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Lower Bounds of Dirichlet Eigenvalues for General Grushin Type Bi-Subelliptic Operators

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Abstract. Let Ω be a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $X=(X_1,X_2,\cdots,X_m)$ be a system of general Grushin type vector fields defined on Ω and the boundary $\partial\Omega$ is non-characteristic for X. For $\Delta_X=\sum_{j=1}^m X_j^2$, we denote λ_k as the k-th eigenvalue for the bi-subelliptic operator Δ_X^2 on Ω. In this paper, by using the sharp sub-elliptic estimates and maximally hypoelliptic estimates, we give the optimal lower bound estimates of λ_k for the operator Δ_X^2.

Key Words: Eigenvalues, degenerate elliptic operators, sub-elliptic estimate, maximally hypoelliptic estimate, bi-subelliptic operator.

AMS Subject Classifications: 35J30, 35J70, 35P15

1 Introduction and main results

Let $X=(X_1,X_2,\cdots,X_m)$ be the system of general Grushin type vector fields, which is defined on an open domain W in $\mathbb{R}^n$ ($n\geq 2$).

Let $J=(j_1,\cdots,j_k)$, $1\leq j_i\leq m$ be a multi-index, $X^J=X_{j_1}X_{j_2}\cdots X_{j_k}$, we denote $|J|=k$ be the length of J, if $|J|=0$, then $X^J=id$. We introduce following function space (cf. [18,21,23]):

$$H_X^2(W)=\{u\in L^2(W)|X^Ju\in L^2(W),\ |J|\leq 2\}.$$

It is well known that $H_X^2(W)$ is a Hilbert space with norm $\|u\|^2_{H_X^2(W)}=\sum_{|J|\leq 2}\|X^Ju\|^2_{L^2(W)}$.

Assume the vector fields $X=(X_1,X_2,\cdots,X_m)$ satisfy Hörmander's condition :

Definition 1.1 (cf. [2,12])**.** We say that $X=(X_1,X_2,\cdots,X_m)$ satisfies the Hörmander's condition in W if there exists a positive integer Q, such that for any $|J|=k\leq Q$, X together with all k-th repeated commutators

$$X_J=[X_{j_1},[X_{j_2},[X_{j_3},\cdots,[X_{j_{k-1}},X_{j_k}]\cdots]]]$$

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span the tangent space at each point of W. Here Q is called the Hörmander index of X in W, which is defined as the smallest positive integer for the Hörmander's condition to be satisfied.

For any bounded open subset $\Omega\subset\subset W$, we define the subspace $H^2_{X,0}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $H^2_X(W)$. Since $\partial\Omega$ is smooth and non characteristic for X, we know that $H^2_{X,0}(\Omega)$ is well defined and also a Hilbert space. In this case, we also say that X satisfies the Hörmander's condition on Ω with Hörmander index $1\le Q<+\infty$. Thus X is a finitely degenerate system of vector fields on Ω and the finitely degenerate elliptic operator $\Delta_X=\sum_{i=1}^m X_i^2$ is a sub-elliptic operator.

The degenerate elliptic operator Δ_X has been studied by many authors, e.g., Hörmander [11], Jerison and Sánchez-Calle [13], Métivier [17], Xu [23]. More results for degenerate elliptic operators can be found in [2–6] and [9,10,12,14].

In this paper, we study the following eigenvalues problem for bi-subelliptic operators in $H^2_{X,0}(\Omega)$:

$$\begin{cases}\Delta_X^2 u=\lambda u & \text{in } \Omega,\\ u=0,\ Xu=0 & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where X will be the following general Grushin type vector fields (see (1.5) and (1.7) below). In this case we know that for each j, X_j is formally skew-adjoint, i.e., $X_j^*=-X_j$. Then there exists a sequence of discrete eigenvalues $\{\lambda_j\}_{j\ge1}$ for the problem (1.1), which satisfying $0<\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lambda_k\cdots$ and $\lambda_k\to+\infty$ as $k\to+\infty$ (see Proposition 2.5 below).

In the classical case, if $X=(\partial_{x_1},\cdots,\partial_{x_n})$, then $\Delta_X^2=\Delta^2$ is the standard bi-harmonic operator. In this case our problem is motivated from the following classical clamped plate problem, namely

$$\begin{cases}\Delta^2 u=\lambda u & \text{in } \Omega,\\ u=\dfrac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega,\end{cases} \tag{1.2}$$

where $\Delta=\partial^2_{x_1}+\partial^2_{x_2}+\cdots+\partial^2_{x_n}$, $\frac{\partial u}{\partial\nu}$ denotes the derivative of u with respect to the outer unit normal vector ν on $\partial\Omega$.

For the eigenvalues of the clamped plate problem (1.2), Agmon [1] and Pleijel [20] showed the following asymptotic formula

$$\lambda_k\sim\frac{16\pi^4}{(B_n vol(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}} \quad \text{as} \quad k\to+\infty, \tag{1.3}$$

where B_n denotes the volume of the unit ball in R^n. In 1985, Levine and Protter [15] proved that

$$\frac{1}{k}\sum_{i=1}^k\lambda_i\ge\frac{n}{n+4}\frac{16\pi^4}{(B_n vol(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}. \tag{1.4}$$

Later in 2012, Cheng and Wei [7] showed that the eigenvalues of the bi-harmonic operator satisfy

$$\begin{aligned}\frac{1}{k}\sum_{i=1}^{k}\lambda_i \geq &\frac{n}{n+4}\frac{16\pi^4}{(B_n vol(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}\\ &+\left(\frac{n+2}{12(n+4)}-\frac{1}{1152n^2(n+4)}\right)\frac{vol(\Omega)}{I(\Omega)}\frac{n}{n+2}\frac{4\pi^4}{(B_n vol(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}\\ &+\left(\frac{1}{576n(n+4)}-\frac{1}{27648n^2(n+2)(n+4)}\right)\left(\frac{vol(\Omega)}{I(\Omega)}\right)^2,\end{aligned}$$

where $I(\Omega)$ is the moment of inertia of Ω.

Next, we consider the situation for the bi-subelliptic operators Δ_X^2. Before we state our results, we need the following concepts:

Definition 1.2. If X satisfies the Hörmander's condition in W with the Hörmander index $Q\geq 1$. Then for each $1\leq j\leq Q$ and $x\in W$, we denote $V_j(x)$ as the subspace of the tangent space $T_x(W)$ spanned by the vector fields X_J with $|J|\leq j$. We say the system of the vector fields X satisfies Métivier's condition on Ω if the dimension of $V_j(x)$ is constant v_j in a neighborhood of each $x\in\overline{\Omega}$, and in this case the Métivier index is defined as

$$v=\sum_{j=1}^{Q} j(v_j-v_{j-1}), \quad \text{here} \quad v_0=0.$$

As it well-known that under the Métivier's condition, we can get the asymptotic estimate for the eigenvalues of sub-elliptic operator $-\Delta_X$ (cf. [17]). However, for most degenerate vector fields X, the Métivier's condition will be not satisfied. Thus we need to introduce the following generalized Métivier index.

