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Vector Solutions with Prescribed Component-Wise Nodes for a Schrödinger System

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Abstract. For the Schrödinger system

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j \quad \text{in } \mathbb{R}^N, \\ u_j(x) \to 0 \quad \text{as } |x| \to \infty, \quad j = 1, \cdots, k, \end{cases}$$

where $k \geq 2$ and $N = 2,3$, we prove that for any $\lambda_j > 0$ and $\beta_{jj} > 0$ and any positive integers p_j, $j = 1,2,\cdots,k$, there exists $b > 0$ such that if $\beta_{ij} = \beta_{ji} \leq b$ for all $i \neq j$ then there exists a radial solution $(u_1, u_2, \cdots, u_k)$ with u_j having exactly $p_j - 1$ zeroes. Moreover, there exists a positive constant C_0 such that if $\beta_{ij} = \beta_{ji} \leq b$ $(i \neq j)$ then any solution obtained satisfies

$$\sum_{i,j=1}^{k} |\beta_{ij}| \int_{\mathbb{R}^N} u_i^2 u_j^2 \leq C_0.$$

Therefore, the solutions exhibit a trend of phase separations as $\beta_{ij} \to -\infty$ for $i \neq j$.

Key Words: Vector solution, prescribed component-wise nodes, Schrödinger system, variational methods.
AMS Subject Classifications: 35A15, 35J10, 35J50

1 Introduction

We consider the coupled Schrödinger system

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j \quad \text{in } \mathbb{R}^N, \\ u_j(x) \to 0 \quad \text{as } |x| \to \infty, \quad j = 1, \ldots, k, \end{cases} \tag{1.1}$$

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where $k\geq 2$ and $N=2,3$. We assume $\lambda_j>0$, $\beta_{jj}>0$, and $\beta_{ij}=\beta_{ji}$ $(j\neq i)$ are constants.

This type of systems arises when one considers standing wave solutions of time-dependent k-coupled Schrödinger systems of the form

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Phi_j=\Delta\Phi_j-V_j(x)\Phi_j+\mu_j|\Phi_j|^2\Phi_j+\Phi_j\sum\limits_{i=1,i\neq j}^{k}\beta_{ij}|\Phi_i|^2 & \text{in } \mathbb{R}^N, \\ \Phi_j=\Phi_j(x,t)\in\mathbb{C} & t>0, \quad j=1,\cdots,k. \end{cases} \tag{1.2}$$

These systems of equations, also known as coupled Gross-Pitaevskii equations, have applications in many physical problems (see [1,27]) in particular in Bose-Einstein condensates theory for multispecies Bose-Einstein condensates (see [10,14,32,40]) which have been studied intensively in the last twenty years. Physically, β_{jj} and β_{ij} $(i\neq j)$ are the intraspecies and interspecies scattering lengths respectively. The sign of the scattering length determines whether the interactions of states are repulsive or attractive. In the attractive case ($\beta_{ij}>0$ for $i\neq j$) the components of a vector solution tend to go along with each other leading to synchronization. And in the repulsive case ($\beta_{ij}<0$ for $i\neq j$) the components tend to segregate component-wisely, leading to phase separations and much more complicated behaviors of solutions.

Mathematical properties of systems of nonlinear Schrödinger equations have been studied extensively in recent years; see, e.g., [2–6,8,10–13,15–23,25,26,28,30,33–39,41–43] and references therein. Phase separation has been proved in several cases with constant potentials such as in the work [4,10–12,30,38,42,43] as the coupling constant β tends to negative infinity in the repulsive case. It is quite natural to assert that due to segregation in the repulsive case the structures of vector solutions are much richer and more complex. In particular, in the repulsive case, multiplicity of positive solutions has been established in [12,38,39,42], multiple non-trivial vector solutions were constructed in [25,26], and multiple sign-changing solutions have been given in [22,23,35]. There has been progress for the mixed coupling cases and, due to the repulsive effects, there exist many distinct types of solutions exhibiting partial synchronization and partial segregation phenomena (see, e.g., [8,31,33,34,37]). Due to the above existing work, we remark that there are new difficulties in dealing with the existence of multiple sign-changing solutions. First, there are many semi-trivial solutions due to systems collapsing, i.e., there are solutions of the form in which one or more components are zero so they are solutions of systems of fewer number of equations. Second, there can exist (infinitely) many positive solutions. For the totally symmetric case ($\lambda_j=\lambda>0$ and $\mu_j=\mu>0$ for all j, and $\beta_{ij}=\beta$ for all $i\neq j$), in [38] radial solutions with domain separations are constructed using variational methods and perturbation methods for k-systems, and in [12,43] minimax method is used to give infinitely many radial positive solutions for 2-systems (see also [39] for generalizations to the k-systems). These radial solutions demonstrate segregation nature. Segregated radial solutions were obtained in repulsive case in [4] by global bifurcation methods for systems (1.1) with $k=2$ establishing the existence of infinitely many branches of radial solutions with the property that a weighted difference between the two components of solutions along the m-th branch has exactly m nodal domains. While these results are all for positive solutions there has been steady progress in constructing sign-changing solutions. In [25,26], we proved the existence of many solutions of non-trivial, sign-changing nature (though the sign-changing property was not established). This was done by constructing invariant sets of the associated negative gradient flow in such a way that neighborhoods of coordinates planes are invariant sets and solutions are constructed outside these neighborhoods. In [22,23], by constructing invariant sets of the gradient flow containing positive or negative cones the existence of multiple sign-changing solutions are established. More precisely, for any positive integer m with $1\le m\le k$, there exist infinitely many solutions such that the first m components are sign-changing and the last $k-m$ components are one-sign functions.

Motivated by these works, one natural question arises. The question is whether we can provide more accurately existence and quantitative information for these vector sign-changing solutions. One special case is to establish existence of solutions with prescribed component-wise number of nodal domains, at least in the radially symmetric case. In this paper we consider radial solutions in a case in which the coupling constants β_{ij} $(i\neq j)$ are slightly less constrained than repulsiveness and our question is more specifically formulated as follows: Given k positive integers $p_1,\cdots,p_k$, does there exist a solution $(u_1,\cdots,u_k)$ such that u_i has exactly p_i-1 zeroes, for $i=1,\cdots,k$? We will give a positive answer to this problem.

Theorem 1.1. *Assume $k\ge 2$, $N=2,3$, $\lambda_j>0$ and $\beta_{jj}>0$ for $j=1,\cdots,k$. Let $p_1,\cdots,p_k$ be any positive integers. Then there exists a constant $b>0$ depending only on λ_j, β_{jj}, and p_j such that for any β_{ij} satisfying $\beta_{ij}\le b$ $(i\neq j)$ (1.1) has a radially symmetric solution $(u_1,\cdots,u_k)$ such that the i^{th} component u_i has exactly p_i-1 simple zeroes. Moreover, there exists a constant C_0 independent of β_{ij} $(i\neq j)$ such that the solutions obtained satisfy*

$$\sum_{i,j=1}^{k}|\beta_{ij}|\int_{\mathbb{R}^N}u_i^2u_j^2\le C_0$$

for all $\beta_{ij}\le b$ $(i\neq j)$.

We remark here that the uniform energy bound should lead these solutions to exhibit a trend of phase separations as $\beta_{ij}\to-\infty$ $(i\neq j)$, though we do not pursuit this further.

As elaborated above, in the repulsive case, since there exist a variety of different types of solutions with very distinct qualitative properties, it is important to know more about the properties of the solutions constructed. Our result gives component-wisely prescribed nodal information. In addition, our result covers the case where the coupling constants are slightly less constrained than repulsiveness and besides nodal information the solutions obtained exhibit a trend of phase separations as $\beta_{ij}\to-\infty$ $(i\neq j)$. The proofs in Section 2 show that b has an expression in terms of $B=\max_j\beta_{jj}$, $\nu=\min_j\lambda_j$, $p=\sum_j p_j$, κ, τ, and μ_0, where κ and τ are constants from Sobolev imbeddings and depend only on ν, and μ_0 is a constant which can be chosen such that it depends only on B, ν, and p. In this way, b depends only on B, ν, and p. The constant C_0 has an expression in terms of B, κ, and μ_0, and therefore it also depends only on B, ν, and p.

