# Regularized Interpolation Driven by Total Variation

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**Abstract.** We explore minimization problems of the form

$$\text{Inf} \left\{ \int_0^1 |u'| + \sum_{i=1}^k |u(a_i) - f_i|^2 + \alpha \int_0^1 |u|^2 \right\},$$

where $u$ is a function defined on $(0,1)$, $(a_i)$ are $k$ given points in $(0,1)$, with $k \geq 2$, $(f_i)$ are $k$ given real numbers, and $\alpha \geq 0$ is a parameter taken to be 0 or 1 for simplicity. The natural functional setting is the Sobolev space $W^{1,1}(0,1)$. When $\alpha = 0$ the Inf is achieved in $W^{1,1}(0,1)$. However, when $\alpha = 1$, minimizers need not exist in $W^{1,1}(0,1)$. One is led to introduce a relaxed functional defined on the space $BV(0,1)$, whose minimizers always exist and can be viewed as generalized solutions of the original ill-posed problem.

**Key Words**: Interpolation, minimization problems, functions of bounded variation, relaxed functional.

**AMS Subject Classifications**: 26B30, 49J45, 65D05

## 1 Introduction

Given $k$ points, with $k \geq 2$,

$$0 < a_1 < a_2 < \cdots < a_k < 1, \tag{1.1}$$

and $k$ real numbers $f_i$, $i = 1, \cdots, k$, the aim is to find a function $u$ defined on $(0,1)$ such that $u(a_i)$ approximates $f_i$ as best as possible, and keeping at the same time some control on the regularity of $u$, measured here in terms of total variation of $u$. For this purpose define the functional

$$F(u) = \int_0^1 |u'| + \sum_{i=1}^k |u(a_i) - f_i|^2, \tag{1.2}$$

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and then minimize $F$. (One may also insert a fidelity parameter in front of the first integral, but we take to be 1 for simplicity). Note that $F$ is well-defined on the Sobolev space $W^{1,1}(0,1)$ since $W^{1,1}(0,1) \subset C([0,1])$, so that $u(a_i)$ makes sense. As is well-known $W^{1,1}(0,1)$ is not a good function space from the point of view of minimization techniques in Functional Analysis. Often, variational problems do *not* admit minimizers in $W^{1,1}(0,1)$. To make up for this "defect" one is usually led to enlarge $W^{1,1}(0,1)$ and replace it by $BV(0,1)$, the space of functions of bounded variation (see e.g., [1,2,5]), where the existence of minimizers is often a matter of routine. The drawback is that the specific functional $F$ is not properly defined on $BV(0,1)$ since the term $u(a_i)$ has no obvious meaning when $u$ has a jump at $a_i$.

In Section 2 we establish that (surprisingly!) the problem

$$\operatorname*{Inf}_{u \in W^{1,1}(0,1)} F(u) \tag{1.3}$$

always admits minimizers. In fact all minimizers are classified with the help of a finite-dimensional auxiliary problem. Given

$$\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{R}^k,$$

set

$$\Phi(\lambda) := \sum_{i=1}^{k-1} |\lambda_{i+1} - \lambda_i| + \sum_{i=1}^{k} |\lambda_i - f_i|^2. \tag{1.4}$$

By convexity

$$m := \min_{\lambda \in \mathbb{R}^k} \Phi(\lambda) \tag{1.5}$$

is achieved by some unique $\lambda$ denoted

$$U = (U_1, \cdots, U_k),$$

and which plays an important role throughout the paper. In this section we never invoke Functional Analysis and the space $BV(0,1)$ is noticeably absent. The existence of minimizers in $W^{1,1}(0,1)$ is derived from an *elementary* computation originally due to T. Sznigir [6,7]. However this "miracle" does not repeat itself: as we are going to see in Section 5 even "mild" pertubations of $F$ need not admit minimizers in $W^{1,1}(0,1)$, and there it will be essential to "relax" the problem and search for minimizers in $BV(0,1)$ using tools of Functional Analysis.

