Analysis in Theory and Applications
Anal. Theory Appl., Vol. **36**, No. 4 (2020), pp. 497-509

DOI: 10.4208/ata.OA-SU2

Local Well-Posedness for the Compressible Nematic Liquid Crystals Flow with Vacuum

Jishan Fan[1], Fucai Li[2,*] and Gen Nakamura[3]

[1] *Department of Applied Mathematics, Nanjing Forestry University, Nanjing, Jiangsu 210037, China*
[2] *Department of Mathematics, Nanjing University, Nanjing, Jiangsu 210093, China*
[3] *Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan*

Received 22 August 2019; Accepted (in revised version) 12 January 2020

Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. In this paper we prove the local well-posedness of strong solutions to the compressible nematic liquid crystals flow with vacuum in a bounded domain $\Omega\subset\mathbb{R}^3$.

Key Words: Liquid crystals, vacuum, Local well-posedness, strong solution.

AMS Subject Classifications: 76D03, 35Q30, 35Q35

1 Introduction

In this paper we consider the following simplified version of the Ericksen-Leslie system modeling the flow of compressible nematic liquid crystals (see [2,3]):

$$\partial_t\rho+\mathrm{div}\,(\rho u)=0, \tag{1.1a}$$
$$\partial_t(\rho u)+\mathrm{div}\,(\rho u\otimes u)+\nabla p(\rho)-\mu\Delta u-(\lambda+\mu)\nabla\mathrm{div}\,u=-\nabla d\cdot\Delta d, \tag{1.1b}$$
$$\partial_t d+u\cdot\nabla d-\Delta d=d|\nabla d|^2\quad\text{in }\ \Omega\times(0,T), \tag{1.1c}$$

with boundary and initial conditions:

$$u=0,\quad \frac{\partial d}{\partial n}=0 \qquad\qquad \text{on }\ \partial\Omega\times(0,T), \tag{1.2a}$$
$$(\rho,u,d)(\cdot,0)=(\rho_0,u_0,d_0)(\cdot)\qquad \text{in }\ \Omega\subset\mathbb{R}^3, \tag{1.2b}$$

where $\rho\geq 0$ is the density of the fluid, $u\in\mathbb{R}^3$ represents velocity field of the fluid, $d\in\mathbb{S}^2$ represents the macroscopic average of the nematic liquid crystals orientation field. The

*Corresponding author. *Email addresses:* `fanjishan@njfu.edu.cn` (J. S. Fan), `fli@nju.edu.cn` (F. C. Li), `nakamuragenn@gmail.com` (G. Nakamura)

parameters μ and λ are shear viscosity and the bulk viscosity coefficients of the fluid, respectively, satisfying the physical conditions:

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0.$$

We assume that the pressure p satisfies the γ-law, i.e., $p =: a\rho^\gamma$ with constants $a > 0$ and $\gamma > 1$. The domain $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, and n is the unit outward normal vector to $\partial\Omega$.

Below let us review some results to the system (1.1a)-(1.1c) briefly. Ding et al. [2] first introduced the system (1.1a)-(1.1c) and studied the low Mach number limit of it, see [9–11] on the recent progress on this topic. Huang, Wang, and Wen [3] (see also [1,5]) showed the local well-posedness of strong solutions with vacuum under the following compatibility condition:

$$-\mu\Delta u_0 - (\lambda+\mu)\nabla \mathrm{div}\, u_0 + \nabla(a\rho_0^\gamma) + \nabla d_0 \cdot \Delta d_0 = \sqrt{\rho_0} g \tag{1.3}$$

for some $g \in L^2(\Omega)$. Jiang, Jiang, and Wang [4] (see also [6]) proved the global existence of weak solutions in $\mathbb{R}^2$. Lin, Lai and Wang [7] established the existence of global weak solutions with finite energy and density satisfying the renormalized continuity equation, provided the initial orientation director field lies in the hemisphere $\mathbb{S}^2_+$.

The purpose of this paper is to establish the local well-posedness of strong solutions of the compressible nematic liquid crystal model (1.1a)-(1.1c) without the compatibility condition (1.3).

We will prove

Theorem 1.1. *Let* $0 \leq \rho_0 \in W^{1,q}$, $(3 < q < 6)$, $u_0 \in H^1_0$, $d_0 \in H^2$ *with* $|d_0| = 1$. *Then the problem* (1.1a)-(1.2b) *has a unique local strong solution* (ρ, u, d) *satisfying*

$$\begin{cases} \rho \in C([0,T];L^2) \cap L^\infty(0,T;W^{1,q}), & \partial_t\rho \in L^\infty(0,T;L^2), \\ \rho u \in C([0,T];L^2),\ u \in L^\infty(0,T;H^1_0) \cap L^2(0,T;H^2), & \sqrt{\rho}\partial_t u \in L^2(0,T;L^2), \\ \sqrt{t}u \in L^\infty(0,T;H^2) \cap L^2(0,T;W^{2,q}), & \sqrt{t}\partial_t u \in L^2(0,T;H^1_0), \\ d \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3),\ \partial_t d \in L^2(0,T;H^1), & \sqrt{t}\partial_t d \in L^\infty(0,T;H^1), \end{cases} \tag{1.4}$$

for some $0 < T \leq \infty$.