Definition 1.3. If X satisfies the Hörmander's condition in W with the Hörmander index $Q\geq 1$. Then for each $1\leq j\leq Q$ and $x\in W$, we denote $V_j(x)$ as the subspace of the tangent space $T_x(W)$ spanned by the vector fields X_J with $|J|\leq j$. We denote that

$$v(x)=\sum_{j=1}^{Q} j(v_j(x)-v_{j-1}(x)), \quad \text{with} \quad v_0(x)=0,$$

where $v_j(x)$ is the dimension of $V_j(x)$. Then we define

$$\tilde{v}=\max_{x\in\overline{\Omega}} v(x),$$

as the generalized Métivier index. It is obvious that $\tilde{v}=v$ if X satisfies the Métivier's condition on Ω.

Recently, in case of X to be some special Grushin vector fields Chen and Zhou [8] obtained lower bound estimates of eigenvalues for the bi-subelliptic operator Δ_X^2. In this paper, we shall study the similar problem for more general Grushin type vector fields X. In the first part of this paper, we shall study the bi-subelliptic operators Δ_X^2 in case of

$$X=(\partial_{x_1},\cdots,\partial_{x_{n-1}},f(\bar{x})\partial_{x_n}), \tag{1.5}$$

where $f(\bar{x})=\sum_{|\alpha|\le s}a_\alpha\bar{x}^\alpha$ is a multivariate polynomial of $\bar{x}$ with order s, $\bar{x}=(x_1,\cdots,x_{n-1})$, $\alpha=(\alpha_1,\cdots,\alpha_{n-1})\in Z_+^{n-1}$, $|\alpha|=\alpha_1+\cdots+\alpha_{n-1}$, a_α are constants. We suppose that

(H_1): If $f(\bar{x})$ has a unique zero point at origin $\bar{x}=0$ in Ω only, and there exists a unique multi-index α_0 with $|\alpha_0|=s_0\le s$, satisfying $\partial_{\bar{x}}^{\alpha_0}f(\bar{x})|_{\bar{x}=0}\neq 0$ and $\partial_{\bar{x}}^{\alpha}f(\bar{x})|_{\bar{x}=0}=0$ for any $|\alpha|<|\alpha_0|$.

Thus we have the following result.

Theorem 1.1. *Let $X=(\partial_{x_1},\cdots,\partial_{x_{n-1}},f(\bar{x})\partial_{x_n})$, $\bar{x}=(x_1,x_2,\cdots,x_{n-1})$. Under the condition (H_1) above, X satisfies the Hörmander's condition with its Hörmander index $Q=s_0+1$, and the generalized Métivier index of X is $\tilde{v}=Q+n-1$. Suppose λ_j is the j-th eigenvalue of the problem* (1.1), *then for all $k\ge 1$,*

$$\sum_{j=1}^{k}\lambda_j\ge\overline{C}(Q)k^{1+\frac{4}{\tilde{v}}}-\frac{C_2(Q)}{C_1(Q)}k, \tag{1.6}$$

where

$$\overline{C}(Q)=\frac{A_Q}{C_1(Q)n^2(n+Q+3)}\left(\frac{(2\pi)^n}{Q\omega_{n-1}|\Omega|_n}\right)^{\frac{4}{n+Q-1}}(n+Q-1)^{\frac{n+Q+3}{n+Q-1}},$$

and

$$A_Q=\begin{cases}\min\{1,n^{\frac{3-Q}{2}}\}, & Q\ge 2,\\ n, & Q=1.\end{cases}$$

Here $C_1(Q),C_2(Q)$ are the constants in Proposition 2.3 below, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, and $|\Omega|_n$ is the volume of Ω.

Remark 1.1. (1) Since $k\lambda_k\ge\sum\limits_{j=1}^{k}\lambda_j$, then Theorem 1.1 shows that the eigenvalues λ_k satisfy

$$\lambda_k\ge\overline{C}(Q)k^{\frac{4}{\tilde{v}}}-\frac{C_2(Q)}{C_1(Q)},\quad \text{for all}\quad k\ge 1.$$

(2) If $Q\ge 1$, we can deduce from Definition 1.3 that $n+Q-1\le\tilde{v}\le nQ$. Thus in our case in Theorem 1.1 $\tilde{v}=n+Q-1$ is the smallest. That means the lower bound estimates (1.6) will be optimal.

(3) If $f(\bar{x})=1$ in Theorem 1.1, then $Q=1$, $\Delta_X^2=\Delta^2$ is the standard bi-harmonic operator. Then $C_1(Q)=1$, $C_2(Q)=0$ and $\overline{C}(Q)=\frac{16\pi^4 n}{n+4}\left(\frac{\omega_{n-1}|\Omega|_n}{n}\right)^{-4/n}$. Thus the result of Theorem 1.1 will be the same to the result of (1.4) in Levine and Protter [15].

In the second part, we shall study the bi-subelliptic operators Δ_X^2 for more general cases, namely

$$X=(\partial_{x_1},\cdots,\partial_{x_{n-p}},f_1(\bar{x}_{(p)})\partial_{x_{n-p+1}},\cdots,f_p(\bar{x}_{(p)})\partial_{x_n}), \tag{1.7}$$

where $\bar{x}_{(p)}=(x_1,\cdots,x_{n-p})$,

$$f_j(\bar{x}_{(p)})=\sum\nolimits_{|\alpha|\le s_j}a_{j\alpha}\bar{x}_{(p)}^{\alpha}, \quad (1\le j\le p<n),$$

are multivariate polynomials of $\bar{x}_{(p)}$ with order s_j. Thus X is more general Grushin type degenerate vector fields with p degenerate directions. We suppose that

(H_2) : For each $j,1\le j\le p<n$, if $f_j(\bar{x}_{(p)})$ has a unique zero point at origin $\bar{x}_{(p)}=0$ in Ω only, and there exists a unique multi-index α_{0j} with $|\alpha_{0j}|=s_{0j}\le s_j$, satisfying $\partial_{\bar{x}_{(p)}}^{\alpha_{0j}}f_j(\bar{x}_{(p)})|_{\bar{x}_{(p)}=0}\neq 0$ and $\partial_{\bar{x}_{(p)}}^{\alpha}f_j(\bar{x}_{(p)})|_{\bar{x}_{(p)}=0}=0$ for any $|\alpha|<|\alpha_{0j}|$.