We make use of an idea based on the Nehari manifold technique, gluing pieces of solutions together to form sing-changing solutions with prescribed component-wise nodal domains. This idea was initiated in Nehari [29] in proving existence of infinitely many sign-changing solutions for a class of ordinary differential equations, and was used independently by Bartsch-Willem in [7] and Cao-Zhu in [9] to study sign-changing solutions of elliptic partial differential equations with radial symmetry. The argument and result in [7] were refined in our own work [24]. For systems of elliptic equations, there are many different variants of the Nehari manifold depending on how the equations are grouped (and/or how the domain is partitioned) to form the constraints in defining the manifold (see [8, 15, 26, 36]). If a Nehari manifold has to be defined by more than one constraints then there will be more than one Lagrangian multipliers and, in many cases, these Lagrangian multipliers can not be shown to be zero and accordingly such a Nehari manifold does not produce a solution. The Nehari manifold defined below consists of $p_1+p_2+\cdots+p_k$ constraints and it is the repulsive nature among components of the system in question which makes it possible for all the Lagrangian multipliers to be zero. In fact, this repulsive nature is the main cause in each key step of the approach.

The idea used in this paper can be easily adapted to prove the same result for the more general system

$$\begin{cases} -\Delta u_j+\lambda_j u_j=\sum\limits_{i=1}^{k}\beta_{ij}|u_i|^q|u_j|^{q-2}u_j & \text{in } \mathbb{R}^N, \\ u_j(x)\to 0 \quad \text{as } \ |x|\to\infty, & j=1,\cdots,k, \end{cases}$$

where $k\geq 2$, $N\geq 2$, $\lambda_j>0$, $\beta_{jj}>0$, $\beta_{ij}=\beta_{ji}$ $(j\neq i)$, and $1<q<\frac{N}{N-2}$ if $N>2$ and $1<q<\infty$ if $N=2$. Furthermore, the arguments of this paper can be generalized to prove similar results for even more general quasilinear systems including k-coupled systems of m-Laplacian equations.

The proof of the main result is contained in Section 2.

2 The proof of the main result

Denote

$$B=\max\{\beta_{jj}|j=1,2,\cdots,k\}. \tag{2.1}$$

Let $H_r^1(\mathbb{R}^N)$ be the subspace of the Sobolev space $H^1(\mathbb{R}^N)$ consisting of all the radially symmetric functions. We shall use the following equivalent norms

$$\|u\|_i=\left(\int_{\mathbb{R}^N}|\nabla u|^2+\lambda_i u^2\right)^{1/2}, \quad i=1,\cdots k,$$

in $H_r^1(\mathbb{R}^N)$. Choose a positive number $\kappa>0$ depending only on the number ν defined as

$$\nu=\min\{\lambda_j | j=1,2,\cdots,k\}, \tag{2.2}$$

such that, for all $i=1,\cdots,k$ and all $u\in H_r^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} u^4 \le \kappa \|u\|_i^4. \tag{2.3}$$

It is known that radially symmetric solutions of (1.1) correspond to critical points of the functional

$$\Phi(u_1,\cdots,u_k)=\frac{1}{2}\sum_{i=1}^k \|u_i\|_i^2-\frac{1}{4}\sum_{i,j=1}^k \beta_{ij}\int_{\mathbb{R}^N} u_i^2u_j^2$$

defined for $(u_1,\cdots,u_k)\in(H_r^1(\mathbb{R}^N))^k$. However, this functional Φ is not the right functional to be used to obtain the desired solution.

For $i=1,\cdots,k$, we divide the positive half axis $[0,+\infty)$ into p_i subintervals and let

$$P_i:\ 0<r_{i,1}<\cdots<r_{i,p_i-1}<+\infty,$$

be the (p_i-1) points of division. Set $\mathbb{P}=(P_1,\cdots,P_k)$ and we shall say that $\mathbb{P}$ is a k-time partition of $[0,+\infty)$ with respect to $(p_1,\cdots,p_k)$, or just a k-time partition for short. We use the notations $r_{i,0}=0$ and $r_{i,p_i}=+\infty$, and set, for $l=1,\cdots,p_i$ and $i=1,\cdots,k$,

$$\Omega_{i,l}(\mathbb{P})=\mathrm{int}\{x\in\mathbb{R}^N | r_{i,l-1}\le|x|<r_{i,l}\}.$$

Denote

$$E_{i,l}=E_{i,l}(\mathbb{P})=\{u\in H_r^1(\mathbb{R}^N)\ |\mathrm{supp}(u)\subset\overline{\Omega_{i,l}(\mathbb{P})}\}$$

and

$$E=E(\mathbb{P})=E_{1,1}\times\cdots\times E_{1,p_1}\times\cdots\times E_{k,1}\times\cdots\times E_{k,p_k},$$

and we shall make use of the functional $J_{\mathbb{P}}:E\to\mathbb{R}$ defined as

$$\begin{aligned}&J_{\mathbb{P}}(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})\\ =&\frac{1}{2}\sum_{i=1}^k\sum_{l=1}^{p_i}\|u_{i,l}\|_i^2-\frac{1}{4}\sum_{i=1}^k\sum_{l=1}^{p_i}\sum_{j=1}^k\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2.\end{aligned}$$

Then E is a subspace of $(H_r^1(\mathbb{R}^N))^p$, where

$$p=p_1+p_2+\cdots+p_k. \tag{2.4}$$

The functional $J_{\mathbb{P}}$ will play the key role in the approach for finding the desired solution. Clearly, for $(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})\in E$, we have

$$J_{\mathbb{P}}(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})=\Phi\Big(\sum_{l=1}^{p_1}u_{1,l},\cdots,\sum_{l=1}^{p_k}u_{k,l}\Big).$$

Choose $u^*=u^*(\mathbb{P})=(u^*_{1,1},\cdots,u^*_{1,p_1},\cdots,u^*_{k,1},\cdots,u^*_{k,p_k})\in E(\mathbb{P})$ such that $u^*_{i,l}\neq 0$ for each subscript index (i,l) and $\text{supp}(u^*_{i,l})\cap\text{supp}(u^*_{j,m})=\emptyset$ if $(i,l)\neq(j,m)$. Multiplying $u^*_{i,l}$ with a positive number if necessary, we may assume that for each (i,l),

$$\|u^*_{i,l}\|_i^2=\beta_{ii}\int_{\mathbb{R}^N}(u^*_{i,l})^4.$$

Then clearly

$$\|u^*_{i,l}\|_i^2=\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}(u^*_{i,l})^2(u^*_{j,m})^2.$$

Define the Nehari type set

$$\begin{aligned}\mathcal{N}_0(\mathbb{P})=\Big\{(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})\ \Big|\ &u_{i,l}\in E_{i,l},\ u_{i,l}\neq 0,\\ &\|u_{i,l}\|_i^2=\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\Big\}.\end{aligned}$$

Then $\mathcal{N}_0(\mathbb{P})$ is nonempty since $u^*\in\mathcal{N}_0(\mathbb{P})$. Define the minimization problem

$$c(\mathbb{P})=\inf J_{\mathbb{P}}|_{\mathcal{N}_0(\mathbb{P})}=\inf_{u\in\mathcal{N}_0(\mathbb{P})}\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|u_{i,l}\|_i^2.$$

We shall make use of the following Nehari type set modified from $\mathcal{N}_0(\mathbb{P})$:

$$\begin{aligned}\mathcal{N}(\mathbb{P})=\Big\{(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})\Big|&u_{i,l}\in E_{i,l},\quad u_{i,l}\neq 0,\\ \|u_{i,l}\|_i^2=\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2,\ &\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|u_{i,l}\|_i^2<2c(\mathbb{P})\Big\}.\end{aligned}$$

Now the above minimization problem can be written as

$$c(\mathbb{P})=\inf J_{\mathbb{P}}|_{\mathcal{N}(\mathbb{P})}.$$

Since

$$c(\mathbb{P})\le\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|u^*_{i,l}\|_i^2$$

and since the choice of u^* is independent of β_{ij} $(i\neq j)$, $c(\mathbb{P})$ has an upper bound independent of β_{ij} $(i\neq j)$.

Note that

$$\int_{\mathbb{R}^N}u_{i,l}^2u_{i,m}^2=0\quad\text{for all }\ u\in E(\mathbb{P}),\text{ all }\ i,\ \text{ and all }\ l,m\in\{1,\cdots,p_i\}\quad\text{with }\ l\neq m.$$

This fact will be repeatedly used in what follows.

Lemma 2.1. *Let $\mathbb{P}$ be fixed and let κ be the number from* (2.3)*. Define*

$$b_{1\mathbb{P}} = \frac{1}{32c(\mathbb{P})\kappa}$$

and assume $\beta_{ij} \le b_{1\mathbb{P}}$ for all $i \neq j$. Then $\mathcal{N}(\mathbb{P})$ is a smooth submanifold of E.