We will prove Theorem 1.1 in the following way: For $\delta > 0$, we choose $0 < \delta \leq \rho_0^\delta \in H^2$ and $u_0^\delta \in H^1_0 \cap H^2$ satisfying

$$\rho_0^\delta \to \rho_0 \quad \text{in } W^{1,q} \quad \text{and} \quad u_0^\delta \to u_0 \quad \text{in } H^1_0 \quad \text{as } \delta \to 0. \tag{1.5}$$

Then it is easy to verify that the problem (1.1a)-(1.2b) has a unique local strong solution $(\rho^\delta, u^\delta, d^\delta)$ in $[0, T_\delta)$.

We further define

$$
\begin{aligned}
M(t) =:1 + \sup_{0\le s\le t} \Big\{ &\|\rho(\cdot,s)\|_{W^{1,q}} + \|u(\cdot,s)\|_{H^1} + \sqrt{s}\|\sqrt{\rho}u_t(\cdot,s)\|_{L^2} \\
&+ \|d(\cdot,s)\|_{H^2} + \sqrt{s}\|d_t(\cdot,s)\|_{H^1}\Big\} + \|u\|_{L^2(0,t;H^2)} \\
&+ \|\sqrt{\rho}u_t\|_{L^2(0,t;L^2)} + \|\sqrt{s}\nabla u_t\|_{L^2(0,t;L^2)} \\
&+ \|d\|_{L^2(0,t;H^3)} + \|d_t\|_{L^2(0,t;H^1)}
\end{aligned} \tag{1.6}
$$

and prove

Theorem 1.2. *For any $t\in[0,T_\delta)$, we have that*

$$M(t)\le C_0(M_0)\exp\Big\{t^{\frac{6-q}{4q}}C(M)\Big\} \tag{1.7}$$

for some nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

It follows from (1.7) that (see [8]):

$$M(t)\le C \tag{1.8}$$

and thus the proof of existence part of Theorem 1.1 is complete by taking $\delta\to 0$ and the standard compactness principle. We present the proof of Theorem 1.2 in Section 2 and the uniqueness part of Theorem 1.1 in Section 3.

2 Proof of Theorem 1.2

Below, for the sake of notational simplicity, we shall drop the superscript "δ" of ρ^δ, u^δ and d^δ. We also ignore to write down the domain Ω in the subsequent integrals.

Testing (1.1b) by u and using (1.1a), we see that

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u|^2\mathrm{d}x+\mu\int|\nabla u|^2\mathrm{d}x+(\lambda+\mu)\int(\operatorname{div}u)^2\mathrm{d}x \\
=&\int p\operatorname{div}u\mathrm{d}x-\int(u\cdot\nabla)d\cdot\Delta d\,\mathrm{d}x.
\end{aligned} \tag{2.1}
$$

Testing (1.1c) by $-\Delta d-d|\nabla d|^2$ and using $d\cdot d_t=0$ and $d\cdot\nabla d=0$, we find that

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla d|^2\mathrm{d}x+\int|\Delta d+d|\nabla d|^2|^2\mathrm{d}x=\int(u\cdot\nabla)d\cdot\Delta d\mathrm{d}x. \tag{2.2}$$

Summing (2.1) and (2.2) up, and rewriting the continuity equation (1.1a) as

$$p_t+u\cdot\nabla p+\gamma p\operatorname{div}u=0, \tag{2.3}$$

we infer that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\rho|u|^2+|\nabla d|^2)\mathrm{d}x+\int(\mu|\nabla u|^2+(\lambda+\mu)(\operatorname{div}u)^2+|\Delta d+d|\nabla d|^2|^2)\mathrm{d}x\\=&\int p\operatorname{div}u\mathrm{d}x=-\frac{1}{\gamma-1}\int p_t\mathrm{d}x,\end{aligned}$$

which gives

$$\begin{aligned}&\int\left(\rho|u|^2+|\nabla d|^2+\frac{2a}{\gamma-1}\rho^\gamma\right)\mathrm{d}x+2\int_0^T\int(|\nabla u|^2+|\Delta d+d|\nabla d|^2|^2)\mathrm{d}x\mathrm{d}t\\\le&\int\left(\rho_0|u_0|^2+|\nabla d_0|^2+\frac{2a}{\gamma-1}\rho_0^\gamma\right)\mathrm{d}x.\end{aligned}\tag{2.4}$$

Testing (1.1c) by d_t and using $d\cdot d_t=0$, we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla d|^2\mathrm{d}x+\int|d_t|^2\mathrm{d}x=-\int u\cdot\nabla d\cdot d_t\mathrm{d}x\\\le&\|u\|_{L^6}\|\nabla d\|_{L^3}\|d_t\|_{L^2}\le C(M)\|d_t\|_{L^2}\\\le&\frac{1}{2}\|d_t\|_{L^2}^2+C(M),\end{aligned}$$

which gives

$$\int_0^t\|d_t\|_{L^2}^2\mathrm{d}s\le C_0(M_0)+C(M)t.\tag{2.5}$$

Applying ∇ to (1.1c), we see that

$$\nabla d_t-\nabla\Delta d+\nabla(u\cdot\nabla d)=\nabla(d|\nabla d|^2).\tag{2.6}$$

