*\in[0,\frac12]$, we have $(k,\ell)=(1,2)$ and $x^*\in\big[0,3/100\big)$. If $k+5=\ell$, from $x^*\ge0$ and $k\ge0$, we have $2x^*+2k+5\ge 5>306/100$, a contradiction!

Case 2. $y^*\in\big(149\pi/100,151\pi/100\big)$. In this case, $2x^*+k+\ell\in\big(894/100,906/100\big)$. If $k+1=\ell$, then $x^*+k\in\big(397/100,403/100\big)$. Since $x^*\in[0,\frac12]$, we have $(k,l)=(4,5)$ and $x^*\in[0,3/100)$. If $k+5=\ell$, we must have $k=0$ and $\ell=5$. Thus, from $x^*\in[0,\frac12]$, we can obtain $2x^*+2k+5\le6<894/100$, a contradiction!

From Case 1 and Case 2, we can see that in the case that $\gamma\le\frac13$, if $0\le k<l<6$ satisfying $(k,l)\in E(1,x^*)$, then $0\le x^*<3/100$, and $(k,l)=(1,2)$ or $(k,l)=(4,5)$.

From above arguments, $e(1,x)=1$ if $x\in[3/100,1/2]$. Using the fact that $e(1,x)=e(1,1-x)$, we also have $e(1,x)=1$ if $x\in[1/2,97/100]$.

Let $K=[3/100,97/100]$. Then $e(1,K)=1$ and $e(1,[0,1))\le 2$. From Lemma 2.1(3), there exist $\varepsilon>0,\delta>0$, such that $e(1,x;\varepsilon,\delta)=1$ if $x\in K$, and $e(1,x;\varepsilon,\delta)\le 2$ if $x\in[0,1)\backslash K$.

In the case that $x\in\big[0,1/2\big]\setminus K$, if $(k,\ell)=(1,2)$, then $x(2)=(x+2)/6\subseteq K$; if $(k,\ell)=(4,5)$, then $x(4)=(x+4)/6\subseteq K$. Using the symmetry (Lemma 4.2), we know that the conditions of Lemma 2.4 hold for $q=1$. Thus $\sigma(1)\le(\sqrt{5}+1)/2$.

If $\gamma>(\sqrt{5}+1)/12$, then $6\gamma>\sigma(1)$. Thus, it suffices to show it is impossible that $\gamma\le(\sqrt{5}+1)/12$ and $e(1)=2$. In fact, if this holds, then from Lemma 3.1,

$$\begin{aligned}
&4\sin^2\Big(\frac{\pi(\ell-k)}{6}\Big)\le\Big(\frac{2\gamma}{6-\gamma}\Big)^2+\Big(\frac{2\gamma}{1-\gamma}\Big)^2\\
\le&\Big(\frac{(\sqrt{5}+1)/6}{6-(\sqrt{5}+1)/12}\Big)^2+\Big(\frac{(\sqrt{5}+1)/6}{1-(\sqrt{5}+1)/12}\Big)^2\le 0.56<4\sin^2\Big(\frac{\pi}{6}\Big),
\end{aligned}$$

which contradicts with $k\neq\ell$. □

4.2 The case $b=5$

Proposition 4.2. *Assume $b=5$ and $e(1)=2$. Then $\sigma(1)<5\gamma$.*

Proof. If $\gamma>2/5$, then $5\gamma>2=e(1)\ge\sigma(1)$. Thus we may assume that $\gamma\le\frac{2}{5}$. From $e(1)=2$, there exist $0\le k<\ell<5$ such that $(k,\ell)\in E(1,x^*)$. Now we will show that $x^*\in(3/20,7/20)$ and $(k,\ell)=(3,4)$.