Proof. For any subscript index (i,l), define $G_{i,l}: E \to \mathbb{R}$ as

$$G_{i,l}(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k}) = \|u_{i,l}\|_i^2 - \sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N} u_{i,l}^2 u_{j,m}^2.$$

Then all the $G_{i,l}$'s are smooth functionals and $\mathcal{N}(\mathbb{P})$ is an open subset of

$$\bigcap_{i=1}^{k}\bigcap_{l=1}^{p_i}[G_{i,l}^{-1}(0)\setminus\{0\}].$$

To see that $\mathcal{N}(\mathbb{P})$ is a smooth submanifold of E, it suffices to show that if $u \in \mathcal{N}(\mathbb{P})$ then the p gradient vectors $\nabla G_{i,l}(u)$'s are linearly independent. Assume $\alpha_{i,l}$'s are p real numbers such that

$$\sum_{i=1}^{k}\sum_{l=1}^{p_i}\alpha_{i,l}\nabla G_{i,l}(u) = 0.$$

Let $u=(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})$. For any subscript index (j,m), taking inner product of this equation with the vector of which all the components are 0 except the $(j,m)^{\rm th}$ component which is assumed to be equal to $u_{j,m}$, we have a linear system of p equations for $\alpha_{i,l}$'s

$$\sum_{i=1}^{k}\sum_{l=1}^{p_i}\Big\langle \frac{\partial}{\partial u_{j,m}} G_{i,l}(u),\, u_{j,m}\Big\rangle \alpha_{i,l} = 0, \quad m=1,\cdots,p_j, \quad j=1,\cdots,k. \tag{2.5}$$

From $u \in \mathcal{N}(\mathbb{P})$ it can be seen that, for all the indices (i,l),

$$\begin{aligned}\Big\langle \frac{\partial}{\partial u_{i,l}} G_{i,l}(u),\, u_{i,l}\Big\rangle &= 2\|u_{i,l}\|_i^2 - 4\beta_{ii}\int_{\mathbb{R}^N} u_{i,l}^4 - 2\sum_{j\neq i}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N} u_{i,l}^2 u_{j,m}^2 \\ &= -2\beta_{ii}\int_{\mathbb{R}^N} u_{i,l}^4.\end{aligned}$$

On the other hand, for all the indices (i,l) and (j,m) with $(j,m) \neq (i,l)$,

$$\Big\langle \frac{\partial}{\partial u_{j,m}} G_{i,l}(u),\, u_{j,m}\Big\rangle = -2\beta_{ij}\int_{\mathbb{R}^N} u_{i,l}^2 u_{j,m}^2.$$

Thus the coefficient matrix of the linear system (2.5) can be expressed as

$$\Big(-2\beta_{ij}\int_{\mathbb{R}^N} u_{i,l}^2 u_{j,m}^2\Big)_{p\times p}.$$

Consider the matrix

$$A=\left(\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\right)_{p\times p},$$

where we regard (i,l) as the row index and (j,m) as the column index. Since, for any $(i,l)^{\text{th}}$ row, the entry on the diagonal minus the sum of the absolute values of the entries off the diagonal can be expressed as

$$\beta_{ii}\int_{\mathbb{R}^N}u_{i,l}^4-\sum_{j\neq i}\sum_{m=1}^{p_j}\left|\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\right|=\|u_{i,l}\|^2-2\sum_{j\neq i,\ \beta_{ij}>0}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2,$$

if $\beta_{ij}\le b_{1\mathbb{P}}$ for all $i\neq j$ we have

$$\begin{aligned}&\beta_{ii}\int_{\mathbb{R}^N}u_{i,l}^4-\sum_{(j,m)\neq(i,l)}\left|\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\right|\\ \ge&\|u_{i,l}\|^2-2b_{1\mathbb{P}}\sum_{j=1}^{k}\sum_{m=1}^{p_j}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\\ \ge&\|u_{i,l}\|^2-2b_{1\mathbb{P}}\kappa\sum_{j=1}^{k}\sum_{m=1}^{p_j}\|u_{i,l}\|_i^2\|u_{j,m}\|_j^2\\ \ge&(1-16b_{1\mathbb{P}}c(\mathbb{P})\kappa)\|u_{i,l}\|^2=\frac{1}{2}\|u_{i,l}\|^2>0.\end{aligned}$$

This implies that A is a diagonally dominant matrix and thus is positively definite by the Gershgorin circle theorem. As a consequence, the coefficient matrix of the linear system (2.5) is negatively definite. Therefore, all the $\alpha_{i,l}$'s are 0 and the $\nabla G_{i,l}(u)$'s are linearly independent. □

To prove that the minimization problem

$$c(\mathbb{P})=\inf J_{\mathbb{P}}|_{\mathcal{N}(\mathbb{P})}$$

is achieved, we introduce an auxiliary minimization problem defined as

$$\tilde{c}(\mathbb{P})=\inf\tilde{J}_{\mathbb{P}}|_{\tilde{\mathcal{N}}(\mathbb{P})},$$

where

$$\tilde{J}_{\mathbb{P}}(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})=\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|u_{i,l}\|_i^2$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(\mathbb{P})=\Big\{&(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})\ \Big|\ u_{i,l}\in E_{i,l},\ u_{i,l}\neq 0,\\ &\|u_{i,l}\|_i^2\le\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2,\ \ \frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|u_{i,l}\|_i^2<2c(\mathbb{P})\Big\}.\end{aligned}$$

Clearly,

$$J_{\mathbb{P}}(u)=\tilde{J}_{\mathbb{P}}(u),$$

if $u=(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})\in\mathcal{N}(\mathbb{P})$, and that implies $\tilde{c}(\mathbb{P})\leq c(\mathbb{P})$.

We shall see that both the infima $\tilde{c}(\mathbb{P})$ and $c(\mathbb{P})$ are achieved, that any minimizer of $\tilde{c}(\mathbb{P})$ is in $\mathcal{N}(\mathbb{P})$, and that $\tilde{c}(\mathbb{P})=c(\mathbb{P})$. The strategy is that we first prove that $\tilde{c}(\mathbb{P})$ is achieved and then show that any minimizer of $\tilde{J}_{\mathbb{P}}|_{\tilde{\mathcal{N}}(\mathbb{P})}$ is indeed in $\mathcal{N}(\mathbb{P})$.

Lemma 2.2. *Let $\mathbb{P}$ be fixed and let $b_{1\mathbb{P}}$ be the number defined in Lemma 2.1 and assume $\beta_{ij}\leq b_{1\mathbb{P}}$ for all $i\neq j$. Then the infimum $\tilde{c}(\mathbb{P})$ is achieved. That is, there exists $u\in\tilde{\mathcal{N}}(\mathbb{P})$ such that*

$$\tilde{J}_{\mathbb{P}}(u)=\tilde{c}(\mathbb{P}).$$

Proof. For any $u=(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})\in\tilde{\mathcal{N}}(\mathbb{P})$, we have

$$\begin{aligned}\|u_{i,l}\|_i^2&\leq\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\\&\leq\beta_{ii}\int_{\mathbb{R}^N}u_{i,l}^4+b_{1\mathbb{P}}\sum_{j\neq i}\sum_{m=1}^{p_j}\int_{\mathbb{R}^N}(u_{i,l})^2(u_{j,m})^2\\&\leq B\kappa\|u_{i,l}\|_i^4+b_{1\mathbb{P}}\kappa\|u_{i,l}\|_i^2\sum_{j\neq i}\sum_{m=1}^{p_j}\|u_{j,m}\|_j^2\\&\leq B\kappa\|u_{i,l}\|_i^4+8b_{1\mathbb{P}}c(\mathbb{P})\kappa\|u_{i,l}\|_i^2=B\kappa\|u_{i,l}\|_i^4+\frac{1}{4}\|u_{i,l}\|_i^2.\end{aligned}$$

Therefore,

$$\frac{3}{4B\kappa}\leq\|u_{i,l}\|_i^2\leq\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2 \tag{2.6}$$

for all $u=(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})\in\tilde{\mathcal{N}}(\mathbb{P})$ and all (i,l).