Testing (2.6) by ∇d_t, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\Delta d|^2\mathrm{d}x+\int|\nabla d_t|^2\mathrm{d}x\\=&\int(\nabla(d|\nabla d|^2)-\nabla(u\cdot\nabla d))\nabla d_t\mathrm{d}x\\\le&C(\|\nabla d\|_{L^6}^3+\|\nabla d\|_{L^\infty}\|\nabla^2d\|_{L^2}^2+\|\nabla u\|_{L^2}\|\nabla d\|_{L^\infty}+\|u\|_{L^6}\|\nabla^2 d\|_{L^3})\|\nabla d_t\|_{L^2}\\\le&C(M)(1+\|\nabla d\|_{L^\infty}+\|\nabla^2d\|_{L^3})\|\nabla d_t\|_{L^2}\\\le&C(M)(1+\|d\|_{H^3}^{\frac{1}{2}})\|\nabla d_t\|_{L^2}\\\le&\frac{1}{2}\|\nabla d_t\|_{L^2}^2+C(M)+C(M)\|d\|_{H^3},\end{aligned}$$

which leads to

$$\int|\Delta d|^2\mathrm{d}x+\int_0^t\|\nabla d_t\|_{L^2}^2\mathrm{d}s\le C_0(M_0)+C(M)t^{\frac{1}{2}}.\tag{2.7}$$

Here we have used the Gagliardo-Nirenberg inequalities

$$\|\nabla d\|_{L^\infty}^2 \le C\|\nabla^2 d\|_{L^2}\|d\|_{H^3}, \tag{2.8a}$$

$$\|\nabla^2 d\|_{L^3}^2 \le C\|\nabla^2 d\|_{L^2}\|d\|_{H^3}. \tag{2.8b}$$

Eq. (1.1b) can be written as

$$-\mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}\, u = f =: -\rho\partial_t u - \rho u\cdot\nabla u - \nabla p(\rho) - \nabla d\cdot\Delta d. \tag{2.9}$$

Then we have

$$\begin{aligned}
\|u\|_{W^{2,q}} \le& C\|f\|_{L^q} \le C\|\rho\partial_t u\|_{L^q} + C\|\rho u\cdot\nabla u\|_{L^q} + C\|\nabla p\|_{L^q} + C\|\nabla d\|_{L^\infty}\|\Delta d\|_{L^q}\\
\le& C\|\rho\|_{L^\infty}^{\frac{5q-6}{4q}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|u_t\|_{L^6}^{\frac{3q-6}{2q}} + C(M)\|u\|_{L^\infty}\|\nabla u\|_{L^q} + C(M)\|d\|_{H^3}^{\frac{3}{2}}\\
\le& C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^2}^{\frac{3q-6}{2q}} + C(M)\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{3}{2}} + C(M)\|d\|_{H^3}^{\frac{3}{2}}\\
\le& C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^2}^{\frac{3q-6}{2q}} + C(M)\|u\|_{H^2}^{\frac{3}{2}} + C(M)\|d\|_{H^3}^{\frac{3}{2}},
\end{aligned}$$

which gives

$$\begin{aligned}
\int_0^t \|u\|_{W^{2,q}}\mathrm{d}s \le& C(M)\int_0^t \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^2}^{\frac{3q-6}{2q}}\mathrm{d}s + C(M)\int_0^t\|u\|_{H^2}^{\frac{3}{2}}\mathrm{d}s + C(M)\int_0^t\|d\|_{H^3}^{\frac{3}{2}}\mathrm{d}s\\
\le& C(M)\int_0^t s^{-\frac{3q-6}{4q}}(\sqrt{s}\|\nabla u_t\|_{L^2})^{\frac{3q-6}{2q}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\mathrm{d}s\\
&+ C(M)\left(\int_0^t\mathrm{d}s\right)^{\frac{1}{4}}\left(\int_0^t\|u\|_{H^2}^2\mathrm{d}s\right)^{\frac{3}{4}} + C(M)\left(\int_0^t\mathrm{d}s\right)^{\frac{1}{4}}\left(\int_0^t\|d\|_{H^3}^2\mathrm{d}s\right)^{\frac{3}{4}}\\
\le& C(M)\left(\int_0^t s^{-\frac{3q-6}{2q}}\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t s\|\nabla u_t\|_{L^2}^2\mathrm{d}s\right)^{\frac{3q-6}{4q}}\left(\int_0^t\|\sqrt{\rho}u_t\|_{L^2}^2\mathrm{d}s\right)^{\frac{6-q}{4q}} + C(M)t^{\frac{1}{4}}\\
\le& C(M)t^{\frac{6-q}{4q}} + C(M)t^{\frac{1}{4}} \le C(M)t^{\frac{6-q}{4q}}
\end{aligned} \tag{2.10}$$

for all $0 < t \le 1$.