In fact, from $(k,\ell)\in E(1,x^*)$ and Lemma 3.1,

$$\begin{aligned}
&\Big|\sin\Big(2\pi\frac{x^*+k}{b}\Big)-\sin\Big(2\pi\frac{x^*+\ell}{b}\Big)\Big|\le\frac{2\gamma}{5-\gamma}\le\frac{2\times\frac{2}{5}}{5-\frac{2}{5}}=\frac{4}{23},\\
&4\sin^2\Big(\frac{\pi(\ell-k)}{5}\Big)\le\Big(\frac{2\gamma}{1-\gamma}\Big)^2+\Big(\frac{2\gamma}{5-\gamma}\Big)^2\le\Big(\frac{2\times\frac{2}{5}}{1-\frac{2}{5}}\Big)^2+\Big(\frac{2\times\frac{2}{5}}{5-\frac{2}{5}}\Big)^2<2.
\end{aligned}$$

Assume that $\ell-k\neq\pm1\mod 5$. Then $\ell-k\in\{2,3\}$. Thus $4\sin^2\big(\pi(k-\ell)/5\big)\ge 4\sin^2\big(2\pi/5\big)>3.618$, a contradiction. Thus $\ell-k=\pm1\mod 5$. Since $\ell>k$, we have $\ell-k=1$ or $\ell-k=4$.

Let $\kappa=\sqrt{2}/2$. We will show that inequality (4.2) in Lemma 4.1 does not hold. In fact, if (4.2) holds, then

$$\begin{aligned}
&1.38<4\sin^2\Big(\frac{\pi}{5}\Big)=4\sin^2\Big(\frac{\pi(k-\ell)}{5}\Big)\\
=&\Big|\sin\Big(2\pi\frac{x^*+k}{5}\Big)-\sin\Big(2\pi\frac{x^*+\ell}{5}\Big)\Big|^2+\Big|\cos\Big(2\pi\frac{x^*+k}{5}\Big)-\cos\Big(2\pi\frac{x^*+\ell}{5}\Big)\Big|^2\\
\le&\Big(\frac{4}{23}\Big)^2+\Big(2\times\frac{\sqrt{2}}{2}\times\frac{2}{5}+\frac{2\times(2/5)^2}{1-\frac{2}{5}}\Big)^2<\Big(\frac{4}{23}\Big)^2+(1.1)^2<1.3,
\end{aligned}$$

a contradiction. Thus the inequality (4.1) in Lemma 4.1 holds. Let $y^* = \pi(2x^* + k + \ell)/5$. We have

$$\begin{aligned}\Big|2\cos(y^*)\sin\Big(\frac{\pi}{5}\Big)\Big| =&\Big|2\cos(y^*)\sin\Big(\frac{\pi(\ell-k)}{5}\Big)\Big|\\ =&\Big|\sin\Big(2\pi\frac{x^*+k}{5}\Big)-\sin\Big(2\pi\frac{x^*+\ell}{5}\Big)\Big|\\ \le&\frac{2\times\frac{2}{5}\times\sqrt{1-(\frac{\sqrt{2}}{2})^2}}{5}+\frac{2\times(\frac{2}{5})^2}{5\times(5-\frac{2}{5})}<0.128.\end{aligned}$$

Thus

$$|\cos(y^*)| \le \frac{0.128}{2\sin(\pi/5)} < 0.11 < \cos\Big(\frac{23\pi}{50}\Big).$$

Since $y^* \in \big[0, 8\pi/5\big]$, we have $y^* \in \big(23\pi/50, 27\pi/50\big) \cup \big(73\pi/50, 77\pi/50\big)$.

Case 1. $y^* \in (23\pi/50, 27\pi/50)$. In this case, $2x^* + k + \ell \in (23/10, 27/10)$. If $\ell - k = 1$, then $x^* + k \in (13/20, 17/20)$, which contradicts the fact that $x^* \in [0, 1/2)$ and k is a nonnegative integer. If $\ell - k = 4$, then $2x^* + 2k + 4 \ge 4 > 27/10$, which also contradicts the fact that $x^* \in [0, 1/2)$ and k is a nonnegative integer.

Case 2. $y^* \in (73\pi/50, 77\pi/50)$. In this case, $2x^* + k + \ell \in (73/10, 77/10)$. If $\ell - k = 1$, then $x^* + k \in (63/20, 67/20)$. Thus $(k, \ell) = (3, 4)$ and $x^* \in (3/20, 7/20)$. If $\ell - k = 4$, then $x^* + k \in (33/20, 37/20)$, which also contradicts the fact that $x^* \in [0, 1/2)$ and k is a nonnegative integer.