Let $\{u_n\}_1^\infty\subset\tilde{\mathcal{N}}(\mathbb{P})$, $u_n=((u_n)_{1,1},\cdots,(u_n)_{1,p_1},\cdots,(u_n)_{k,1},\cdots,(u_n)_{k,p_k})$, be a minimizing sequence for $\tilde{c}(\mathbb{P})$. Then

$$\tilde{c}(\mathbb{P})=\lim_{n\to\infty}\tilde{J}_{\mathbb{P}}(u_n).$$

Passing to a subsequence if necessary, we may assume that, for each (i,l),

$$(u_n)_{i,l}\to u_{i,l}\quad\text{as }\ n\to\infty$$

weakly in $E_{i,l}$, strongly in $L^4(\mathbb{R}^N)$, and a.e. in $\mathbb{R}^N$. Set

$$u=(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k}).$$

Then the weak convergence implies that

$$\tilde{c}(\mathbb{P})=\lim_{n\to\infty}\tilde{J}_{\mathbb{P}}(u_n)=\lim_{n\to\infty}\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|(u_n)_{i,l}\|_i^2\geq\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|u_{i,l}\|_i^2=\tilde{J}_{\mathbb{P}}(u).$$

The $L^4(\mathbb{R}^N)$ strong convergence together with (2.6) implies that

$$\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2=\lim_{n\to\infty}\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}(u_n)_{i,l}^2(u_n)_{j,m}^2\geq\frac{3}{4B\kappa},$$

and, as a consequence $u_{i,l}\neq 0$ for all (i,l). In addition, we have

$$\begin{aligned}\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2&=\lim_{n\to\infty}\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}(u_n)_{i,l}^2(u_n)_{j,m}^2\\&\geq\liminf_{n\to\infty}\|(u_n)_{i,l}\|_i^2\geq\|u_{i,l}\|_i^2.\end{aligned}$$

Since u_n is a minimizing sequence,

$$\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|u_{i,l}\|_i^2\leq\lim_{n\to\infty}\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|(u_n)_{i,l}\|_i^2<2c(\mathbb{P}).$$

Then $u\in\tilde{\mathcal{N}}(\mathbb{P})$ and $\tilde{c}(\mathbb{P})\leq\tilde{J}_{\mathbb{P}}(u)$. Therefore, $\tilde{c}(\mathbb{P})=\tilde{J}_{\mathbb{P}}(u)$ and the infimum $\tilde{c}(\mathbb{P})$ is achieved at u. $\square$

Lemma 2.3. *Let $\mathbb{P}$ be fixed and let B, κ be the numbers from* (2.1) *and* (2.3) *respectively. Define*

$$b_{2\mathbb{P}}=\frac{3}{32c(\mathbb{P})\kappa[64Bc(\mathbb{P})\kappa+3]}.$$

If $\beta_{ij}\leq b_{2\mathbb{P}}$ for all $i\neq j$ then any minimizer of $\tilde{J}_{\mathbb{P}}|_{\tilde{\mathcal{N}}(\mathbb{P})}$ is in $\mathcal{N}(\mathbb{P})$. As a consequence,

$$c(\mathbb{P})=\tilde{c}(\mathbb{P})$$

and $c(\mathbb{P})$ is achieved.

Proof. Let $u=(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})\in\tilde{\mathcal{N}}(\mathbb{P})$ be any minimizer of $\tilde{J}_{\mathbb{P}}|_{\tilde{\mathcal{N}}(\mathbb{P})}$. Suppose, for a contradiction, $u\notin\mathcal{N}(\mathbb{P})$. Then for some (i,l) the strict inequality

$$\|u_{i,l}\|_i^2<\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2$$

holds. For convenience, we denote by $\mathcal{T}_1$ the set of indices (i,l) for which the last inequality holds and by $\mathcal{T}_2$ the set of remaining indices. Then for $(i,l)\in\mathcal{T}_2$ we have

$$\|u_{i,l}\|_i^2=\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2.$$

If $\mathcal{T}_2=\emptyset$ then u lies in the interior of $\tilde{\mathcal{N}}(\mathbb{P})$ and, for small $\epsilon>0$, $(1-\epsilon)u\in\tilde{\mathcal{N}}(\mathbb{P})$ and

$$\tilde{J}_{\mathbb{P}}((1-\epsilon)u)<\tilde{J}_{\mathbb{P}}(u).$$

In this way we come to a contradiction.

Now assume $\mathcal{T}_2 \neq \emptyset$ and thus u lies on the boundary of $\tilde{\mathcal{N}}(\mathbb{P})$. It is quite interesting that even though the point u lies on the boundary of $\tilde{\mathcal{N}}(\mathbb{P})$ it can be shifted inward to a point $v=(v_{1,1},\cdots,v_{1,p_1},\cdots,v_{k,1},\cdots,v_{k,p_k})\in\mathrm{int}(\tilde{\mathcal{N}}(\mathbb{P}))$ such that

$$\tilde{J}_{\mathbb{P}}(v)<\tilde{J}_{\mathbb{P}}(u).$$

This v can be defined as

$$v_{i,l}=\begin{cases} (1-\epsilon)^{1/2}u_{i,l}, & \text{if } (i,l)\in\mathcal{T}_1,\\ \left(1+\dfrac{3}{64Bc(\mathbb{P})\kappa}\epsilon\right)^{1/2}u_{i,l}, & \text{if } (i,l)\in\mathcal{T}_2,\end{cases}$$

where $\epsilon>0$ is a number small enough such that

$$\|v_{i,l}\|_i^2<\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}v_{i,l}^2v_{j,m}^2 \quad \text{for } (i,l)\in\mathcal{T}_1. \tag{2.7}$$

Clearly, $v_{i,l}\neq 0$. Now we show that, for $(i,l)\in\mathcal{T}_2$, the inequality

$$\|v_{i,l}\|_i^2<\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}v_{i,l}^2v_{j,m}^2 \tag{2.8}$$

is also valid. This inequality can be rewritten as

$$\|u_{i,l}\|_i^2<(1-\epsilon)\sum_{(j,m)\in\mathcal{T}_1}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2+\left(1+\frac{3}{64Bc(\mathbb{P})\kappa}\epsilon\right)\sum_{(j,m)\in\mathcal{T}_2}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2. \tag{2.9}$$

Note that $(i,l)\in\mathcal{T}_2$ implies

$$\|u_{i,l}\|_i^2=\sum_{(j,m)\in\mathcal{T}_1}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2+\sum_{(j,m)\in\mathcal{T}_2}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2.$$

Inserting the lase equation into (2.9), we see that (2.8) is equivalent to

$$\left(1+\frac{64Bc(\mathbb{P})\kappa}{3}\right)\sum_{(j,m)\in\mathcal{T}_1}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2<\|u_{i,l}\|_i^2.$$

(2.8) then follows since the left side of the above inequality does not exceed

$$\left(1+\frac{64Bc(\mathbb{P})\kappa}{3}\right)8b_{2\mathbb{P}}c(\mathbb{P})\kappa\|u_{i,l}\|_i^2=\frac{1}{4}\|u_{i,l}\|_i^2.$$

The definition of v implies

$$\begin{aligned}\tilde{J}_{\mathbb{P}}(v)=&\frac{1}{4}(1-\epsilon)\sum_{(i,l)\in\mathcal{T}_1}\|u_{i,l}\|_i^2+\frac{1}{4}\Big(1+\frac{3}{64Bc(\mathbb{P})\kappa}\epsilon\Big)\sum_{(i,l)\in\mathcal{T}_2}\|u_{i,l}\|_i^2\\ =&\tilde{J}_{\mathbb{P}}(u)-\frac{\epsilon}{4}\sum_{(i,l)\in\mathcal{T}_1}\|u_{i,l}\|_i^2+\frac{3\epsilon}{256Bc(\mathbb{P})\kappa}\sum_{(i,l)\in\mathcal{T}_2}\|u_{i,l}\|_i^2.\end{aligned}$$

In view of (2.6), we have

$$\tilde{J}_{\mathbb{P}}(v)\le\tilde{J}_{\mathbb{P}}(u)-\frac{3\epsilon}{16B\kappa}+\frac{3\epsilon}{32B\kappa}=\tilde{J}_{\mathbb{P}}(u)-\frac{3\epsilon}{32B\kappa}<\tilde{J}_{\mathbb{P}}(u),$$

and as a consequence

$$\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|v_{i,l}\|_i^2<\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|u_{i,l}\|_i^2<2c(\mathbb{P}),$$

which together with (2.7) and (2.8) implies $v=(v_{1,1},\cdots,v_{1,p_1},\cdots,v_{k,1},\cdots,v_{k,p_k})\in\tilde{\mathcal{N}}(\mathbb{P})$. We have arrived at a contradiction since $v\in\tilde{\mathcal{N}}(\mathbb{P})$, $\tilde{J}_{\mathbb{P}}(v)<\tilde{J}_{\mathbb{P}}(u)$, and u is a minimizer of $\tilde{J}_{\mathbb{P}}|_{\tilde{\mathcal{N}}(\mathbb{P})}$.