Using the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^\infty} \le C\|\nabla u\|_{L^2}^{\frac{2q-6}{5q-6}}\|u\|_{W^{2,q}}^{\frac{3q}{5q-6}}, \tag{2.11}$$

we observe that

$$\begin{aligned}
&\int_0^t\|\nabla u\|_{L^\infty}\mathrm{d}s \le C(M)\int_0^t\|u\|_{W^{2,q}}^{\frac{3q}{5q-6}}\mathrm{d}s\\
\le& C\left(\int_0^t\mathrm{d}s\right)^{\frac{2q-6}{5q-6}}\left(\int_0^t\|u\|_{W^{2,q}}\mathrm{d}s\right)^{\frac{3q}{5q-6}}\\
\le& C(M)t^{\frac{2q-6}{5q-6}}\cdot t^{\frac{6-q}{4q}\cdot\frac{3q}{5q-6}} = C(M)t^{\frac{6-q}{4q}}.
\end{aligned} \tag{2.12}$$

Testing (1.1a) by ρ^{m-1}, we see that

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^m\mathrm{d}x=-\int\mathrm{div}\,(\rho u)\rho^{m-1}\mathrm{d}x=\int\rho u\nabla\rho^{m-1}\mathrm{d}x=-\frac{m-1}{m}\int\rho^m\mathrm{div}\,u\mathrm{d}x,$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^m}\le\|\mathrm{div}\,u\|_{L^\infty}\|\rho\|_{L^m},$$

and thus

$$\begin{aligned}\|\rho\|_{L^m}\le&\|\rho_0\|_{L^m}\exp\left(\int_0^t\|\mathrm{div}\,u\|_{L^\infty}\mathrm{d}s\right)\\ \le&\|\rho_0\|_{L^m}\exp\left\{t^{\frac{6-q}{4q}}C(M)\right\},\quad 2\le m<\infty.\end{aligned}\tag{2.13}$$

For $m=\infty$, (2.13) still holds.

Taking ∇ to (1.1a), testing the result by $|\nabla\rho|^{q-2}\nabla\rho$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^q}\le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q}+C\|\rho\|_{L^\infty}\|\nabla\mathrm{div}\,u\|_{L^q},$$

which implies

$$\begin{aligned}\|\nabla\rho\|_{L^q}\le&C\left(\|\nabla\rho_0\|_{L^q}+\int_0^t\|\rho\|_{L^\infty}\|\nabla\mathrm{div}\,u\|_{L^q}\mathrm{d}s\right)\exp\left(\int_0^t\|\nabla u\|_{L^\infty}\mathrm{d}s\right)\\ \le&C\Big(1+C(M)t^{\frac{6-q}{4q}}\Big)\exp\left\{t^{\frac{6-q}{4q}}C(M)\right\}\\ \le&C_0(M_0)\exp\left\{t^{\frac{6-q}{4q}}C(M)\right\}.\end{aligned}\tag{2.14}$$

Testing (1.1b) by u_t, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}\,u)^2)\mathrm{d}x+\int\rho|u_t|^2\mathrm{d}x\\ =&-\int\rho u\cdot\nabla u\cdot u_t\mathrm{d}x+\int p\mathrm{div}\,u_t\mathrm{d}x+\int\left(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\right):\nabla u_t\mathrm{d}x\\ =:&I_1+I_2+I_3.\end{aligned}\tag{2.15}$$

We bound I_1, I_2 and I_3 as follows.

$$\begin{aligned}|I_1|&\le\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}\\ &\le C(M)\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\\ &\le C(M)\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{H^2}^{\frac{1}{2}}\\ &\le\frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2+C(M)\|u\|_{H^2},\end{aligned}$$

$$
\begin{aligned}
I_2 &= \frac{\mathrm{d}}{\mathrm{d}t}\int p\mathrm{div}\, u\mathrm{d}x - \int p_t \mathrm{div}\, u\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int p\mathrm{div}\, u\mathrm{d}x + \int (u\cdot\nabla p + \gamma p \mathrm{div}\, u)\mathrm{div}\, u\mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int p\mathrm{div}\, u\mathrm{d}x + (\|u\|_{L^6}\|\nabla p\|_{L^3} + \gamma\|p\|_{L^\infty}\|\mathrm{div}\, u\|_{L^2})\|\mathrm{div}\, u\|_{L^2} \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int p\mathrm{div}\, u\mathrm{d}x + C(M).
\end{aligned}
$$

Here we have used the equality (2.3). And for the term I_3, it holds

$$
\begin{aligned}
I_3 =&\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right) : \nabla u\mathrm{d}x - \int \partial_t \left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right) : \nabla u\mathrm{d}x \\
\le&\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right) : \nabla u\mathrm{d}x + C\|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^2}\|\nabla u\|_{L^2} \\
\le&\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right) : \nabla u\mathrm{d}x + C(M)\|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^2}.
\end{aligned}
$$

Inserting the above estimates into (2.15), integrating over $(0,t)$,and using (2.7), we have

$$
\begin{aligned}
&\|\nabla u\|_{L^2}^2 + \int_0^t \|\sqrt{\rho}u_t\|_{L^2}^2\mathrm{d}s \\
\le&C_0(M_0) + C(M)t^{\frac{1}{2}} + C(M)t + C\|\nabla d\|_{L^4}^2 \\
\le&C_0(M_0) + C(M)t^{\frac{1}{2}} + C\|\nabla^2 d\|_{L^2} \\
\le&C_0(M_0) + C(M)t^{\frac{1}{2}} \le C_0(M_0)\exp\left\{t^{\frac{6-q}{4q}}C(M)\right\}.
\end{aligned} \tag{2.16}
$$