In Section 3 we introduce the relaxed functional $F_r$ of $F$, which is much better suited to minimization problems involving the functional $F$. We start with the standard abstract formulation, namely $F_r$ is defined for every $v \in BV(0,1)$ by

$$F_r(v) := \operatorname{Inf} \liminf_{n \to \infty} F(v_n), \tag{1.6}$$

where the Inf in (1.6) is taken over all sequences $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$. The main result, Theorem 3.1, provides an *explicit* formula for $F_r$. The major

obstacle stems from the fact that $u(a)$ is not well-defined when $u \in BV(0,1)$; however, $u$ admits at every point $a \in (0,1)$ limits from the left and from the right, which enter in the formula for $F_r$. Theorem 4.1 provides a complete description of all minimizers of $F_r$ on $BV(0,1)$. It turns out that $F_r$ admits *many more* minimizers than the original functional $F$, even when $F_r$ is restricted to $W^{1,1}(0,1)$.

In Section 5 we consider a mild perturbation of $F$, and we show that the corresponding minimizing problems differ significantly from those associated with $F$. Set

$$G(u) = F(u) + \int_0^1 |u|^2 = \int_0^1 |u'| + \sum_{i=1}^k |u(a_i) - f_i|^2 + \int_0^1 |u|^2, \qquad (1.7)$$

where $u \in W^{1,1}(0,1)$. Our initial goal is to investigate the minimization problem

$$A = \operatorname*{Inf}_{u \in W^{1,1}} G(u). \qquad (1.8)$$

As we are going to see the infimum in (1.8) need *not* be achieved and we will replace it by a relaxed problem defined on $BV(0,1)$ as we have done in Section 3. It is easy to check that the relaxed functional $G_r$ of $G$ is given by

$$G_r(v) = F_r(v) + \int_0^1 |v|^2, \qquad \forall v \in BV(0,1), \qquad (1.9)$$

so that $G_r$ is strictly convex on $BV(0,1)$ and it is lower semicontinuous in the sense that for every sequence $(v_n) \subset BV(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ as $n \to \infty$, we have

$$\liminf_{n \to \infty} G_r(v_n) \geq G_r(v).$$

Consequently

$$B = \operatorname*{Inf}_{v \in BV} G_r(v) \qquad (1.10)$$

is uniquely achieved and we denote by $\bar{v} \in BV(0,1)$, its unique minimizer.

The bottom line is that we have replaced Problem (1.8) which need *not* have a solution by Problem (1.10) which *always* admits a unique solution $\bar{v}$. Moreover, *if* Problem (1.8) admits a minimizer, it must coincide with $\bar{v}$. Therefore $\bar{v}$ may be viewed as *the generalized solution* of Problem (1.8). In addition, $\bar{v}$ has a very simple structure and can be computed via a *finite-dimensional* convex minimization problem.

## 2 The functional $F$ and its minimizers on $W^{1,1}$

The main result in this section is

**Theorem 2.1** (T. Sznigir [6,7]). *We have*

$$m = \operatorname*{Inf}_{u \in W^{1,1}} F(u), \qquad (2.1)$$

*where m has been defined in* (1.5), *and the* Inf *in* (2.1) *is achieved. More precisely* $u \in W^{1,1}(0,1)$ *is a minimizer if and only if it satisfies the following three conditions:*

$$u \text{ is monotone on each interval } (a_i, a_{i+1}), \quad i = 1, \cdots, k-1, \tag{2.2a}$$

$$u(a_i) = U_i, \quad i = 1, \cdots k, \tag{2.2b}$$

$$u(x) = U_1, \quad \forall x \in [0, a_1] \quad and \quad u(x) = U_k, \quad \forall x \in [a_k, 1]. \tag{2.2c}$$

*Proof.* Given $u \in W^{1,1}(0,1)$ we have

$$\int_0^1 |u'| \geq \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} |u'| \geq \sum_{i=1}^{k-1} |u(a_{i+1}) - u(a_i)|, \tag{2.3}$$

with equalities if and only if:

$$u \text{ is monotone on each interval } \quad (a_i, a_{i+1}), \tag{2.4a}$$

$$u \text{ is constant on } (0, a_1) \text{ and on } \quad (a_k, 1). \tag{2.4b}$$

Thus

$$F(u) \geq \sum_{i=1}^{k-1} |u(a_{i+1}) - u(a_i)| + \sum_{i=1}^{k} |u(a_i) - f_i|^2.$$