Thus, in the case that $\gamma \in (0, 2/5]$, if $0 \le k < \ell < 5$ satisfying $(k, \ell) \in E(1, x^*)$, then $x^* \in (3/20, 7/20)$ and $(k, \ell) = (3, 4)$.

From above arguments, $e(1, x) = 1$ if $x \in [0, 3/20] \cup [7/20, 1/2]$. Using the fact that $e(1, x) = e(1, 1 - x)$, we have $e(1, x) = 1$ if $x \in [1/2, 13/20] \cup [17/20, 1]$.

Let $K = [0, 3/20] \cup [7/20, 13/20] \cup [17/20, 1]$. Then $e(1, K) = 1$ and $e(1, [0, 1)) \le 2$. From Lemma 2.1, there exist $\varepsilon, \delta > 0$, such that $e(1, x; \varepsilon, \delta) = 1$ if $x \in K$, and $e(1, x; \varepsilon, \delta) \le 2$ if $x \in [0, 1)\backslash K$.

If $x \in (3/20, 1/4)$, we have $x(3) = (x + 3)/5 \in (7/20, 13/20) \subseteq K$. If $x \in [1/4, 7/20)$, we have $x(4) = (x + 4)/5 \in [17/20, 1) \subseteq K$. From Lemma 2.4, we have $\sigma(1) \le (\sqrt{5} + 1)/2$.

If $\gamma > (\sqrt{5} + 1)/10$, then $5\gamma > \sigma(1)$. Thus, it suffices to show it is impossible that $\gamma \le (\sqrt{5} + 1)/10$ and $e(1) = 2$. In fact, if this holds, then from Lemma 3.1,

$$\begin{aligned}&4\sin^2\Big(\frac{\pi(\ell-k)}{5}\Big) \le \Big(\frac{2\gamma}{5-\gamma}\Big)^2+\Big(\frac{2\gamma}{1-\gamma}\Big)^2\\ \le&\Big(\frac{(\sqrt{5}+1)/5}{5-(\sqrt{5}+1)/10}\Big)^2+\Big(\frac{(\sqrt{5}+1)/5}{1-(\sqrt{5}+1)/10}\Big)^2 < 0.9348 < 4\sin^2\Big(\frac{\pi}{5}\Big),\end{aligned}$$

which contradicts with $k \ne \ell$. □

4.3 The case $b=4$

Proposition 4.3. *Assume that $b=4$ and $e(1)=2$. Then $\sigma(1)<4\gamma$.*

Proof. If $\gamma>\frac{1}{2}$, then $4\gamma>2=e(1)\geq\sigma(1)$. Thus we may assume that $\gamma\leq\frac{1}{2}$. From $e(1)=2$, there exist $0\leq k<\ell<4$ such that $(k,\ell)\in E(1,x^*)$. Now we will show that $x^*\in(9/25,1/2]$ and $(k,\ell)=(2,3)$.

In fact, from $(k,\ell)\in E(1,x^*)$ and Lemma 3.1,

$$\Big|\sin\Big(2\pi\frac{x^*+k}{4}\Big)-\sin\Big(2\pi\frac{x^*+\ell}{4}\Big)\Big|\leq\frac{2\gamma}{4-\gamma}\leq\frac{2\times\frac{1}{2}}{4-\frac{1}{2}}=\frac{2}{7}. \tag{4.6}$$

Let $\kappa=\frac{1}{3}$. We will show that the inequality (4.2) in Lemma 4.1 does not hold. In fact, if (4.2) holds, then

$$\begin{aligned}2=&4\sin^2\Big(\frac{\pi}{4}\Big)\leq4\sin^2\Big(\frac{\pi(k-\ell)}{4}\Big)\\ =&\Big|\sin\Big(2\pi\frac{x^*+k}{4}\Big)-\sin\Big(2\pi\frac{x^*+\ell}{4}\Big)\Big|^2+\Big|\cos\Big(2\pi\frac{x^*+k}{4}\Big)-\cos\Big(2\pi\frac{x^*+\ell}{4}\Big)\Big|^2\\ \leq&\Big(\frac{2}{7}\Big)^2+\Big(2\times\frac{1}{3}\times\frac{1}{2}+\frac{2\times(1/2)^2}{1-\frac{1}{2}}\Big)^2<\Big(\frac{2}{7}\Big)^2+\Big(\frac{4}{3}\Big)^2<2.\end{aligned}$$