Applying ∂_t to (1.1b) and using (1.1a), we infer that

$$
\begin{aligned}
&\rho\partial_t^2 u + \rho u\cdot\nabla u_t - \mu\Delta u_t - (\lambda+\mu)\nabla\mathrm{div}\, u_t \\
=&-\nabla p_t + \mathrm{div}\,(\rho u)(u_t + u\cdot\nabla u) - \rho u_t\cdot\nabla u \\
&-\partial_t \mathrm{div}\left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right).
\end{aligned} \tag{2.17}
$$

Testing (2.23) by u_t and using (1.1a), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\mathrm{d}x + \int(\mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div}\, u_t)^2)\mathrm{d}x \\
=&\int p_t \mathrm{div}\, u_t\mathrm{d}x - \int \rho u\nabla|u_t|^2\mathrm{d}x - \int \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)\mathrm{d}x \\
&-\int \rho u_t\cdot\nabla u\cdot u_t\mathrm{d}x + \int \partial_t\left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right) : \nabla u_t\mathrm{d}x \\
=:&\sum_{i=4}^{8} I_i.
\end{aligned} \tag{2.18}
$$

We bound I_i $(i=4,\cdots,8)$ as follows.

$$\begin{aligned}
|I_4| &= \left|\int (u\cdot\nabla p+\gamma p \operatorname{div} u)\operatorname{div} u_t \mathrm{d}x\right| \\
&\le \big(\|u\|_{L^6}\|\nabla p\|_{L^3}+\gamma\|p\|_{L^\infty}\|\operatorname{div} u\|_{L^2}\big)\|\operatorname{div} u_t\|_{L^2} \\
&\le C(M)\|\nabla u_t\|_{L^2} \le \frac{\mu}{16}\|\nabla u_t\|_{L^2}^2 + C(M), \\
|I_5| &\le \|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2} \le C(M)\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u_t\|_{L^2} \\
&\le C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\sqrt{\rho}u_t\|_{L^6}^{\frac12}\|\nabla u_t\|_{L^2} \le C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac32} \\
&\le \frac{\mu}{16}\|\nabla u_t\|_{L^2}^2 + C(M)\|\sqrt{\rho}u_t\|_{L^2}^2, \\
|I_6| &\le \|\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}^2\|u_t\|_{L^6} + \|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla^2 u\|_{L^2}\|u_t\|_{L^6} \\
&\quad\ + \|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2} \\
&\le C(M)(\|\nabla u\|_{L^3}^2+\|u\|_{H^2})\|\nabla u_t\|_{L^2} \\
&\le C(M)(\|\nabla u\|_{L^2}\|u\|_{H^2}+\|u\|_{H^2})\|\nabla u_t\|_{L^2} \\
&\le C(M)\|u\|_{H^2}\|\nabla u_t\|_{L^2} \le \frac{\mu}{16}\|\nabla u_t\|_{L^2}^2 + C(M)\|u\|_{H^2}^2, \\
|I_7| &\le \|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^4}^2 \le \|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\sqrt{\rho}u_t\|_{L^6}^{\frac32} \\
&\le C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac32} \le \frac{\mu}{16}\|\nabla u_t\|_{L^2}^2 + C(M)\|\sqrt{\rho}u_t\|_{L^2}^2, \\
|I_8| &\le C\|\nabla u_t\|_{L^2}\|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^2}\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}^2\|\nabla d_t\|_{L^2}^2 \\
&\le \frac{\mu}{16}\|\nabla u_t\|_{L^2}^2 + C(M)\|d\|_{H^3}\|\nabla d_t\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates into (2.18) gives

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\mathrm{d}x + \frac{11}{16}\mu\int |\nabla u_t|^2\mathrm{d}x \\
\le& C(M)+C(M)\|\sqrt{\rho}u_t\|_{L^2}^2 + C(M)\|u\|_{H^2}^2 + C(M)\|d\|_{H^3}\|\nabla d_t\|_{L^2}^2.
\end{aligned}\tag{2.19}$$

Multiplying the above inequality by t, we obtain

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\left(t\int \rho u_t^2\mathrm{d}x\right) + \frac{11}{16}\mu t\int |\nabla u_t|^2\mathrm{d}x \\
\le& \frac12\int \rho|u_t|^2\mathrm{d}x + C(M)t + C(M)t\int \rho|u_t|^2\mathrm{d}x + C(M)t\|u\|_{H^2}^2 \\
&+ C(M)\|d\|_{H^3}\cdot t\|\nabla d_t\|_{L^2}^2,
\end{aligned}\tag{2.20}$$

which implies that

$$t\int \rho|u_t|^2\mathrm{d}x + \int_0^t s\|\nabla u_t\|_{L^2}^2\mathrm{d}s \le C_0(M_0)\exp\left\{t^{\frac{6-q}{4q}}C(M)\right\}. \tag{2.21}$$