Therefore, any minimizer u of $\tilde{J}_{\mathbb{P}}|_{\tilde{\mathcal{N}}(\mathbb{P})}$ is in $\mathcal{N}(\mathbb{P})$. For such a u we have

$$c(\mathbb{P})\le J_{\mathbb{P}}(u)=\tilde{J}_{\mathbb{P}}(u)=\tilde{c}(\mathbb{P})\le c(\mathbb{P}).$$

Therefore, $c(\mathbb{P})=\tilde{c}(\mathbb{P})$ and $c(\mathbb{P})$ is achieved. □

Lemma 2.4. *Let $\mathbb{P}$ be fixed and let $b_{2\mathbb{P}}$ be the number defined in Lemma 2.3 and assume $\beta_{ij}\le b_{2\mathbb{P}}$ for all $i\neq j$. Let u be any minimizer of $J_{\mathbb{P}}$ constrained on $\mathcal{N}(\mathbb{P})$. Then*

$$\nabla J_{\mathbb{P}}(u)=0.$$

Proof. Since, by Lemma 2.1, $\mathcal{N}(\mathbb{P})$ is a smooth manifold and since the assumption that u is a minimizer of $J_{\mathbb{P}}$ constrained on $\mathcal{N}(\mathbb{P})$ implies

$$\nabla(J_{\mathbb{P}}|_{\mathcal{N}(\mathbb{P})})(u)=0,$$

there are p real numbers $\alpha_{i,l}$'s acting as Lagrangian multipliers such that

$$\nabla J_{\mathbb{P}}(u)+\sum_{i=1}^{k}\sum_{l=1}^{p_i}\alpha_{i,l}\nabla G_{i,l}(u)=0,$$

where $G_{i,l}$'s are the functionals defined in Lemma 2.1. Taking inner product of this equation with the vector of which all the components are 0 except the $(j,m)^{\text{th}}$ component which is taken to be $u_{j,m}$, we arrive at the same linear system as (2.5). According to the proof of Lemma 2.1, we have $\alpha_{i,l}=0$ for all (i,l). Then the conclusion follows. □

Now we consider the minimization problem

$$c=\inf_{\mathbb{P}} c(\mathbb{P}).$$

The infimum is taken over all the k-time partitions $\mathbb{P}=(P_1,\cdots,P_k)$ with P_i having (p_i-1) points of division. By (2.6), we see that $c(\mathbb{P})\geq\frac{3p}{16B\kappa}$ for any $\mathbb{P}$ and, as a consequence,

$$c\geq\frac{3p}{16B\kappa}.$$

We will show that c is achieved by a k-time partition $\mathbb{P}$.

Fix a k-time partition $\mathbb{P}_0$ and a point $u_0\in E(\mathbb{P}_0)$ having the same properties as u^* mentioned above. That is, $u_0=((u_0)_{1,1},\cdots,(u_0)_{1,p_1},\cdots,(u_0)_{k,1},\cdots,(u_0)_{k,p_k})$ belongs to $\mathcal{N}(\mathbb{P}_0)$ and satisfies

$$\begin{cases} (u_0)_{i,l}\neq 0 & \text{for all } \ (i,l),\\ \operatorname{supp}((u_0)_{i,l})\cap\operatorname{supp}((u_0)_{j,m})=\emptyset, & \text{if } \ (i,l)\neq(j,m),\\ \|(u_0)_{i,l}\|_i^2=\beta_{ii}\int_{\mathbb{R}^N}((u_0)_{i,l})^4 & \text{for all } \ (i,l). \end{cases}$$

Then fix a number μ_0 such that

$$\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\|(u_0)_{i,l}\|_i^2<\mu_0. \tag{2.10}$$

This provides an upper bound μ_0 independent of β_{ij} $(i\neq j)$ for c since

$$c\leq c(\mathbb{P}_0)<\mu_0.$$

Let B and κ be the numbers from (2.1) and (2.3) respectively, and define

$$b_0=\frac{3}{32\kappa\mu_0(64B\kappa\mu_0+3)}. \tag{2.11}$$

Lemma 2.5. *Assume $\beta_{ij}\leq b_0$ for all $i\neq j$. Then there exists a k-time partition $\mathbb{P}$ and a $u\in\mathcal{N}(\mathbb{P})$ such that*

$$c=c(\mathbb{P})=J_{\mathbb{P}}(u).$$

Proof. Let $\{\mathbb{P}_n\}$ be a sequence of k-time partitions such that

$$c=\lim_{n\to\infty}c(\mathbb{P}_n).$$

We may assume that $c(\mathbb{P}_n)<\mu_0$ for all n since $c<\mu_0$. Then for all n

$$b_{2\mathbb{P}_n}=\frac{3}{32c(\mathbb{P}_n)\kappa[64Bc(\mathbb{P}_n)\kappa+3]}>\frac{3}{32\kappa\mu_0(64B\kappa\mu_0+3)}=b_0.$$

Since $\beta_{ij}\le b_0$ for all $i\neq j$, by Lemma 2.3 there exists $u_{\mathbb{P}_n}\in\mathcal{N}(\mathbb{P}_n)$ such that

$$c(\mathbb{P}_n)=J_{\mathbb{P}_n}(u_{\mathbb{P}_n}).$$

For simplicity, we write $u_n=u_{\mathbb{P}_n}$. Suppose

$$\mathbb{P}_n=(P_{n1},\cdots,P_{nk})$$

and

$$P_{ni}:\ 0<(r_n)_{i,1}<\cdots<(r_n)_{i,p_i-1}<+\infty.$$

We claim that, for all i, the sequence $\{(r_n)_{i,p_i-1}\}_{n=1}^\infty$ is bounded above and the sequence $\{(r_n)_{i,l}-(r_n)_{i,l-1}\}_{n=1}^\infty$ is bounded below by a positive number if $l\le p_i-1$. Since $J_{\mathbb{P}_n}(u_n)=c(\mathbb{P}_n)<\mu_0$, we have

$$\sum_{i=1}^k\sum_{l=1}^{p_i}\|(u_n)_{i,l}\|_i^2=\sum_{i=1}^k\sum_{l=1}^{p_i}\sum_{j=1}^k\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}(u_n)_{i,l}^2(u_n)_{j,m}^2<4\mu_0.$$

Then, for any index (i,l),

$$\begin{aligned}\|(u_n)_{i,l}\|_i^2=&\beta_{ii}\int_{\mathbb{R}^N}(u_n)_{i,l}^4+\sum_{j\neq i}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}(u_n)_{i,l}^2(u_n)_{j,m}^2\\ \le& B\int_{\mathbb{R}^N}(u_n)_{i,l}^4+b_0\kappa\|(u_n)_{i,l}\|_i^2\sum_{j\neq i}\sum_{m=1}^{p_j}\|(u_n)_{j,m}\|_j^2\\ \le& B\int_{\mathbb{R}^N}(u_n)_{i,l}^4+4b_0\kappa\mu_0\|(u_n)_{i,l}\|_i^2<B\int_{\mathbb{R}^N}(u_n)_{i,l}^4+\frac{1}{8}\|(u_n)_{i,l}\|_i^2.\end{aligned}$$

Using the Strauss inequality

$$|u(x)|\le C|x|^{-\frac{N-1}{2}}\|u\|_i,\quad u\in H_r^1(\mathbb{R}^N),$$

we estimate as

$$\begin{aligned}\|(u_n)_{i,p_i}\|_i^2\le&2B\int_{\mathbb{R}^N}(u_n)_{i,p_i}^4\le C\|(u_n)_{i,p_i}\|_i^2\int_{\mathbb{R}^N}|x|^{-(N-1)}(u_n)_{i,p_i}^2\\ \le& C\|(u_n)_{i,p_i}\|_i^4[(r_n)_{i,p_i-1}]^{-(N-1)}.\end{aligned}$$

Here and in the sequel, we use C to denote a positive constant whose exact value is irrelevant. Since $(u_n)_{i,p_i}\neq 0$ and the sequence $\{\|(u_n)_{i,p_i}\|_i\}$ is bounded, the above inequality implies that $\{(r_n)_{i,p_i-1}\}_{n=1}^\infty$ is bounded. If $l\le p_i-1$ then by the Hölder inequality and the Sobolev inequality

$$\begin{aligned}\|(u_n)_{i,l}\|_i^2\le&2B\int_{\mathbb{R}^N}(u_n)_{i,l}^4\le 2B\left(\int_{\mathbb{R}^N}(u_n)_{i,l}^6\right)^{2/3}|\Omega_{i,l}(\mathbb{P}_n)|^{1/3}\\ \le& C\|(u_n)_{i,l}\|_i^4[((r_n)_{i,l})^N-((r_n)_{i,l-1})^N]^{1/3}\\ \le& C\|(u_n)_{i,l}\|_i^4[(r_n)_{i,p_i-1}]^{(N-1)/3}[(r_n)_{i,l}-(r_n)_{i,l-1}]^{1/3}.\end{aligned}$$

Since we have already seen that $\{(r_n)_{i,p_i-1}\}$ is bounded above, we have

$$\|(u_n)_{i,l}\|_i^2 \le C\|(u_n)_{i,l}\|_i^4[(r_n)_{i,l}-(r_n)_{i,l-1}]^{1/3},$$

which together with the boundedness of $\{\|(u_n)_{i,l}\|_i\}$ implies that $\{(r_n)_{i,l}-(r_n)_{i,l-1}\}_{n=1}^\infty$ is bounded below by a positive number.