Thus we have

Theorem 1.2. *Under the condition (H_2) above, the vector fields X satisfies the Hörmander's condition with its Hörmander index $Q=\max\{s_{01},s_{02},\cdots,s_{0p}\}+1$, and the generalized Métivier index $\tilde{v}=n+\sum_{j=1}^{p}s_{0j}$. Suppose λ_j is the j-th eigenvalue of the problem* (1.1)*, then for all $k\ge 1$,*

$$\sum_{j=1}^{k}\lambda_j\ge \widehat{C}(Q)k^{1+\frac{4}{\tilde{v}}}-\frac{C_4(Q)}{C_3(Q)}k, \tag{1.8}$$

where

$$\widehat{C}(Q)=\frac{2^n}{5C_3(Q)n^{\frac{6+\tilde{v}}{2}}}\left(\frac{\tilde{v}}{\omega_{n-1}\prod\limits_{j=1}^{p}(s_{0j}+1)}\right)^{\frac{4+\tilde{v}}{\tilde{v}}}\left(\frac{(2\pi)^n}{|\Omega|_n}\right)^{\frac{4}{\tilde{v}}},$$

where $\tilde{v}=n+\sum_{j=1}^{p}s_{0j}$, $C_3(Q)$ and $C_4(Q)$ are the corresponding sub-elliptic estimate constants in Proposition 2.4, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω.

Remark 1.2. Since $k\lambda_k\ge\sum_{j=1}^{k}\lambda_j$, then Theorem 1.2 shows that the eigenvalues λ_k satisfy

$$\lambda_k\ge \widehat{C}(Q)k^{\frac{4}{\tilde{v}}}-\frac{C_4(Q)}{C_3(Q)}, \quad \text{for all} \quad k\ge 1.$$

Our paper is organized as follows. In Section 2, we introduce some preliminaries about subelliptic estimates and discreteness of the Dirichlet eigenvalues for the operator $-\Delta_X^2$. In Section 3, we prove Theorem 1.1. Finally, we prove Theorem 1.2 in Section 4.

2 Preliminaries

Proposition 2.1. Let the system of vector fields $X=(X_1,\cdots,X_m)$ satisfies the Hörmander's condition on Ω with its Hörmander index $Q\geq 1$, then the following estimate

$$\left\||\nabla|^{\frac{2}{Q}}u\right\|^2_{L^2(\Omega)}\leq C(Q)\|\Delta_X u\|^2_{L^2(\Omega)}+\widetilde{C}(Q)\|u\|^2_{L^2(\Omega)} \tag{2.1}$$

holds for all $u\in C_0^\infty(\Omega)$, where $\nabla=(\partial_{x_1},\cdots,\partial_{x_m})$, $|\nabla|^{\frac{2}{Q}}$ is a pseudo-differential operator with the symbol $|\xi|^{\frac{2}{Q}}$, the constants $C(Q)>0$, $\widetilde{C}(Q)\geq 0$ depending on Q.

Proof. Refer to [12] and [21], the subelliptic operator $\Delta_X=\sum_{i=1}^m X_i^2$ satisfies the following sub-elliptic estimate for any $u\in C_0^\infty(\Omega)$,

$$\|u\|_{(2\epsilon)}\leq C_1\|\Delta_X u\|_{L^2(\Omega)}+C_2\|u\|_{L^2(\Omega)},$$

with $\epsilon=\frac{1}{Q}$, where $\|u\|_{(2\epsilon)}$ is the Sobolev norm of order 2ϵ. On the other hand, we have

$$\begin{aligned}\|u\|_{(\frac{2}{Q})}&=\left(\int_n(1+|\xi|^2)^{\frac{2}{Q}}|\widehat{u}(\xi)|^2d\xi\right)^{\frac{1}{2}}\\&\geq\left(\int_n|\xi|^{\frac{4}{Q}}|\widehat{u}(\xi)|^2d\xi\right)^{\frac{1}{2}}\\&=\left\||\nabla|^{\frac{2}{Q}}u\right\|_{L^2(n)}=\left\||\nabla|^{\frac{2}{Q}}u\right\|_{L^2(\Omega)}.\end{aligned}$$

By using the Cauchy-Schwarz inequality we get the following estimate

$$\left\||\nabla|^{\frac{2}{Q}}u\right\|^2_{L^2(\Omega)}\leq C(Q)\|\Delta_X u\|^2_{L^2(\Omega)}+\widetilde{C}(Q)\|u\|^2_{L^2(\Omega)}.$$

Thus, we complete the proof. □

Proposition 2.2. (cf. [19, 21] and [22]) Let the system of vector fields $X=(X_1,\cdots,X_m)$ satisfies the Hörmander's condition on Ω, then the operator $\Delta_X=\sum\limits_{i=1}^m X_i^2$ is maximally hypo-elliptic, i.e., there exists a constant $C>0$, such that for any $u\in C_0^\infty(\Omega)$ we have the following maximally hypo-elliptic estimate

$$\sum_{|\alpha|\leq 2}\|X^\alpha u\|^2_{L^2(\Omega)}\leq C(\|\Delta_X u\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)}),$$

where $\alpha=(\alpha_1,\cdots,\alpha_m)$ is a multi-index with $|\alpha|=\alpha_1+\cdots+\alpha_m$ and $X^\alpha=X_1^{\alpha_1}\cdots X_m^{\alpha_m}$.

Proposition 2.3. Let $X=(\partial_{x_1},\cdots,\partial_{x_{n-1}},f(\bar{x})\partial_{x_n})$, $\bar{x}=(x_1,x_2,\cdots,x_{n-1})$. Here $f(\bar{x})$ is a multivariate polynomial and satisfies the condition (H_1) above. Then X satisfies the Hörmander's condition with its Hörmander index $Q\geq 1$, and we can deduce the following sub-elliptic estimate

$$\sum_{j=1}^{n-1}\|\partial_{x_j}^2 u\|_{L^2(\Omega)}^2+\Big\||\partial_{x_n}|^{\frac{2}{Q}}u\Big\|_{L^2(\Omega)}^2\leq C_1(Q)\|\Delta_X u\|_{L^2(\Omega)}^2+C_2(Q)\|u\|_{L^2(\Omega)}^2, \tag{2.2}$$

for all $u\in C_0^\infty(\Omega)$, where $|\partial_{x_n}|^{\frac{2}{Q}}$ is a pseudo-differential operator with the symbol $|\xi_n|^{\frac{2}{Q}}$, $C_1(Q)>0$, $C_2(Q)\geq 0$ are constants depending on Q.