A contradiction. Thus (4.1) in Lemma 4.1 holds. Let $y^*=\pi(2x^*+k+\ell)/4$. We have

$$\begin{aligned}\Big|2\cos(y^*)\sin\Big(\frac{\pi}{4}\Big)\Big|\leq&\Big|2\cos(y^*)\sin\Big(\frac{\pi(\ell-k)}{4}\Big)\Big|\\ =&\Big|\sin\Big(2\pi\frac{x^*+k}{4}\Big)-\sin\Big(2\pi\frac{x^*+\ell}{4}\Big)\Big|\\ \leq&\frac{2\times\frac{1}{2}\times\sqrt{1-(\frac{1}{3})^2}}{4}+\frac{2\times(\frac{1}{2})^2}{4\times(4-\frac{1}{2})}=\frac{\sqrt{2}}{6}+\frac{1}{28}<0.28.\end{aligned}$$

Thus $|\cos(y^*)|\leq0.28/\sqrt{2}<0.2<\cos(43\pi/100)$. Since $y^*\in[0,3\pi/2]$, we have $y^*\in\big(43\pi/100,57\pi/100\big)\cup\big(143\pi/100,3\pi/2\big)$.

If $y^*\in(43\pi/100,57\pi/100)$, then $2x^*+k+\ell\in(43/25,57/25)$. In this case, we have $k+\ell=2$ so that $(k,\ell)=(0,2)$ and $x^*\in[0,7/50)$. If $y^*\in(143\pi/100,3\pi/2]$, then $2x^*+k+\ell\in(143/25,6]$. In this case, we have $k+\ell=5$ so that $(k,\ell)=(2,3)$ and $x^*\in(9/25,1/2]$.

In the case that $\gamma\leq\frac{1}{2}$, we have $(0,2)\notin E(1,x)$ for all $x\in[0,\frac{1}{2}]$. In fact, assume that $(0,2)\in E(1,x)$. Then there exist $\mathbf{k}=\{k_n\}_{n=1}^{\infty}$ and $\mathbf{l}=\{\ell_n\}_{n=1}^{\infty}$, such that $S(x,\mathbf{k})-S(x,\mathbf{l})=0$, where $k_1=0,\ell_1=2$.

Let $x_n=(x+4k_2+\cdots+4^{n-1}k_n)/4^n$, $y_n=(x+2+4\ell_2+\cdots+4^{n-1}\ell_n)/4^n$. Then

$$\begin{aligned}&\big|\cos(2\pi x_1)-\cos(2\pi y_1)+\gamma(\cos(2\pi x_2)-\cos(2\pi y_2))\big|\\=&\Big|-\sum_{n=3}^{\infty}\gamma^{n-1}\big(\cos(2\pi x_n)-\cos(2\pi y_n)\big)\Big|\\\le&2\sum_{n=2}^{\infty}\gamma^n\le\frac{2\gamma^2}{1-\gamma}\le\frac{2\times(\frac{1}{2})^2}{1-\frac{1}{2}}=1.\end{aligned}$$

Notice that

$$\begin{aligned}\cos(2\pi x_2)-\cos(2\pi y_2)&=\cos\Big(\frac{\pi x}{8}+\frac{k_2\pi}{2}\Big)-\cos\Big(\frac{\pi(x+2)}{8}+\frac{\ell_2\pi}{2}\Big)\\=-2\sin\Big(\frac{\pi(x+1)}{8}+\frac{(k_2+\ell_2)\pi}{4}\Big)&\sin\Big(-\frac{\pi}{8}+\frac{(k_2-\ell_2)\pi}{4}\Big)\ge-2\sin\frac{3\pi}{8}.\end{aligned}$$