Passing to subsequences if necessary, we may assume that as $n\to\infty$, for each (i,l),

$$(r_n)_{i,l}\to r_{i,l},$$

and

$$(u_n)_{i,l}\to u_{i,l}$$

weakly in $H_r^1(\mathbb{R}^N)$, strongly in $L^4(\mathbb{R}^N)$, and a.e. in $\mathbb{R}^N$. The above discussions show that

$$P_i:\ 0<r_{i,1}<\cdots<r_{i,p_i-1}<+\infty$$

is a partition with (p_i-1) points of division of $[0,+\infty)$ and $\mathbb{P}=(P_1,\cdots,P_k)$ is a k-time partition. Clearly,

$$u_{i,l}\in E_{i,l}(\mathbb{P}).$$

Using the same argument as in the proof of Lemma 2.2, we see that

$$\sum_{j=1}^k\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2=\lim_{n\to\infty}\sum_{j=1}^k\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}(u_n)_{i,l}^2(u_n)_{j,m}^2\ge\frac{3}{4B\kappa},$$

and

$$\sum_{j=1}^k\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\ge\|u_{i,l}\|_i^2.$$

Moreover,

$$\begin{aligned}\tilde{J}_{\mathbb{P}}(u)&=\frac{1}{4}\sum_{i=1}^k\sum_{l=1}^{p_i}\|u_{i,l}\|_i^2\le\liminf_{n\to\infty}\frac{1}{4}\sum_{i=1}^k\sum_{l=1}^{p_i}\|(u_n)_{i,l}\|_i^2\\&=\liminf_{n\to\infty}\tilde{J}_{\mathbb{P}_n}(u_n)=\liminf_{n\to\infty}J_{\mathbb{P}_n}(u_n)=\lim_{n\to\infty}c(\mathbb{P}_n)=c\le c(\mathbb{P}).\end{aligned}$$

Therefore, $u\in\tilde{\mathcal{N}}(\mathbb{P})$. Since

$$c\le c(\mathbb{P})=\tilde{c}(\mathbb{P})\le\tilde{J}_{\mathbb{P}}(u)\le c,$$

by Lemma 2.3 we have $u\in\mathcal{N}(\mathbb{P})$ and $c=c(\mathbb{P})=J_{\mathbb{P}}(u)$. □

Lemma 2.6. *Let b_0 be the number defined above and assume $\beta_{ij}\le b_0$ for all $i\ne j$. For any $u=(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})\in\mathcal{N}(\mathbb{P})$, define a function $F:\mathbb{R}^p\to\mathbb{R}$ as*

$$F(s_{1,1},\cdots,s_{1,p_1},\cdots,s_{k,1},\cdots,s_{k,p_k})=\Phi\left(\sum_{l=1}^{p_1}s_{1,l}u_{1,l},\ \cdots,\ \sum_{l=1}^{p_k}s_{k,l}u_{k,l}\right).$$

Then $(1,\cdots,1,\cdots,1,\cdots,1)$ is a strict local maximizer of F.

Proof. From the definitions of F and Φ we have

$$\begin{aligned}&F(s_{1,1},\cdots,s_{1,p_1},\cdots,s_{k,1},\cdots,s_{k,p_k})\\=&\frac{1}{2}\sum_{i=1}^{k}\sum_{l=1}^{p_i}s_{i,l}^2\|u_{i,l}\|_i^2-\frac{1}{4}\sum_{i=1}^{k}\sum_{l=1}^{p_i}\sum_{j=1}^{k}\sum_{m=1}^{p_j}s_{i,l}^2s_{j,m}^2\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2.\end{aligned}$$

The first and second partial derivatives of F are given by

$$\begin{aligned}\frac{\partial F}{\partial s_{i,l}}&=s_{i,l}\|u_{i,l}\|_i^2-s_{i,l}^3\beta_{ii}\int_{\mathbb{R}^N}u_{i,l}^4-\sum_{j\neq i}\sum_{m=1}^{p_j}s_{i,l}s_{j,m}^2\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2,\\ \frac{\partial^2 F}{\partial s_{i,l}^2}&=\|u_{i,l}\|_i^2-3s_{i,l}^2\beta_{ii}\int_{\mathbb{R}^N}u_{i,l}^4-\sum_{j\neq i}\sum_{m=1}^{p_j}s_{j,m}^2\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2,\end{aligned}$$

and

$$\frac{\partial^2 F}{\partial s_{i,l}\partial s_{j,m}}=-2s_{i,l}s_{j,m}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\quad\text{for}\ \ (j,m)\neq(i,l).$$

Since $u=(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})\in\mathcal{N}(\mathbb{P})$, it follows that

$$\frac{\partial F}{\partial s_{i,l}}\Big|_{(1,\cdots,1,\cdots,1,\cdots,1)}=0\quad\text{for all}\ \ (i,l)$$

and

$$\frac{\partial^2 F}{\partial s_{i,l}\partial s_{j,m}}\Big|_{(1,\cdots,1,\cdots,1,\cdots,1)}=-2\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\quad\text{for all}\ \ (i,l)\ \ \text{and}\ \ (j,m).$$

According to the proof of Lemma 2.1, the matrix

$$\left(\frac{\partial^2 F}{\partial s_{i,l}\partial s_{j,m}}\Big|_{(1,\cdots,1,\cdots,1,\cdots,1)}\right)_{p\times p}$$

is negatively definite. Therefore, $(1,\cdots,1,\cdots,1,\cdots,1)$ is a strict local maximizer of F. □

Let τ be a positive number depending only on the number ν in (2.2) such that for all $i=1,\cdots,k$ and all $u\in H_r^1(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N}u^6\right)^{1/3}\leq\tau\|u\|_i^2. \tag{2.12}$$

Set

$$B^*=\max\{B,\,b_0\}, \tag{2.13}$$

where B and b_0 are the numbers in (2.1) and (2.11) respectively. With the constants ν, p, μ_0, b_0, τ, and B^* from (2.2), (2.4), (2.10), (2.11), (2.12), and (2.13) respectively, we define

$$b=\min\left\{b_0,\,\frac{\nu}{288B^*p^2\tau^3\mu_0^2}\right\}.$$

Proof of Theorem 1.1. We first assume that $\beta_{ij}\le b_0$ for all $i\neq j$. By Lemma 2.5 we choose a k-time partition $\mathbb{P}=(P_1,\cdots,P_k)$ with

$$P_i\colon\ 0<r_{i,1}<\cdots<r_{i,p_i-1}<+\infty$$

and $u\in\mathcal{N}(\mathbb{P})$ such that

$$c=c(\mathbb{P})=J_{\mathbb{P}}(u)=\Phi(u_1,\cdots,u_k),$$

where

$$u_i=\sum_{l=1}^{p_i}u_{i,l},\quad i=1,2,\cdots,k.$$

Replacing $u_{i,l}$ with $(-1)^{l-1}|u_{i,l}|$ if necessary, we may assume that $(-1)^{l-1}u_{i,l}\ge 0$. Then, since $\nabla J_{\mathbb{P}}(u)=0$ by Lemma 2.4, $u=(u_{1,1},\cdots,u_{1,p_1},\cdots,u_{k,1},\cdots,u_{k,p_k})$ is a solution of the system

$$\begin{cases} -\Delta u_{i,l}+\lambda_i u_{i,l}=\sum\limits_{j=1}^{k}\sum\limits_{m=1}^{p_j}\beta_{ij}u_{i,l}u_{j,m}^2 & \text{in } \Omega_{i,l}(\mathbb{P}),\\ u_{i,l}\neq 0 & \text{in } \Omega_{i,l}(\mathbb{P}),\\ (-1)^{l-1}u_{i,l}\ge 0 & \text{in } \Omega_{i,l}(\mathbb{P}),\\ u_{i,l}=0 & \text{in } \mathbb{R}^N\setminus\Omega_{i,l}(\mathbb{P}). \end{cases} \tag{2.14}$$

For any n, multiplying the above equation with $(u_{i,l})^{3^n}$ and taking integral yields

$$\begin{aligned}&\frac{1}{3^n}\int_{\mathbb{R}^N}|\nabla(u_{i,l})^{\frac{3^n+1}{2}}|^2+\lambda_i\int_{\mathbb{R}^N}(u_{i,l})^{3^n+1}\\ \le& B^*\Big(\int_{\mathbb{R}^N}(u_{i,l})^{3^n+3}\Big)^{\frac{3^n+1}{3^n+3}}\sum_{j=1}^{k}\sum_{m=1}^{p_j}\Big(\int_{\mathbb{R}^N}(u_{j,m})^{3^n+3}\Big)^{\frac{2}{3^n+3}},\end{aligned}$$

which implies

$$\Big(\int_{\mathbb{R}^N}(u_{i,l})^{3^{n+1}+3}\Big)^{1/3}\le 3^nB^*\tau\Big(\int_{\mathbb{R}^N}(u_{i,l})^{3^n+3}\Big)^{\frac{3^n+1}{3^n+3}}\sum_{j=1}^{k}\sum_{m=1}^{p_j}\Big(\int_{\mathbb{R}^N}(u_{j,m})^{3^n+3}\Big)^{\frac{2}{3^n+3}}.$$