Proof. From the Plancherel's formula, we have

$$\begin{aligned}\Big\||\partial_{x_n}|^{\frac{2}{Q}}u\Big\|_{L^2(\Omega)}^2&=\Big\||\xi_n|^{\frac{2}{Q}}\widehat{u}\Big\|_{L^2(\mathbb{R}^n)}^2\leq\Big\||\xi|^{\frac{2}{Q}}\widehat{u}\Big\|_{L^2(\mathbb{R}^n)}^2\\&=\Big\||\nabla|^{\frac{2}{Q}}u\Big\|_{L^2(\mathbb{R}^n)}^2=\Big\||\nabla|^{\frac{2}{Q}}u\Big\|_{L^2(\Omega)}^2.\end{aligned} \tag{2.3}$$

Also, from the maximally hypo-elliptic estimate of Proposition 2.2 we can deduce that

$$\sum_{j=1}^{n-1}\|\partial_{x_j}^2 u\|_{L^2(\Omega)}^2\leq\sum_{|\alpha|\leq 2}\|X^\alpha u\|_{L^2(\Omega)}^2\leq C(\|\Delta_X u\|_{L^2(\Omega)}^2+\|u\|_{L^2(\Omega)}^2). \tag{2.4}$$

Combining (2.1), (2.3) and (2.4) we can deduce that

$$\sum_{j=1}^{n-1}\|\partial_{x_j}^2 u\|_{L^2(\Omega)}^2+\Big\||\partial_{x_n}|^{\frac{2}{Q}}u\Big\|_{L^2(\Omega)}^2\leq C_1(Q)\|\Delta_X u\|_{L^2(\Omega)}^2+C_2(Q)\|u\|_{L^2(\Omega)}^2.$$

Thus, we complete the proof. □

Proposition 2.4. Let $X=(\partial_{x_1},\cdots,\partial_{x_{n-p}},f_1(\bar{x}_{(p)})\partial_{x_{n-p+1}},\cdots,f_p(\bar{x}_{(p)})\partial_{x_n})$, $\bar{x}_{(p)}=(x_1,x_2,\cdots,x_{n-p})$. Here $f_j(\bar{x}_{(p)})$ (for $1\leq j\leq p<n$) are multivariate polynomials which satisfying the condition $(H2)$ above. Then X satisfies the Hörmander's condition with its Hörmander index $Q\geq 1$, and we get the following sub-elliptic estimate

$$\sum_{i=1}^{n-p}\|\partial_{x_i}^2 u\|_{L^2(\Omega)}^2+\sum_{j=1}^{p}\Big\||\partial_{x_{n-p+j}}|^{\frac{2}{s_{0j}+1}}u\Big\|_{L^2(\Omega)}^2\leq C_3(Q)\|\Delta_X u\|_{L^2(\Omega)}^2+C_4(Q)\|u\|_{L^2(\Omega)}^2, \tag{2.5}$$

for all $u\in C_0^\infty(\Omega)$, where $|\partial_{x_j}|^{\frac{2}{r}}$ is a pseudo-differential operator with the symbol $|\xi_j|^{\frac{2}{r}}$, and the constants $C_3(Q)>0$, $C_4(Q)\geq 0$ depending on Q.

Proof. We consider the system of vector fields $\tilde{X}=(\partial_{x_1},\cdots,\partial_{x_{n-p}}, f_j(\bar{x}_{(p)})\partial_{x_{n-p+j}})$ (for $1\le j\le p<n$) defined on the projection $\Omega_{x'_j}$ of Ω on the direction $x'_j=(x_1,\cdots,x_{n-p},x_{n-p+j})$. Similar to Proposition 2.3, for all j $(1\le j\le p)$, we have

$$\sum_{i=1}^{n-p}\|\partial^2_{x_i}u\|^2_{L^2(\Omega_{x'_j})}+\Big\||\partial_{x_{n-p+j}}|^{\frac{2}{s_{0j}+1}}u\Big\|^2_{L^2(\Omega_{x'_j})}\le\widehat{C}_1(Q)\|\Delta_{\tilde{X}}u\|^2_{L^2(\Omega_{x'_j})}+\widehat{C}_2(Q)\|u\|^2_{L^2(\Omega_{x'_j})}.$$

Then for all j $(1\le j\le p)$, we have

$$\sum_{i=1}^{n-p}\|\partial^2_{x_i}u\|^2_{L^2(\Omega)}+\Big\||\partial_{x_{n-p+j}}|^{\frac{2}{s_{0j}+1}}u\Big\|^2_{L^2(\Omega)}\le\widehat{C}_1(Q)\|\Delta_{\tilde{X}}u\|^2_{L^2(\Omega)}+\widehat{C}_2(Q)\|u\|^2_{L^2(\Omega)}. \tag{2.6}$$

By using the Cauchy-Schwarz inequality and Proposition 2.2, there exists a constant $C_3>0$ such that

$$\|\Delta_{\tilde{X}}u\|^2_{L^2(\Omega)}\le C_3\sum_{|\alpha|\le 2}\|X^\alpha u\|^2_{L^2(\Omega)}\le C_3C(\|\Delta_X u\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)}),$$

where C is given in Proposition 2.2. Finally, we get the following sub-elliptic estimate from (2.6)

$$\sum_{i=1}^{n-p}\|\partial^2_{x_i}u\|^2_{L^2(\Omega)}+\sum_{j=1}^{p}\Big\||\partial_{x_{n-p+j}}|^{\frac{2}{s_{0j}+1}}u\Big\|^2_{L^2(\Omega)}\le C_3(Q)\|\Delta_X u\|^2_{L^2(\Omega)}+C_4(Q)\|u\|^2_{L^2(\Omega)}.$$

Thus, we complete the proof. $\square$

Next, for the Dirichlet eigenvalues problem (1.1), we have

Proposition 2.5. The Dirichlet eigenvalues problem (1.1) has a sequence of discrete eigenvalues $\{\lambda_j\}_{j\ge1}$, which satisfying $0<\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lambda_k\cdots$ and $\lambda_k\to+\infty$ as $k\to+\infty$. Also, the corresponding eigenfunctions $\{\phi_k(x)\}_{k\ge1}$ constitute an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H^2_{X,0}(\Omega)$.

The proof of Proposition 2.5 depends the following lemma:

Lemma 2.1. *If $u\in H^2_{X,0}(\Omega)$, then for $1\le j\le m$, $X_ju\in H^1_{X,0}(\Omega)$.*

Proof. Since $u\in H^2_{X,0}(\Omega)$, we have $X_i(X_ju)\in L^2(\Omega)$ for any $1\le i,j\le m$, and $(X_ju)\in L^2(\Omega)$. That implies $X_ju\in H^1_X(\Omega)$. Now, $u\in H^2_{X,0}(\Omega)$, then there exists a sequence $\varphi_i\in C_0^\infty(\Omega)$ which converges to u in $H^2_{X,0}(\Omega)$. That means $X_j\varphi_i\to X_ju$ in $H^1_X(\Omega)$. Observe that $X_j\varphi_i\in H^1_{X,0}(\Omega)$ and $H^1_{X,0}(\Omega)$ is a Hilbert space, thus we have $X_ju\in H^1_{X,0}(\Omega)$. $\square$

Proof of Proposition 2.5. We know that the definition domain of Δ_X^2 is

$$dom(\Delta_X^2)=\{u\in H^2_{X,0}(\Omega)|\Delta_X^2 u\in L^2(\Omega)\}.$$

Thus, for X_j to be formally skew-adjoint, then for any function $u\in C_0^\infty(\Omega)$ and $v\in dom(\Delta_X^2)$, we have

$$\begin{aligned}\int_\Omega u\Delta_X^2 v dx&=\int_\Omega v\Delta_X^2 u dx\\ &=\int_\Omega v\Delta_X(\Delta_X u)dx=\sum_{j=1}^m\int_\Omega v\cdot X_j^2(\Delta_X u)dx.\end{aligned}$$