It follows from (2.9) that

$$\begin{aligned}\|u\|_{H^2} \leq& C\|f\|_{L^2} \leq C\|\rho u_t + \rho u\cdot\nabla u + \nabla p + \nabla d\cdot\Delta d\|_{L^2}\\ \leq& C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + C\|\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}\\ &\quad + C\|\nabla p\|_{L^2} + C\|\nabla d\|_{L^\infty}\|\Delta d\|_{L^2}\\ \leq& C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + C(M)\|\nabla u\|_{L^3} + C(M) + C(M)\|\nabla d\|_{L^\infty}\\ \leq& C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + C(M)\|\nabla u\|_{L^2}^{\frac12}\cdot\|u\|_{H^2}^{\frac12} + C(M) + C(M)\|\nabla d\|_{L^\infty},\end{aligned}$$

which yields

$$\|u\|_{H^2} \leq C(M) + C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + C(M)\|\nabla d\|_{L^\infty},$$

whence

$$\|u\|_{L^2(0,t;H^2)} \leq C_0(M_0)\exp\Big\{t^{\frac{6-q}{4q}}C(M)\Big\}. \tag{2.22}$$

Taking ∂_t to (1.1c), testing the result by d_t, and using $d\cdot d_t = 0$, we observe that

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int |d_t|^2\mathrm{d}x + \int|\nabla d_t|^2\mathrm{d}x\\ =&-\int u_t\cdot\nabla d\cdot d_t\mathrm{d}x - \int u\cdot\nabla d_t\cdot d_t\mathrm{d}x + \int |d_t|^2|\nabla d|^2\mathrm{d}x\\ \leq&\|u_t\|_{L^6}\|\nabla d\|_{L^3}\|d_t\|_{L^2} + \|u\|_{L^6}\|\nabla d_t\|_{L^2}\|d_t\|_{L^3} + \|\nabla d\|_{L^6}^2\|d_t\|_{L^6}\|d_t\|_{L^2}\\ \leq& C\|\nabla d\|_{L^3}\|\nabla u_t\|_{L^2}\|d_t\|_{L^2} + C(M)\|\nabla d_t\|_{L^2}\|d_t\|_{L^3} + C(M)\|d_t\|_{L^6}\|d_t\|_{L^2}\\ \leq&\frac12\|\nabla d_t\|_{L^2}^2 + C\|\nabla d\|_{L^3}\|\nabla u_t\|_{L^2}\|d_t\|_{L^2} + C(M)\|d_t\|_{L^2}^2,\end{aligned}$$

which gives

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int|d_t|^2\mathrm{d}x + \frac12\int|\nabla d_t|^2\mathrm{d}x \leq C\|\nabla d\|_{L^3}\|\nabla u_t\|_{L^2}\|d_t\|_{L^2} + C(M)\|d_t\|_{L^2}^2.$$

Multiplying the above inequality by t, using (2.21), (2.5), and (2.7), we obtain

$$t\int|d_t|^2\mathrm{d}x + \int_0^t s\|\nabla d_t\|_{L^2}^2\mathrm{d}s \leq C_0(M_0)\exp\Big\{t^{\frac{6-q}{4q}}C(M)\Big\}. \tag{2.23}$$

Taking ∂_t to (1.1c), testing the result by $-\Delta d_t$, and using (2.5) and (2.7), we have

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla d_t|^2\mathrm{d}x + \int|\Delta d_t|^2\mathrm{d}x\\ =&\int(u_t\cdot\nabla d + u\cdot\nabla d_t)\Delta d_t\mathrm{d}x - \int(d_t|\nabla d|^2 + d\partial_t|\nabla d|^2)\Delta d_t\mathrm{d}x\\ \leq&(\|u_t\|_{L^6}\|\nabla d\|_{L^3} + \|u\|_{L^6}\|\nabla d_t\|_{L^3})\|\Delta d_t\|_{L^2}\\ &\quad + (\|d_t\|_{L^6}\|\nabla d\|_{L^6}^2 + \|\nabla d\|_{L^6}\|\nabla d_t\|_{L^3})\|\Delta d_t\|_{L^2}\\ \leq&\frac12\|\Delta d_t\|_{L^2}^2 + C\|\nabla d\|_{L^3}^2\|\nabla u_t\|_{L^2}^2 + C(M)(\|\nabla d_t\|_{L^2}^2 + \|d_t\|_{L^2}^2).\end{aligned}$$

Multiplying the above inequality by t, using (2.21), (2.5), (2.7), and (2.23), we have

$$t\int|\nabla d_t|^2\mathrm{d}x\le C_0(M_0)\exp\Big\{t^{\frac{6-q}{4q}}C(M)\Big\}. \tag{2.24}$$