Thus

$$\cos(2\pi x_1)-\cos(2\pi y_1)\le 1-\gamma\big(\cos(2\pi x_2)-\cos(2\pi y_2)\big)\le 1+\sin\frac{3\pi}{8}<1.93.$$

From the inequality (4.6), we have

$$\begin{aligned}4=&4\sin^2\Big(\frac{2\pi}{4}\Big)=4\sin^2\Big(\frac{\pi(\ell-k)}{4}\Big)\\=&\big|\sin(2\pi x_1)-\sin(2\pi y_1)\big|^2+\big|\cos(2\pi x_1)-\cos(2\pi y_1)\big|^2\\<&\Big(\frac{2}{7}\Big)^2+1.93^2<3.81.\end{aligned}$$

A contradiction. Thus, in the case that $\gamma\le 1/2$, if $0\le k<\ell<4$ satisfying $(k,\ell)\in E(1,x^*)$, then $x^*\in\big(9/25,1/2\big]$ and $(k,\ell)=(2,3)$.

From the above arguments, $e(1,x)=1$ if $x\in\big[0,9/25\big]$. Using the fact that $e(1,x)=e(1,1-x)$, we have $e(1,x)=1$ if $x\in\big[16/25,1\big]$.

Let $K=\big[0,9/25\big]\cup\big[16/25,1\big]$. Then $e(1,K)=1$ and $e(1,[0,1))\le 2$. From Lemma 2.1, there exist $\varepsilon,\delta>0$, such that $e(1,x;\varepsilon,\delta)=1$ if $x\in K$, and $e(1,x;\varepsilon,\delta)\le 2$ if $x\in[0,1)\backslash K$.

In the case that $x\in\big(9/25,1/2\big]$, we have $x(3)=(x+3)/4\in\big[16/25,1\big]\subseteq K$. From Lemma 2.4, we have $\sigma(1)\le(\sqrt{5}+1)/2$.

If $\gamma>(\sqrt{5}+1)/8$, then $4\gamma>\sigma(1)$. Thus, it suffices to show it is impossible that $\gamma\le(\sqrt{5}+1)/8$ and $e(1)=2$. In fact, if this holds, then from Lemma 3.1,

$$\begin{aligned}&4\sin^2\Big(\frac{\pi(\ell-k)}{4}\Big)\le\Big(\frac{2\gamma}{4-\gamma}\Big)^2+\Big(\frac{2\gamma}{1-\gamma}\Big)^2\\\le&\Big(\frac{(\sqrt{5}+1)/4}{4-(\sqrt{5}+1)/8}\Big)^2+\Big(\frac{(\sqrt{5}+1)/4}{1-(\sqrt{5}+1)/8}\Big)^2<1.897<4\sin^2\Big(\frac{\pi}{4}\Big),\end{aligned}$$

which contradicts with $k\neq\ell$. □

4.4 The case $b=3$

Proposition 4.4. *Assume $b=3$ and $e(1)=2$. Then $\sigma(1)<3\gamma$.*

Proof. If $\gamma>2/3$, then $3\gamma>2=e(1)\geq\sigma(1)$. Thus we may assume that $\gamma\leq 2/3$. From $e(1)=2$, there exist $0\leq k<\ell<3$ such that $(k,\ell)\in E(1,x^*)$.

Let $y^*=\pi(2x^*+k+\ell)/3$. From $(k,\ell)\in E(1,x^*)$ and Lemma 3.1,

$$\begin{aligned}&2\Big|\cos(y^*)\Big|\Big|\sin\Big(\frac{\pi(\ell-k)}{3}\Big)\Big|\\ =&\Big|\sin\Big(2\pi\frac{x^*+k}{3}\Big)-\sin\Big(2\pi\frac{x^*+\ell}{3}\Big)\Big|\\ \leq&\frac{2\gamma}{3-\gamma}\leq\frac{4/3}{3-2/3}=\frac{4}{7}.\end{aligned}$$

Thus $|\cos(y^*)|\leq 4/(7\sqrt{3})<\cos(39\pi/100)$. Hence $y^*\in(39\pi/100,61\pi/100)$. Since $y^*\in[0,4\pi/3]$, we have $2x^*+k+\ell\in(117/100,183/100)$. Thus $(k,\ell)=(0,1)$ and $x^*\in(17/200,83/200)$.