Since $v\in H^2_{X,0}\subset H^1_{X,0}(\Omega)$, and from the result of Lemma 2.1, $X_jv\in H^1_{X,0}(\Omega)$. Then the equation above gives

$$\int_\Omega u\Delta_X^2 v dx=-\sum_{j=1}^m\int_\Omega X_jv\cdot X_j(\Delta_X u)dx=\sum_{j=1}^m\int_\Omega X_j^2 v\cdot(\Delta_X u)dx,$$

that gives the following Green formula:

$$\int_\Omega u\Delta_X^2 v dx=\int_\Omega \Delta_X u\cdot\Delta_X v dx,\quad \text{for}\quad u\in H^2_{X,0}(\Omega),\quad v\in dom(\Delta_X^2). \tag{2.7}$$

On the other hand, for $u\in H^2_{X,0}(\Omega)$,

$$\|u\|^2_{H^2_X}=\|u\|^2_{L^2(\Omega)}+\sum_{i=1}^m\|X_iu\|^2_{L^2(\Omega)}+\sum_{i,j=1}^m\|X_iX_ju\|^2_{L^2(\Omega)}.$$

Thus we have

$$\|u\|_{H^2_X}\ge\|u\|_{L^2(\Omega)}+\sum_{j=1}^m\|X_j^2u\|_{L^2(\Omega)}\ge\|\Delta_X u\|_{L^2(\Omega)}. \tag{2.8}$$

By maximally hypoellipticity of Δ_X (also see Proposition 2.2 above), we have following estimate for any $u\in H^2_{X,0}(\Omega)$,

$$\|u\|^2_{H^2_X}=\sum_{|\alpha|\le 2}\|X^\alpha u\|^2_{L^2(\Omega)}\le C(\|\Delta_X u\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)}). \tag{2.9}$$

Furthermore, the Poincaré inequality gives

$$\|u\|^2_{L^2(\Omega)}\le C_1\|Xu\|^2_{L^2(\Omega)}\le C_1|(\Delta_X u,u)|\le C_1\|\Delta_X u\|_{L^2(\Omega)}\cdot\|u\|_{L^2(\Omega)}.$$

Thus for any $0<\epsilon<1$ there is $C_\epsilon>0$, such that

$$\|\Delta_X u\|_{L^2(\Omega)}\cdot\|u\|_{L^2(\Omega)}\le C_\epsilon\|\Delta_X u\|^2_{L^2(\Omega)}+\epsilon\|u\|^2_{L^2(\Omega)}.$$

That means from (2.9) that there exists $C_2>0$, such that

$$\|u\|^2_{H^2_X} \le C_2\|\Delta_X u\|^2_{L^2(\Omega)}. \tag{2.10}$$

Hence from (2.8) and (2.10) one has for any $u\in H^2_{X,0}(\Omega)$,

$$\|\Delta_X u\| \le \|u\|_{H^2_X} \le C_3\|\Delta_X u\|. \tag{2.11}$$

Thus we define that

$$[u,\varphi]=(\Delta_X u,\Delta_X\varphi), \tag{2.12}$$

then $[\cdot,\cdot]$ is another inner product, and $H^2_{X,0}(\Omega)$ with this inner product is complete.

Now, we choose $u,v\in dom(\Delta^2_X)$, then

$$(\Delta^2_X u,v)=(\Delta_X u,\Delta_X v)=(\Delta^2_X v,u).$$

Hence, Δ^2_X is symmetric operator in $dom(\Delta^2_X)$. Also

$$(\Delta^2_X u,u)=(\Delta_X u,\Delta_X u)\ge 0,$$

which implies that Δ^2_X is positive in $dom(\Delta^2_X)$.

Next, for any given $f\in L^2(\Omega)$ and any $\varphi\in H^2_{X,0}(\Omega)$, we define a functional $f(\varphi)=(f,\varphi)$. Since

$$|(f,\varphi)|\le\|f\|_{L^2(\Omega)}\cdot\|\varphi\|_{L^2(\Omega)}\le\|f\|_{L^2(\Omega)}\cdot\|\varphi\|_{H^2_X(\Omega)},$$

then the functional (f,φ) is a continuous linear functional on Hilbert space $H^2_{X,0}(\Omega)$. By Riesz representation theorem, there exists a unique $u\in H^2_{X,0}(\Omega)$ such that

$$(f,\varphi)=[u,\varphi]=(\Delta_X u,\Delta_X\varphi).$$

Thus the Green formula (2.7) gives that

$$(\Delta^2_X u,\varphi)=(\Delta_X u,\Delta_X\varphi)=(f,\varphi) \tag{2.13}$$

holds for any $\varphi\in C^\infty_0(\Omega)$. That implies $\Delta^2_X u=f$, i.e., $u\in dom(\Delta^2_X)$. This proves the existence of the resolvent operator $R:=(\Delta^2_X)^{-1}$, and $Rf=u$.

On the other hand, if we choose $\varphi=u$ in (2.13), then $(Rf,f)=(u,f)=\|\Delta_X u\|^2_{L^2(\Omega)}\ge 0$. R is positive in $L^2(\Omega)$. Meanwhile we have

$$\|Rf\|^2_{L^2(\Omega)}=\|u\|^2_{L^2(\Omega)}\le C\|f\|_{L^2(\Omega)}\|Rf\|_{L^2(\Omega)},$$

this implies that R is bounded in $L^2(\Omega)$. In order to prove the operator R is self-adjoint, it suffices to prove that R is symmetric, i.e.,

$$(Rf,g)=(f,Rg) \quad \text{for all} \quad f,g\in L^2(\Omega).$$

Let $Rf=u$, $Rg=v$, and choosing $\varphi=v$ in (2.13), we obtain

$$(\Delta_X u,\Delta_X v)=(f,Rg).$$

Since the left hand side is symmetric in u and v, we conclude that the right side is symmetric in f and g. That implies that R is symmetric. Also, we know that the operator $R^{-1}:=\Delta_X^2$ is a self-adjoint on $dom(\Delta_X^2)$.