It follows from (1.1c) and (2.7) that

$$\begin{aligned}\|d\|_{H^3}\le&C\|d\|_{H^1}+\|\nabla\Delta d\|_{L^2}\\ \le&C\|d\|_{H^1}+\|\nabla(d_t+u\cdot\nabla d-d|\nabla d|^2)\|_{L^2}\\ \le&C+C\|\nabla d_t\|_{L^2}+C\|u\|_{L^6}\|\nabla^2 d\|_{L^3}+C\|\nabla u\|_{L^2}\|\nabla d\|_{L^\infty}\\ &+C\|\nabla d\|_{L^6}^3+C\|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^2}\\ \le&C+C\|\nabla d_t\|_{L^2}+C(M)\|d\|_{H^3}^{\frac12}+C(M),\end{aligned}$$

which implies

$$\|d\|_{H^3}\le C+C(M)+C\|\nabla d_t\|_{L^2},$$

whence

$$\begin{aligned}\|d\|_{L^2(0,t;H^3)}\le&Ct+C(M)t+C(M)t^{\frac12}\le C(M)t^{\frac12}\\ \le&C_0(M_0)\exp\Big\{t^{\frac{6-q}{4q}}C(M)\Big\}.\end{aligned} \tag{2.25}$$

Combining (2.16) and (2.21)-(2.25), we conclude that (1.7) holds true. This completes the proof of Theorem 1.2. □

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since the existence part has been given, we only need to show the uniqueness part. Let (ρ_i,u_i,d_i), $(i=1,2)$ be the two strong solutions satisfying (1.4) with the same initial data.

We denote

$$(\rho,u,d)=:(\rho_1-\rho_2,u_1-u_2,d_1-d_2).$$

Then it is easy to verify that

$$\partial_t\rho+u_2\cdot\nabla\rho+\rho\,\mathrm{div}\,u_2+\rho_1\mathrm{div}\,u+u\cdot\nabla\rho_1=0, \tag{3.1}$$

$$\begin{aligned}&\rho_1\partial_t u+\rho_1u_1\cdot\nabla u-\mu\Delta u-(\lambda+\mu)\nabla\mathrm{div}\,u\\ &=-\rho_1u\cdot\nabla u_2-\rho(\partial_t u_2+u_2\cdot\nabla u_2)-\nabla(p(\rho_1)-p(\rho_2))\\ &\quad-\mathrm{div}\left(\nabla d_1\odot\nabla d_1-\nabla d_2\odot\nabla d_2-\frac12|\nabla d_1|^2\mathbb{I}_3+\frac12|\nabla d_2|^2\mathbb{I}_3\right),\end{aligned} \tag{3.2}$$

$$\partial_t d+u_1\nabla d+u\cdot\nabla d_2-\Delta d=d_1|\nabla d_1|^2-d_2|\nabla d_2|^2. \tag{3.3}$$

Testing (3.1) by ρ and using (1.4), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^2\mathrm{d}x=&-\int(u_2\nabla\rho+\rho\mathrm{div}\,u_2+\rho_1\mathrm{div}\,u+u\cdot\nabla\rho_1)\rho\mathrm{d}x\\=&-\int\left(\frac{1}{2}\rho^2\mathrm{div}\,u_2+\rho_1\mathrm{div}\,u\rho+u\nabla\rho_1\rho\right)\mathrm{d}x\\\le&C\|\nabla u_2\|_{L^\infty}\|\rho\|_{L^2}^2+C\|\rho_1\|_{L^\infty}\|\nabla u\|_{L^2}\|\rho\|_{L^2}+C\|u\|_{L^6}\|\nabla\rho_1\|_{L^3}\|\rho\|_{L^2}\\\le&C\|\nabla u_2\|_{L^\infty}\|\rho\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\rho\|_{L^2},\end{aligned}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^2}\le C\|\nabla u_2\|_{L^\infty}\|\rho\|_{L^2}+C\|\nabla u\|_{L^2}.\tag{3.4}$$

By the Gronwall inequality, we get

$$\|\rho\|_{L^2}\le C\int_0^t\|\nabla u\|_{L^2}\mathrm{d}s.\tag{3.5}$$

Testing (3.2) by u, using (1.1a), (1.4), and (3.5), we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\rho_1|u|^2\mathrm{d}x+\int_0^t\int(\mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}\,u)^2)\mathrm{d}x\mathrm{d}s\right)\\&\quad+\frac{1}{2}\int(\mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}\,u)^2)\mathrm{d}x\\\le&C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C\|\partial_t u_2\|_{L^6}\|u\|_{L^6}\|\rho\|_{L^{\frac{3}{2}}}+C\|u_2\|_{L^6}\|\nabla u_2\|_{L^6}\|u\|_{L^6}\|\rho\|_{L^2}\\&\quad+C\|p(\rho_1)-p(\rho_2)\|_{L^2}\|\nabla u\|_{L^2}+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla d\|_{L^2}\|\nabla u\|_{L^2}\\\le&C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C\|\nabla u_{2t}\|_{L^2}\|\nabla u\|_{L^2}\|\rho\|_{L^2}+C\|u_2\|_{H^2}\|\nabla u\|_{L^2}\|\rho\|_{L^2}\\&\quad+C\|\rho\|_{L^2}\|\nabla u\|_{L^2}+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla d\|_{L^2}\|\nabla u\|_{L^2}\\\le&C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C(\|\nabla u_{2t}\|_{L^2}+\|u\|_{H^2}+1)\|\nabla u\|_{L^2}\int_0^t\|\nabla u\|_{L^2}\mathrm{d}s\\&\quad+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla u\|_{L^2}\|\nabla d\|_{L^2}\\\le&\frac{\mu}{16}\|\nabla u\|_{L^2}^2+C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C(\|\nabla u_{2t}\|_{L^2}^2+\|u\|_{H^2}^2+1)\left(\int_0^t\|\nabla u\|_{L^2}\mathrm{d}s\right)^2\\&\quad+C(\|\nabla d_1\|_{L^\infty}^2+\|\nabla d_2\|_{L^\infty}^2)\|\nabla d\|_{L^2}^2\\\le&\frac{\mu}{16}\|\nabla u\|_{L^2}^2+C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C(t\|\nabla u_{2t}\|_{L^2}^2+\|u\|_{H^2}^2+1)\int_0^t\|\nabla u\|_{L^2}^2\mathrm{d}s\\&\quad+C(\|\nabla d_1\|_{L^\infty}^2+\|\nabla d_2\|_{L^\infty}^2)\|\nabla d\|_{L^2}^2.\end{aligned}\tag{3.6}$$