From above arguments, $e(1,x)=1$ if $x\in\big[0,17/200\big]\cup\big[83/200,1/2\big]$. Using the fact that $e(1,x)=e(1,1-x)$, $e(1,x)=1$ if $x\in\big[1/2,117/200\big]\cup\big[183/200,1\big]$.

Let $K_1=\big[0,17/200\big]\cup\big[83/200,117/200\big]\cup\big[183/200,1\big]$. From Lemma 2.1, there exist $\varepsilon>0,\delta>0$, such that $e(1,x;\varepsilon,\delta)=1$ if $x\in K_1$, and $e(1,x;\varepsilon,\delta)\leq 2$ if $x\in[0,1)\backslash K_1$.

If $x\in\big(17/200,51/200\big)$, we have $x(0)=x/3\in(17/600,17/200)\subseteq K_1$.

If $x\in\big[51/200,83/200\big)$, we have $x(1)=(x+1)/3\in(251/600,283/600)\subseteq K_1$.

From Lemma 2.4, we have $\sigma(1)\leq(\sqrt{5}+1)/2$. Thus, if $\gamma>(\sqrt{5}+1)/6$, then $3\gamma>\sigma(1)$.

Now we will show: if $\gamma\leq(\sqrt{5}+1)/6$ and $(k,\ell)=(0,1)$, then $x^*\in\big(23/200,77/200\big)$. In fact, from $(k,\ell)\in E(1,x^*)$ and Lemma 3.1,

$$\begin{aligned}&2\Big|\sin\Big(\frac{\pi(k-\ell)}{3}\Big)\cos(y^*)\Big|\\ =&\Big|\sin\Big(2\pi\frac{x^*+k}{3}\Big)-\sin\Big(2\pi\frac{x^*+\ell}{3}\Big)\Big|\\ \leq&\frac{2\gamma}{3-\gamma}\leq\frac{(\sqrt{5}+1)/3}{3-(\sqrt{5}+1)/6}\leq 0.4384.\end{aligned}$$

Thus $|\cos(y^*)|\leq 0.4384/\sqrt{3}<\cos(0.41\pi)$. Hence $y^*\in(0.41\pi,0.59\pi)$. By the definition of y^*, we have $2x^*+k+\ell\in(1.23,1.77)$. Combining this with $(k,\ell)=(0,1)$, we have $x^*\in\big(23/200,77/200\big)$.

Now we will show: if $\gamma\leq(\sqrt{5}+1)/6$ and $x\in\big(23/200,1/8\big]$, then $(0,1)\notin E(1,x)$.

In fact, assume that $(0,1)\in E(1,x)$. Then from Lemma 3.1,

$$\begin{aligned}
&3=4\sin^2\Big(\frac{\pi}{3}\Big)\\
=&\Big|\cos\Big(2\pi\frac{x+0}{3}\Big)-\cos\Big(2\pi\frac{x+1}{3}\Big)\Big|^2+\Big|\sin\Big(2\pi\frac{x+0}{3}\Big)-\sin\Big(2\pi\frac{x+1}{3}\Big)\Big|^2\\
\le&\Big(\sqrt{3}\sin\Big(\pi\frac{2x+1}{3}\Big)\Big)^2+\Big(\frac{2\gamma}{3-\gamma}\Big)^2\\
\le&\Big(\sqrt{3}\sin\frac{5\pi}{12}\Big)^2+\Big(\frac{(\sqrt{5}+1)/3}{3-(\sqrt{5}+1)/6}\Big)^2\le 2.9913.
\end{aligned}$$

A contradiction. If $x\in\big[3/8,77/200\big)$, then $0\le\sin((2x+1)\pi/3)\le\sin(7\pi/12)=\sin(5\pi/12)$. Thus, using the same argument, we can see that: if $\gamma\le(\sqrt{5}+1)/6$ and $x\in\big[3/8,77/200\big)$, then $(0,1)\notin E(1,x)$.