Similarly, we can prove that the inverse operator $(\Delta_X^2+\alpha\cdot id)^{-1}$ exists and is bounded for any $\alpha\geq 0$. We see that $-\alpha$ is a regular value of Δ_X^2, hence $\mathrm{spec}(\Delta_X^2)\subset(0,+\infty)$. Moreover, we can deduce that $R:L^2(\Omega)\to H^2_{X,0}(\Omega)$ is continuous, this is because that

$$\|Rf\|^2_{H^2_X}\leq C(\|\Delta_X(Rf)\|^2_{L^2(\Omega)})\leq C(f,Rf)\leq C\|f\|_{L^2(\Omega)}\|Rf\|_{L^2(\Omega)}\leq C\|f\|_{L^2(\Omega)}\|Rf\|_{H^2_X(\Omega)}.$$

By using the subelliptic estimate, we know that $H^2_{X,0}$ can be continuously embedded into the standard Sobolev space $H^{\frac{2}{Q}}(\Omega)$, and $H^{\frac{2}{Q}}(\Omega)$ can be compactly embedded into $L^2(\Omega)$. Hence R is a compact operator from $L^2(\Omega)$ to $L^2(\Omega)$. By spectral theory we know that R has positive discrete eigenvalues μ_i, $\mu_1\geq\mu_2\geq\cdots\geq\mu_k\geq\cdots$ and $\mu_k\to 0$ as $k\to+\infty$; and the corresponding eigenfunctions ϕ_i of R form an orthonormal basis of $L^2(\Omega)$, namely

$$R\phi_i=\mu_i\phi_i.$$

That means the eigenfunctions $\{\phi_i\}_{i\geq 1}$ will be the orthogonal basis of $H^2_{X,0}(\Omega)$. Finally we let $\lambda_i=\mu_i^{-1}$, then λ_i are the Dirichlet eigenvalues of Δ_X^2 which will be discrete and satisfying $0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_k\leq\cdots$, and $\lambda_k\to+\infty$ as $k\to+\infty$. The proof of Proposition 2.5 is completed. $\square$

3 Proof of Theorem 1.1

Lemma 3.1 (cf. [3,16])**.** *For the system of vector fields $X=(X_1,\cdots,X_m)$, if $\{\phi_j\}_{j=1}^k$ are the set of orthonormal eigenfunctions corresponding to the eigenvalues $\{\lambda_j\}_{j=1}^k$. Define*

$$\Phi(x,y)=\sum_{j=1}^k\phi_j(x)\phi_j(y).$$

Then for $\widehat{\Phi}(z,y)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}\Phi(x,y)e^{-ix\cdot z}dx$ to be the partial Fourier transformation of $\Phi(x,y)$ with respect to the x-variable, we have

$$\int_\Omega\int_{\mathbb{R}^n}\left|\widehat{\Phi}(z,y)\right|^2dzdy=k\quad\text{and}\quad\int_\Omega\left|\widehat{\Phi}(z,y)\right|^2dy\leq(2\pi)^{-n}|\Omega|_n.$$

Lemma 3.2 (cf. [8]). *Let f be a real-valued function defined on $\mathbb{R}^n$ with $0\le f\le M_1$, and for $Q\in\mathbb{N}^+$,*

$$\int_{\mathbb{R}^n}\left(\sum_{j=1}^{n-1}z_j^2+|z_n|^{\frac{2}{Q}}\right)^2 f(z)dz\le M_2.$$

Then

$$\int_{\mathbb{R}^n} f(z)dz\le \frac{(QM_1\omega_{n-1})^{\frac{4}{n+Q+3}}}{n+Q-1}\left(\frac{n(n+Q+3)}{A_Q}\right)^{\frac{n+Q-1}{n+Q+3}} M_2^{\frac{n+Q-1}{n+Q+3}},$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, and

$$A_Q=\begin{cases}\min\{1,n^{\frac{3-Q}{2}}\}, & Q\ge 2,\\ n, & Q=1.\end{cases}$$

Proof of Theorem 1.1. From the results of Proposition 2.5, let $\{\lambda_k\}_{k\ge1}$ be a sequence of the eigenvalues for the problem (1.1), and $\{\phi_k(x)\}_{k\ge1}$ be the corresponding eigenfunctions, then $\{\phi_k(x)\}_{k\ge1}$ constitute an orthogonal basis of $H^2_{X,0}(\Omega)$.

Let

$$\Phi(x,y)=\sum_{j=1}^{k}\phi_j(x)\phi_j(y),$$

by Cauchy-Schwarz inequality we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_\Omega\left(\sum_{j=1}^{n-1}z_j^2+|z_n|^{\frac{2}{Q}}\right)^2\left|\widehat{\Phi}(z,y)\right|^2dydz\\ \le& n\int_{\mathbb{R}^n}\int_\Omega\left(\sum_{j=1}^{n-1}z_j^4+|z_n|^{\frac{4}{Q}}\right)\left|\widehat{\Phi}(z,y)\right|^2dydz. \qquad (3.1)\end{aligned}$$

Next, by using integration-by-parts, we have

$$\begin{aligned}\sum_{j=1}^{k}\lambda_j&=\sum_{j=1}^{k}\int_\Omega\lambda_j\phi_j(x)\cdot\phi_j(x)dx=\sum_{j=1}^{k}\int_\Omega\Delta_X^2\phi_j(x)\cdot\phi_j(x)dx\\ &=\sum_{j=1}^{k}\int_\Omega X(\Delta_X\phi_j(x))\cdot X\phi_j(x)dx=\sum_{j=1}^{k}\int_\Omega\Delta_X\phi_j(x)\cdot\Delta_X\phi_j(x)dx\\ &=\int_\Omega\int_\Omega\sum_{j=1}^{k}\left|\Delta_X\phi_j(x)\phi_j(y)\right|^2dxdy=\int_\Omega\int_\Omega|\Delta_X\Phi(x,y)|^2dxdy. \qquad (3.2)\end{aligned}$$

Then by using Plancherel's formula and Proposition 2.3, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{j=1}^{n-1} z_j^2+|z_n|^{\frac{2}{Q}}\right)^2\left|\widehat{\Phi}(z,y)\right|^2 dydz\\
\leq& n\int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{j=1}^{n-1} z_j^4+|z_n|^{\frac{4}{Q}}\right)\left|\widehat{\Phi}(z,y)\right|^2 dydz\\
=& n\int_{n}\int_{\Omega}\left(\sum_{j=1}^{n-1}|\partial_{x_j}^2\Phi(x,y)|^2+\left||\partial_{x_n}|^{\frac{2}{Q}}\Phi(x,y)\right|^2\right)dydx\\
=& n\int_{\Omega}\int_{\Omega}\left(\sum_{j=1}^{n-1}|\partial_{x_j}^2\Phi(x,y)|^2+\left||\partial_{x_n}|^{\frac{2}{Q}}\Phi(x,y)\right|^2\right)dydx\\
\leq& n\left[C_1(Q)\int_{\Omega}\int_{\Omega}|\Delta_X\Phi(x,y)|^2dxdy+C_2(Q)\int_{\Omega}\int_{\Omega}|\Phi(x,y)|^2dxdy\right]. \qquad (3.3)
\end{aligned}
$$