Testing (3.3) by d and using (1.4), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |d|^2\mathrm{d}x+\int |\nabla d|^2\mathrm{d}x\\
\le&\|u_1\|_{L^6}\|\nabla d\|_{L^2}\|d\|_{L^3}+\|u\|_{L^6}\|\nabla d_2\|_{L^3}\|d\|_{L^2}+\|d\|_{L^2}^2\|\nabla d_1\|_{L^\infty}^2\\
&\quad+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla d\|_{L^2}\|d\|_{L^2}\\
\le&C\|\nabla d\|_{L^2}\|d\|_{L^3}+C\|d\|_{L^2}^2+C\|\nabla d_1\|_{L^\infty}^2\|d\|_{L^2}^2\\
&\quad+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla d\|_{L^2}\|d\|_{L^2}+\frac{\mu}{16}\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{3.7}$$

Testing (3.3) by $-\Delta d$ and using (1.4), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla d|^2\mathrm{d}x+\int |\Delta d|^2\mathrm{d}x\\
\le&\|u_1\|_{L^6}\|\nabla d\|_{L^3}\|\Delta d\|_{L^2}+\|u\|_{L^6}\|\nabla d_2\|_{L^3}\|\Delta d\|_{L^2}\\
&\quad+C\|d\|_{L^6}\|\Delta d\|_{L^2}\|\nabla d_1\|_{L^6}^2+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla d\|_{L^2}\|\Delta d\|_{L^2}\\
\le&C_1\mu\|\nabla u\|_{L^2}^2+\frac{1}{2}\|\Delta d\|_{L^2}^2+C\|\nabla d\|_{L^2}^2+C\|d\|_{L^2}^2\\
&\quad+C(\|\nabla d_1\|_{L^\infty}^2+\|\nabla d_2\|_{L^\infty}^2)\|\nabla d\|_{L^2}^2.
\end{aligned}\tag{3.8}$$

Doing (3.6)$\times 8C_1$+(3.7)+(3.8) and using the Gronwall inequality, we arrive at

$$\rho=0,\quad u=0\quad\text{and}\quad d=0.\tag{3.9}$$

This completes the proof of Theorem 1.1. □

Acknowledgements

Fan is supported by NSFC (Grant No. 11971234). Li is supported by NSFC (Grant No. 11671193) and A Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

References

[1] Y. Chu, X.-G. Liu and X. Liu, Strong solutions to the compressible liquid crystal system, Pac. J. Math., 257 (2012), 37–52.

[2] S. Ding, J. Huang, H. Wen and R. Zi, Incompressible limit of the compressible nematic liquid crystal flow, J. Funct. Anal., 264 (2013), 1711–1756.

[3] T. Huang, C. Wang and H. Wen, Strong solutions of the compressible nematic liquid crystal flow, J. Differential Equations, 252 (2012), 2222–2265.

[4] F. Jiang, S. Jiang and D. Wang, On multi-dimensional compressible flow of nematic liquid crystals with large initial energy in a bounded domain, J. Funct. Anal., 265 (2013), 3369–3397.

[5] X. Li and B. Guo, Well-posedness for the three-dimensional compressible liquid crystal flows, Discrete Contin. Dyn. Syst. Ser. S, 9 (2016), 1913–1937.
[6] F. Lin and C. Wang, Recent developments of analysis for hydrodynamic flow of nematic liquid crystals, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 372 (2014), 20130361.
[7] J. Lin, B. Lai and C. Wang, Global finite energy weak solutions to the compressible nematic liquid crystal flow in dimension three, SIAM J. Math. Anal., 47 (2015), 2952–2983.
[8] G. Metivier and S. Schochet, The incompressible limit of the non-isentropic Euler equations, Arch. Ration. Mech. Anal., 158 (2001), 61–90.
[9] G. Qi and J. Xu, The low Mach number limit for the compressible flow of liquid crystals, Appl. Math. Comput., 297 (2017), 39–49.
[10] X. Yang, Uniform well-posedness and low Mach number limit to the compressible nematic liquid flows in a bounded domain, Nonlinear Anal., 120 (2015), 118–126.
[11] L. Zeng, G. Ni and X. Ai, Low Mach number limit of global solutions to 3-D compressible nematic liquid crystal flows with Dirichlet boundary condition, Math. Methods Appl. Sci., 42 (2019), 2053–2068.