Summing up with respect to (i,l) and then taking the $(3^n+1)^{\text{th}}$ root, we see that

$$\Big(\sum_{i=1}^{k}\sum_{l=1}^{p_i}\int_{\mathbb{R}^N}(u_{i,l})^{3^{n+1}+3}\Big)^{\frac{1}{3^{n+1}+3}}\le(3^nB^*p\tau)^{\frac{1}{3^n+1}}\Big(\sum_{i=1}^{k}\sum_{l=1}^{p_i}\int_{\mathbb{R}^N}(u_{i,l})^{3^n+3}\Big)^{\frac{1}{3^n+1}}.$$

Then an iteration process can be used to deduce that

$$\Big(\sum_{i=1}^{k}\sum_{l=1}^{p_i}\int_{\mathbb{R}^N}(u_{i,l})^{3^{n+1}+3}\Big)^{\frac{1}{3^{n+1}+3}}\le 3^{r_n}(B^*p\tau)^{s_n}\Big(\sum_{i=1}^{k}\sum_{l=1}^{p_i}\int_{\mathbb{R}^N}(u_{i,l})^{6}\Big)^{t_n},$$

where

$$r_n=\frac{n}{3^n+1}+\frac{3(n-1)}{3^n+1}+\frac{3^2(n-2)}{3^n+1}+\cdots+\frac{3^{n-1}}{3^n+1},$$
$$s_n=\frac{1}{3^n+1}+\frac{3}{3^n+1}+\frac{3^2}{3^n+1}+\cdots+\frac{3^{n-1}}{3^n+1},$$

and

$$t_n=\frac{3^{n-1}}{3^n+1}.$$

Since $r_n<1$ we arrive at

$$\Big(\sum_{i=1}^k\sum_{l=1}^{p_i}\int_{\mathbb{R}^N}(u_{i,l})^{3^{n+1}+3}\Big)^{\frac{1}{3^{n+1}+3}}\le 3(B^*p\tau)^{s_n}\Big(\sum_{i=1}^k\sum_{l=1}^{p_i}\int_{\mathbb{R}^N}(u_{i,l})^6\Big)^{t_n}.$$

In view of $\lim_{n\to\infty}s_n=\frac{1}{2}$ and $\lim_{n\to\infty}t_n=\frac{1}{3}$, letting $n\to\infty$ yields

$$\begin{aligned}\max_{1\le i\le k}\max_{1\le l\le p_i}\|u_{i,l}\|_{L^\infty(\mathbb{R}^N)}&\le 3(B^*p\tau)^{1/2}\sum_{i=1}^k\sum_{l=1}^{p_i}\Big(\int_{\mathbb{R}^N}(u_{i,l})^6\Big)^{1/3}\\&\le 3(B^*p)^{1/2}\tau^{3/2}\sum_{i=1}^k\sum_{l=1}^{p_i}\|u_{i,l}\|_i^2\le 12(B^*p)^{1/2}\tau^{3/2}\mu_0.\end{aligned}\tag{2.15}$$

Write the equation in (2.14) in the form

$$-\Delta u_{i,l}+\left(\lambda_i-\sum_{j\neq i}\sum_{m=1}^{p_j}\beta_{ij}(u_{j,m})^2\right)u_{i,l}=\beta_{ii}(u_{i,l})^3.$$

We now assume that $\beta_{ij}\le b$ for all $i\neq j$. Using (2.15) we see that

$$\lambda_i-\sum_{j\neq i}\sum_{m=1}^{p_j}\beta_{ij}(u_{j,m})^2\ge\nu-144bp^2B^*\tau^3\mu_0^2\ge\nu/2>0.$$

Then the elliptic regularity theory implies that $u_{i,l}$ is smooth in $\Omega_{i,l}(\mathbb{P})$, and by the maximum principle

$$(-1)^{l-1}u_{i,l}(r)>0\quad\text{for }\ r\in(r_{i,l-1},r_{i,l}),\tag{2.16}$$

and

$$(-1)^{l-1}\lim_{r\to r_{i,l-1}+}\frac{\partial u_{i,l}}{\partial r}(r)>0,\quad(-1)^{l-1}\lim_{r\to r_{i,l}-}\frac{\partial u_{i,l}}{\partial r}(r)<0.\tag{2.17}$$

Here we have written $u_{i,l}(|x|)=u_{i,l}(x)$. Then u_i has exactly p_i-1 nodes

$$r_{i,1},\ r_{i,2},\ \cdots,\ r_{i,p_i-1}.$$

We show that $(u_1,\cdots,u_k)$ is a solution of (1.1). If this is not the case, then $\Phi'(u_1,\cdots,u_k)\neq 0$ and there exists $(\phi_1,\cdots,\phi_k)\in[C^\infty_{0,\,r}(\mathbb{R}^N)]^k$ such that

$$\langle \Phi'(u_1,\cdots,u_k),(\phi_1,\cdots,\phi_k)\rangle=-2, \tag{2.18}$$

where $C^\infty_{0,\,r}(\mathbb{R}^N)$ is the subspace of $C^\infty_0(\mathbb{R}^N)$ consisting of all the radially symmetric functions. In view of (2.16), (2.17), and (2.18), we can choose a number $\delta\in(0,1)$ such that if $0\le\epsilon\le\delta$ and if $s=(s_{1,1},\cdots,s_{1,p_1},\cdots,s_{k,1},\cdots,s_{k,p_k})\in\mathbb{R}^p$ satisfies $|s_{i,l}-1|\le\delta$ for all (i,l) then

$$\left\langle \Phi'\left(\sum_{l=1}^{p_1}s_{1,l}u_{1,l}+\epsilon\phi_1,\cdots,\sum_{l=1}^{p_k}s_{k,l}u_{k,l}+\epsilon\phi_k\right),(\phi_1,\cdots,\phi_k)\right\rangle<-1$$

and the function $\sum_{l=1}^{p_i}s_{i,l}u_{i,l}+\epsilon\phi_i$ has exactly p_i-1 zeros, denoted by,

$$0<r_{i,1}(s,\epsilon)<\cdots<r_{i,p_i-1}(s,\epsilon)<+\infty,$$

which depend continuously on s and ϵ.

Set

$$Q=\{s=(s_{1,1},\cdots,s_{1,p_1},\cdots,s_{k,1},\cdots,s_{k,p_k})\in\mathbb{R}^p\,|\,|s_{i,l}-1|\le\delta\}.$$

and let F be the function defined in Lemma 2.6. Decreasing δ if necessary, we may assume that $(1,\cdots,1,\cdots,1\cdots,1)$ is a strict global maximizer of $F|_Q$. Choose a function $h\in C^\infty(Q)$ such that $0\le h(s)\le 1$ for all $s\in Q$, $h(s)=0$ in a neighborhood of ∂Q, and $h(1,\cdots,1,\cdots,1\cdots,1)=1$.

Now, the functions $\sum_{l=1}^{p_i}s_{i,l}u_{i,l}+\delta h(s)\phi_i$, $i=1,2,\cdots,k$, have the following properties: for any $s\in Q$,

$$\left\langle \Phi'\left(\sum_{l=1}^{p_1}s_{1,l}u_{1,l}+\delta h(s)\phi_1,\,\cdots,\,\sum_{l=1}^{p_k}s_{k,l}u_{k,l}+\delta h(s)\phi_k\right),\,(\phi_1,\cdots,\phi_k)\right\rangle<-1, \tag{2.19}$$

the function $\sum_{l=1}^{p_i}s_{i,l}u_{i,l}+\delta h(s)\phi_i$ has exactly p_i-1 zeros, denoted by,

$$P_i(s):\ 0<r_{i,1}(s)<\cdots<r_{i,p_i-1}(s)<+\infty,$$

and $r_{i,l}(s)$ is a continuous function for any (i,l). Let $\mathbb{P}(s)=(P_1(s),\cdots,P_k(s))$ be the k-time partition formed by all the $r_{i,l}(s)$. Define the map $H:Q\to\mathbb{R}^p$ as

$$H(s)=(H_{1,1}(s),\cdots,H_{1,p_1}(s),\cdots,H_{k,1}(s),\cdots,H_{k,p_k}(s)),$$

where

$$H_{i,l}(s)=\|U_{i,l}(s)\|_i^2-\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}U^2_{i,l}(s)U^2_{j,m}(s)$$

and

$$U_{i,l}(s)=\left(\sum_{l'=1}^{p_i}s_{i,l'}u_{i,l'}+\delta h(s)\phi_i\right)\chi_{\Omega_{i,l}(\mathbb{P}(s))}.$$

Here, for $l=1,\cdots,p_i$ and $i=1,\cdots,k$,

$$\Omega_{i,l}(\mathbb{P}(s))=\mathrm{int}\{x\in\mathbb{R}^N|r_{i,l-1}(s)\le|x|<r_{i,l}(s)\},$$

$\chi_{\Omega_{i,l}(\mathbb{P}(s))}(x)=1$ for $x\in\Omega_{i,l}(\mathbb{P}(s))$, $\chi_{\Omega_{i,l}(\mathbb{P}(s))}(x)=0$ for $x\in\mathbb{R}^N\setminus\Omega_{i,l}(\mathbb{P}(s))$, and we use again the conventions $r_{i,0}(s)=0$ and $r_{i,p_i}(s)=+\infty$. Since the $r_{i,l}$'s are continuous functions, H is a continuous map.