From the above arguments, in the case that $\gamma\le(\sqrt{5}+1)/6$, if $0\le k<\ell<3$ satisfying $(k,\ell)\in E(1,x^*)$, then $x^*\in(1/8,3/8)$ and $(k,\ell)=(0,1)$.

Let $K_2=\big[0,1/8\big]\cup\big[3/8,5/8\big]\cup\big[7/8,1\big]$. From Lemma 2.1, there exist $\varepsilon,\delta>0$, such that $e(1,x;\varepsilon,\delta)=1$ if $x\in K_2$, and $e(1,x;\varepsilon,\delta)\le 2$ if $x\in[0,1)\backslash K_2$.

In the case that $x\in\big(1/8,3/8\big)$, we have $x(0)=x/3\in\big(1/24,1/8\big)\subseteq K_2$, and $x(1)=(x+1)/3\in\big(3/8,11/24\big)\subseteq K_2$. From Lemma 2.3, $\sigma(1)\le\sqrt{2}$.

If $\gamma>\sqrt{2}/3$, then $3\gamma>\sqrt{2}\ge\sigma(1)$. Thus, it suffices to show that if $\gamma\le\sqrt{2}/3$, then it is impossible that $e(1)=2$. In fact, assume that there exists $x\in(\frac{1}{8},\frac{3}{8})$ satisfying $(0,1)\in E(1,x)$. From Lemma 2.1, we know that there exist $\mathbf{k}=\{k_n\}_{n=1}^{\infty}$ and $\mathbf{l}=\{\ell_n\}_{n=1}^{\infty}$ such that $S(x,\mathbf{k})-S(x,\mathbf{l})=0$, where $k_1=0,\ \ell_1=1$.

Let $x_n=(x+3k_2+\cdots+3^{n-1}k_n)/3^n$, $y_n=(x+1+3\ell_2+\cdots+3^{n-1}\ell_n)/3^n$. We have

$$\begin{aligned}
&\big|\cos(2\pi x_1)-\cos(2\pi y_1)+\gamma(\cos(2\pi x_2)-\cos(2\pi y_2))\big|\\
=&\Big|-\sum_{n=3}^{\infty}\gamma^{n-1}\big(\cos(2\pi x_n)-\cos(2\pi y_n)\big)\Big|\le 2\sum_{n=2}^{\infty}\gamma^n\le\frac{2\gamma^2}{1-\gamma}.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\cos(2\pi x_2)-\cos(2\pi y_2)\\
=&\cos\frac{2\pi(x+3k_2)}{9}-\cos\frac{2\pi(x+1+3\ell_2)}{9}\\
\ge&\cos\Big(\frac{2\pi x}{9}+\frac{2\pi}{3}\Big)-\cos\Big(\frac{2\pi x}{9}+\frac{2\pi}{9}\Big)\\
=&-2\sin\Big(\frac{2\pi x}{9}+\frac{4\pi}{9}\Big)\sin\Big(\frac{2\pi}{9}\Big)\ge-2\sin\Big(\frac{2\pi}{9}\Big)>-1.3.
\end{aligned}$$

Thus $\cos(2\pi x_1) - \cos(2\pi y_1) \leq 2\gamma^2/(1-\gamma) + 1.3\gamma$. Hence

$$\begin{aligned}
3 =& 4\sin^2\left(\frac{\pi}{3}\right) = \left|\cos(2\pi x_1) - \cos(2\pi y_1)\right|^2 + \left|\sin(2\pi x_1) - \sin(2\pi y_1)\right|^2 \\
\leq& \Big(\frac{2\gamma^2}{1-\gamma} + 1.3\gamma\Big)^2 + \Big(2\Big|\cos\Big(\frac{\pi(2x+1)}{3}\Big)\Big|\sin\Big(\frac{\pi}{3}\Big)\Big)^2 \\
\leq& \Big(\frac{2\times(\sqrt{2}/3)^2}{1-\sqrt{2}/3} + 1.3\times\frac{\sqrt{2}}{3}\Big)^2 + \Big(\sqrt{3}\cos(\frac{5\pi}{12})\Big)^2 < 2.3140.
\end{aligned}$$

A contradiction. □

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