Thus from (3.2) and Lemma 3.1 above, we can deduce that

$$\int_{n}\int_{\Omega}\left(\sum_{j=1}^{n-1} z_j^2+|z_n|^{\frac{2}{Q}}\right)^2\left|\widehat{\Phi}(z,y)\right|^2 dydz\leq n\left(C_1(Q)\sum_{j=1}^{k}\lambda_j+C_2(Q)k\right).$$

Next, we choose

$$f(z)=\int_{\Omega}\left|\widehat{\Phi}(z,y)\right|^2dy,\quad M_1=(2\pi)^{-n}|\Omega|_n,\quad M_2=n\left(C_1(Q)\sum_{j=1}^{k}\lambda_j+C_2(Q)k\right).$$

Then from the result of Lemma 3.2, we know that for any $k\geq 1$,

$$
\begin{aligned}
&k\\
\leq&\frac{Q\omega_{n-1}(2\pi)^{-n}|\Omega|_n}{n+Q-1}\left(\frac{n(n+Q+3)}{(2\pi)^{-n}|\Omega|_n QA_Q\omega_{n-1}}\right)^{\frac{n+Q-1}{n+Q+3}}\left(n\left(C_1(Q)\sum_{j=1}^{k}\lambda_j+C_2(Q)k\right)\right)^{\frac{n+Q-1}{n+Q+3}}.
\end{aligned}
$$

This means, for any $k\geq 1$,

$$\sum_{j=1}^{k}\lambda_j\geq\widetilde{C}(Q)k^{1+\frac{4}{\tilde{v}}}-\frac{C_2(Q)}{C_1(Q)}k,$$

with

$$\widetilde{C}(Q)=\frac{A_Q}{C_1(Q)n^2(n+Q+3)}\left(\frac{(2\pi)^n}{Q\omega_{n-1}|\Omega|_n}\right)^{\frac{4}{n+Q-1}}(n+Q-1)^{\frac{n+Q+3}{n+Q-1}}.$$

The proof of Theorem 1.1 is completed. $\square$

4 Proof of Theorem 1.2

Lemma 4.1. *Let f be a real-valued function defined on $\mathbb{R}^n$ with $0\le f\le M_1$, and for $p,q\in\mathbb{N}^+$,*

$$\int_{\mathbb{R}^n}\left(\sum_{i=1}^{n-p} z_i^2+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\right)^2 f(z)dz\le M_2.$$

Then

$$\int_{\mathbb{R}^n} f(z)dz\le \frac{\omega_{n-1}\prod\limits_{j=1}^{p}(s_{0j}+1)}{\tilde{v}}\left(\frac{5n^{\frac{4+\tilde{v}}{2}}}{2^n}\right)^{\frac{\tilde{v}}{4+\tilde{v}}} M_1^{\frac{4}{4+\tilde{v}}} M_2^{\frac{\tilde{v}}{4+\tilde{v}}},$$

where $\tilde{v}=n+\sum_{j=1}^{p}s_{0j}$, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Proof. First, we choose R such that

$$\int_{\mathbb{R}^n}\left(\sum_{i=1}^{n-p} z_i^2+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\right)^2 g(z)dz=M_2,$$

where

$$g(z)=\begin{cases} M_1, & \sum\limits_{i=1}^{n-p} z_i^2+\sum\limits_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\le R^2,\\ 0, & \sum\limits_{i=1}^{n-p} z_i^2+\sum\limits_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}> R^2.\end{cases}$$

Then

$$\left[\left(\sum_{i=1}^{n-p} z_i^2+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\right)^2-R^4\right](f(z)-g(z))\ge 0.$$

Hence we have

$$R^4\int_{\mathbb{R}^n}(f(z)-g(z))dz\le\int_{\mathbb{R}^n}\left(\sum_{i=1}^{n-p} z_i^2+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\right)^2(f(z)-g(z))dz\le 0.$$

That means

$$\int_{\mathbb{R}^n} f(z)dz\le\int_{\mathbb{R}^n} g(z)dz. \tag{4.1}$$

Now we have

$$M_2=\int_{\mathbb{R}^n}\left(\sum_{i=1}^{n-p} z_i^2+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\right)^2 g(z)dz=M_1\int_{\widetilde{B}_R}\left(\sum_{i=1}^{n-p} z_i^2+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\right)^2 dz,$$

where

$$\widetilde{B}_R=\left\{z\in\mathbb{R}^n,\quad \sum_{i=1}^{n-p} z_i^2+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\le R^2\right\}.$$

Next, we change the variables as follows,

$$z_i=z_i' \quad (i=1,2,\cdots,n-p),\quad z_{n-p+j}=sgn(z'_{n-p+j})|z'_{n-p+j}|^{s_{0j}+1},\quad (j=1,2,\cdots,p).$$

Then we have the following determinant of Jacobian,

$$\left|\det\Big(\frac{\partial(z_1,\cdots,z_n)}{\partial(z_1',\cdots,z_n')}\Big)\right|=\prod_{j=1}^{p}(s_{0j}+1)|z'_{n-p+j}|^{s_{0j}}.$$

Hence

$$\begin{aligned}
M_2=&M_1\int_{\widetilde{B}_R}\left(\sum_{i=1}^{n-p} z_i^2+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\right)^2 dz\\
=&M_1\prod_{j=1}^{p}(s_{0j}+1)\int_{B_R}|z|^4\prod_{j=1}^{p}|z_{n-p+j}|^{s_{0j}}dz\\
\ge&M_1\prod_{j=1}^{p}(s_{0j}+1)\int_{A_R}|z|^4\prod_{j=1}^{p}|z_{n-p+j}|^{s_{0j}}dz,
\end{aligned}$$

where

$$B_R=\{z\in\mathbb{R}^n,\,|z|\le R\},\quad A_R=\left\{z\in\mathbb{R}^n,\,|z_j|\le\frac{R}{\sqrt{n}},\,j=1,\cdots,n\right\}.$$

By a direct calculation, we have

$$\begin{aligned}
&\int_{A_R}|z|^4\prod_{j=1}^{p}|z_{n-p+j}|^{s_{0j}}dz\\
\ge&\int_{A_R}|z_1|^4\prod_{j=1}^{p}|z_{n-p+j}|^{s_{0j}}dz\\
=&2\int_0^{\frac{R}{\sqrt{n}}}|z_1|^4dz_1\times\prod_{j=1}^{p}\left(2\int_0^{\frac{R}{\sqrt{n}}}|z_{n-p+j}|^{s_{0j}}dz_{n-p+j}\right)\times\left(2\int_0^{\frac{R}{\sqrt{n}}}1dz\right)^{n-p-1}\\
=&\frac{2^n}{5}\frac{1}{\prod\limits_{j=1}^{p}(s_{0j}+1)}n^{-\frac{n+4+\sum_{j=1}^{p}s_{0j}}{2}}R^{n+4+\sum_{j=1}^{p}s_{0j}}=\frac{2^n}{5}\frac{1}{\prod\limits_{j=1}^{p}(s_{0j}+1)}n^{-\frac{4+\tilde{v}}{2}}R^{4+\tilde{v}}.
\end{aligned}$$