We want to prove that there exists $s\in Q$ such that $H(s)=0$. For this we need to study the images of H on the boundary of Q. Suppose $s\in\partial Q$. Then $h(s)=0$ and $U_{j,m}(s)=s_{j,m}u_{j,m}$ for all j and m. For $s_{i,l}=1-\delta$, we have

$$\begin{aligned}
&H_{i,l}(s)\\
=&(1-\delta)^2\|u_{i,l}\|_i^2-(1-\delta)^2\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}s_{j,m}^2\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\\
\ge&(1-\delta)^2\|u_{i,l}\|_i^2-(1-\delta)^4\beta_{ii}\int_{\mathbb{R}^N}(u_{i,l})^4-(1-\delta)^4\sum_{j\ne i,\ \beta_{ij}<0}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\\
&-(1-\delta)^2(1+\delta)^2\sum_{j\ne i,\ \beta_{ij}>0}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\\
=&\delta(1-\delta)^2(2-\delta)\|u_{i,l}\|_i^2-4\delta(1-\delta)^2\sum_{j\ne i,\ \beta_{ij}>0}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2.
\end{aligned}$$

Since

$$\sum_{j\ne i,\ \beta_{ij}>0}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\le b\kappa\|u_{i,l}\|_i^2\sum_{j=1}^{k}\sum_{m=1}^{p_j}\|u_{j,m}\|_j^2\le 4b\kappa\mu_0\|u_{i,l}\|_i^2\le\frac{1}{4}\|u_{i,l}\|_i^2, \tag{2.20}$$

we see that

$$H_{i,l}(s)\ge\delta(1-\delta)^3\|u_{i,l}\|_i^2>0.$$

In the same way, if $s_{i,l}=1+\delta$ then

$$\begin{aligned}
&H_{i,l}(s)\\
=&(1+\delta)^2\|u_{i,l}\|_i^2-(1+\delta)^2\sum_{j=1}^{k}\sum_{m=1}^{p_j}\beta_{ij}s_{j,m}^2\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2\\
\le&-\delta(1+\delta)^2(2+\delta)\|u_{i,l}\|_i^2+4\delta(1+\delta)^2\sum_{j\ne i,\ \beta_{ij}>0}\sum_{m=1}^{p_j}\beta_{ij}\int_{\mathbb{R}^N}u_{i,l}^2u_{j,m}^2,
\end{aligned}$$

which together with (2.20) yields

$$H_{i,l}(s)\le-\delta(1+\delta)^2(2+\delta)\|u_{i,l}\|_i^2+\delta(1+\delta)^2\|u_{i,l}\|_i^2=-\delta(1+\delta)^3\|u_{i,l}\|_i^2<0.$$

Using a degree theory argument (or by the Miranda theorem), we see that there exists $s\in Q$ such that $H_{i,l}(s)=0$ for all (i,l). That is,

$$U(s):=(U_{1,1}(s),\cdots,U_{1,p_1}(s),\cdots,U_{k,1}(s),\cdots,U_{k,p_k}(s))\in\mathcal{N}(\mathbb{P}(s)).$$

Fix such an s in what follows. Then the definition of c implies

$$c\le c(\mathbb{P}(s))\le J_{\mathbb{P}(s)}(U(s))=\Phi\left(\sum_{l=1}^{p_1}s_{1,l}u_{1,l}+\delta h(s)\phi_1,\,\cdots,\,\sum_{l=1}^{p_k}s_{k,l}u_{k,l}+\delta h(s)\phi_k\right).$$

On the other hand, in view of (2.19), the Taylor expansion yields

$$\begin{aligned}
&\Phi\left(\sum_{l=1}^{p_1}s_{1,l}u_{1,l}+\delta h(s)\phi_1,\,\cdots,\,\sum_{l=1}^{p_k}s_{k,l}u_{k,l}+\delta h(s)\phi_k\right)\\
=&\Phi\left(\sum_{l=1}^{p_1}s_{1,l}u_{1,l},\,\cdots,\,\sum_{l=1}^{p_k}s_{k,l}u_{k,l}\right)\\
&+\delta h(s)\int_0^1\Big\langle\Phi'\Big(\sum_{l=1}^{p_1}s_{1,l}u_{1,l}+\theta\delta h(s)\phi_1,\,\cdots,\,\sum_{l=1}^{p_k}s_{k,l}u_{k,l}+\theta\delta h(s)\phi_k\Big),(\phi_1,\cdots,\phi_k)\Big\rangle d\theta\\
\le&\Phi\left(\sum_{l=1}^{p_1}s_{1,l}u_{1,l},\,\cdots,\,\sum_{l=1}^{p_k}s_{k,l}u_{k,l}\right)-\delta h(s).
\end{aligned}$$

Combining the last two inequalities, we have

$$c\le\Phi\left(\sum_{l=1}^{p_1}s_{1,l}u_{1,l},\,\cdots,\,\sum_{l=1}^{p_k}s_{k,l}u_{k,l}\right)-\delta h(s).$$

If $s_{i,l}=1$ for all (i,l) then we have a contradiction as

$$c\le\Phi\left(\sum_{l=1}^{p_1}u_{1,l},\,\cdots,\,\sum_{l=1}^{p_k}u_{k,l}\right)-\delta h(1,\cdots,1,\cdots,1,\cdots,1)=c-\delta.$$

If $s_{i,l}\neq1$ for some (i,l), then using the fact that $(1,\cdots,1,\cdots,1\cdots,1)$ is a strict global maximizer of $F|_Q$ we have

$$c\le\Phi\left(\sum_{l=1}^{p_1}s_{1,l}u_{1,l},\,\cdots,\,\sum_{l=1}^{p_k}s_{k,l}u_{k,l}\right)<\Phi\left(\sum_{l=1}^{p_1}u_{1,l},\,\cdots,\,\sum_{l=1}^{p_k}u_{k,l}\right)=c,$$

which is also a contradiction. Therefore, $(u_1,\cdots,u_k)$ is such a solution of (1.1) that u_i has exactly p_i-1 simple zeroes

$$r_{i,1},\,r_{i,2},\,\cdots,\,r_{i,p_i-1}.$$

At last, we estimate the quantity

$$\sum_{i,j=1}^{k}|\beta_{ij}|\int_{\mathbb{R}^N}u_i^2u_j^2.$$

Using the equality

$$\sum_{i=1}^{k}\|u_i\|_i^2=\sum_{i,j=1}^{k}\beta_{ij}\int_{\mathbb{R}^N}u_i^2u_j^2,$$

we have

$$\sum_{i,j=1}^{k}|\beta_{ij}|\int_{\mathbb{R}^N}u_i^2u_j^2=2\sum_{\beta_{ij}>0}\beta_{ij}\int_{\mathbb{R}^N}u_i^2u_j^2-\sum_{i=1}^{k}\|u_i\|_i^2.$$

Therefore,

$$\sum_{i,j=1}^{k}|\beta_{ij}|\int_{\mathbb{R}^N}u_i^2u_j^2\le 2B^*\kappa\sum_{i,j=1}^{k}\|u_i\|_i^2\|u_j\|_j^2=2B^*\kappa\left(\sum_{i=1}^{k}\|u_i\|_i^2\right)^2,$$

which together with the fact that

$$\frac{1}{4}\sum_{i=1}^{k}\|u_i\|_i^2<\mu_0$$

implies

$$\sum_{i,j=1}^{k}|\beta_{ij}|\int_{\mathbb{R}^N}u_i^2u_j^2\le 32B^*\kappa\mu_0^2.$$

The proof is completed. □

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