Then we have

$$M_2\ge\frac{2^nM_1}{5}n^{-\frac{4+\tilde{v}}{2}}R^{4+\tilde{v}}. \tag{4.2}$$

From the definition of $g(z)$, we know that

$$\begin{aligned}\int_{\mathbb{R}^n} g(z)dz=&M_1\int_{\widetilde{B}_R} dz=M_1\prod_{j=1}^{p}(s_{0j}+1)\int_{B_R}\prod_{j=1}^{p}|z_{n-p+j}|^{s_{0j}}dz\\ &\le M_1\prod_{j=1}^{p}(s_{0j}+1)\int_{B_R}|z|^{\sum\limits_{j=1}^{p}s_{0j}}dz=M_1\prod_{j=1}^{p}(s_{0j}+1)\int_0^R\omega_{n-1}r^{n-1+\sum\limits_{j=1}^{p}s_{0j}}dr\\ &=\frac{M_1\omega_{n-1}\prod\limits_{j=1}^{p}(s_{0j}+1)}{n+\sum\limits_{j=1}^{p}s_{0j}}R^{n+\sum\limits_{j=1}^{p}s_{0j}}=\frac{M_1\omega_{n-1}\prod\limits_{j=1}^{p}(s_{0j}+1)}{\tilde{v}}R^{\tilde{v}}.\end{aligned}\tag{4.3}$$

From (4.1), (4.2) and (4.3), we obtain

$$\int_{\mathbb{R}^n} f(z)dz\le\int_{\mathbb{R}^n} g(z)dz\le\frac{\omega_{n-1}\prod\limits_{j=1}^{p}(s_{0j}+1)}{\tilde{v}}\left(\frac{5n^{\frac{4+\tilde{v}}{2}}}{2^n}\right)^{\frac{\tilde{v}}{4+\tilde{v}}}M_1^{\frac{4}{4+\tilde{v}}}M_2^{\frac{\tilde{v}}{4+\tilde{v}}},$$

where $\tilde{v}=n+\sum_{j=1}^{p}s_{0j}$. Lemma 4.1 is proved. □

Proof of Theorem 1.2. Let $\{\lambda_k\}_{k\ge1}$ be a sequence of the eigenvalues for the problem (1.1), $\{\phi_k(x)\}_{k\ge1}$ be the corresponding eigenfunctions. Then $\{\phi_k(x)\}_{k\ge1}$ constitute an orthogonal basis of $H^2_{X,0}(\Omega)$.

Let $\Phi(x,y)=\sum_{j=1}^{k}\phi_j(x)\phi_j(y)$. Thus, by using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{j=1}^{n-p}z_j^2+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\right)^2\left|\widehat{\Phi}(z,y)\right|^2dydz\\ \le& n\int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{j=1}^{n-p}z_j^4+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{4}{s_{0j}+1}}\right)\left|\widehat{\Phi}(z,y)\right|^2dydz.\end{aligned}\tag{4.4}$$

Similar to the result of (3.2), we obtain that

$$\sum_{j=1}^{k}\lambda_j=\int_{\Omega}\int_{\Omega}|\Delta_X\Phi(x,y)|^2dxdy.\tag{4.5}$$

Then by using Plancherel's formula and Proposition 2.4, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{j=1}^{n-p}z_j^2+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\right)^2\left|\widehat{\Phi}(z,y)\right|^2dydz\\ \le& n\int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{j=1}^{n-p}z_j^4+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{4}{s_{0j}+1}}\right)\left|\widehat{\Phi}(z,y)\right|^2dydz\end{aligned}$$

$$
\begin{aligned}
&= n\int_{\mathbb{R}^n}\int_\Omega \left(\sum_{j=1}^{n-p}|\partial_{x_j}^2\Phi(x,y)|^2+\sum_{j=1}^{p}\left||\partial_{x_{n-p+j}}|^{\frac{2}{s_{0j}+1}}\Phi(x,y)\right|^2\right)dydx\\
&= n\int_{\Omega}\int_\Omega \left(\sum_{j=1}^{n-p}|\partial_{x_j}^2\Phi(x,y)|^2+\sum_{j=1}^{p}\left||\partial_{x_{n-p+j}}|^{\frac{2}{s_{0j}+1}}\Phi(x,y)\right|^2\right)dydx\\
&\le n\left[C_3(Q)\int_\Omega\int_\Omega|\Delta_X\Phi(x,y)|^2dxdy+C_4(Q)\int_\Omega\int_\Omega|\Phi(x,y)|^2dxdy\right].
\end{aligned}
$$

Thus from (4.5) and Lemma 3.1 above, we can deduce that

$$
\int_{\mathbb{R}^n}\int_\Omega\left(\sum_{j=1}^{n-p}z_j^2+\sum_{j=1}^{p}|z_{n-p+j}|^{\frac{2}{s_{0j}+1}}\right)^2\left|\widehat{\Phi}(z,y)\right|^2dydz\le n\left(C_3(Q)\sum_{j=1}^{k}\lambda_i+C_4(Q)k\right).
$$

Finally, we choose

$$
f(z)=\int_\Omega|\widehat{\Phi}(z,y)|^2dy,\quad M_1=(2\pi)^{-n}|\Omega|_n,\quad M_2=n\left(C_3(Q)\sum_{i=1}^{k}\lambda_i+C_4(Q)k\right).
$$

Then from the Lemma 4.1, we have for any $k\ge1$,

$$
k\le\frac{\omega_{n-1}\prod_{j=1}^{p}(s_{0j}+1)}{\tilde{v}}\left((2\pi)^{-n}|\Omega|_n\right)^{\frac{4}{4+\tilde{v}}}\left(\frac{5n^{\frac{4+\tilde{v}}{2}}}{2^n}\right)^{\frac{\tilde{v}}{4+\tilde{v}}}\left(n\left(C_3(Q)\sum_{j=1}^{k}\lambda_j+C_4(Q)k\right)\right)^{\frac{\tilde{v}}{4+\tilde{v}}}.
$$

This means, for any $k\ge1$,

$$
\sum_{j=1}^{k}\lambda_j\ge\widehat{C}(Q)k^{1+\frac{4}{\tilde{v}}}-\frac{C_4(Q)}{C_3(Q)}k,
$$

where $\tilde{v}=n+\sum_{j=1}^{p}s_{0j}$, and

$$
\widehat{C}(Q)=\frac{2^n}{5C_3(Q)n^{\frac{6+\tilde{v}}{2}}}\left(\frac{\tilde{v}}{\omega_{n-1}\prod_{j=1}^{p}(s_{0j}+1)}\right)^{\frac{4+\tilde{v}}{\tilde{v}}}\left(\frac{(2\pi)^n}{|\Omega|_n}\right)^{\frac{4}{\tilde{v}}}.
$$

The proof of Theorem 1.2 is completed. □

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