Letting $\lambda_i = u(a_i)$, $i = 1, \cdots, k$, we see that, for every $u \in W^{1,1}(0,1)$,

$$F(u) \geq \min_{\lambda \in \mathbb{R}^k} \Phi(\lambda) = m. \tag{2.5}$$

If $u \in W^{1,1}(0,1)$ satisfies (2.2a)-(2.2c) we have

$$F(u) = \sum_{i=1}^{k-1} |U_{i+1} - U_i| + \sum_{i=1}^{k} |U_i - f_i|^2 = m,$$

so that $u$ is a minimizer for (2.1). Conversely if $u \in W^{1,1}(0,1)$ is such that $F(u) = m$ then (2.4a) and (2.4b) hold. Moreover $u(a_i) = \lambda_i$ is a minimizer in (1.5), and by uniqueness we have $u(a_i) = U_i$ for $i = 1, \cdots, k$. □

**Remark 2.1.** In view of the abundance of minimizers for $F$ in $W^{1,1}(0,1)$ one may wonder whether some of them are "preferred" e.g., in the sense that they are "stable" with respect to pertubations. The minimizer $u_\ell$ of $F$ which is obtained by *linear* interpolation (i.e., $u_\ell$ is linear on each interval $(a_i, a_{i+1})$ is definitely a good candidate. Here are three "natural" perturbed functionals:

$$F_{1,\varepsilon}(u) = \varepsilon \int_0^1 |u'|^2 + F(u), \qquad\qquad u \in H^1(0,1), \qquad \varepsilon > 0,$$

$$F_{2,p}(u) = \int_0^1 |u'|^p + \sum_{i=1}^{k} |u(a_i) - f_i|^2, \qquad u \in W^{1,p}(0,1), \quad p > 1,$$

$$F_{3,\varepsilon}(u) = \varepsilon \int_0^1 |u''|^2 + F(u), \qquad\qquad u \in H^2(0,1), \qquad \varepsilon > 0.$$

It is easy to see that each one admits a unique minimizer. T. Sznigir [6,7] has established that as $\varepsilon \to 0$ (resp. $p \searrow 1$) the minimizers of $F_{1,\varepsilon}$ (resp. $F_{2,p}$) converge to $u_\ell$. By contrast the minimizers of $F_{3,\varepsilon}$ converge as $\varepsilon \to 0$ to the solution $\hat{u}$ of a variational inequality corresponding to

$$\min \left\{ \int_0^1 |u''|^2; \, u \in H^2(0,1) \text{ and satisfies } (2.2a) - (2.2c) \right\}.$$

The function $\hat{u}$ belongs to $C^1([0,1])$ (while $u_\ell \notin C^1$) and $\hat{u}$ is a piecewise cubic function on each interval $(a_i, a_{i+1})$, $i = 1, \cdots, k-1$, see [6,7].

## 3 The relaxed functional $F_r$ on $BV$

As usual the relaxed functional $F_r$ is defined for every $v \in BV(0,1)$ by

$$F_r(v) := \text{Inf} \liminf_{n \to \infty} F(v_n), \tag{3.1}$$

where the Inf in (3.1) is taken over all sequences $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ (the choice of $L^2$ is just a matter of convenience–one can replace it by any $L^p$, $1 \le p < \infty$).

The main result in this section is an *explicit* formula for $F_r$, but first some notation. Given $v \in BV(0,1)$ and $a \in (0,1)$ we denote by $j(v)(a)$ the jump interval of $v$ at $a$, i.e.,

$$j(v)(a) = [\min(v(a-0), v(a+0)), \max(v(a-0), v(a+0))]. \tag{3.2}$$

We also set

$$\varphi(t) = \begin{cases} t^2, & \text{if } 0 \le t \le 1, \\ 2t - 1, & \text{if } t > 1. \end{cases} \tag{3.3}$$

**Theorem 3.1.** *For every $v \in BV(0,1)$, we have*

$$F_r(v) = \int_0^1 |v'| + \sum_{i=1}^k \varphi \left( \text{dist}(f_i, j(v)(a_i)) \right), \tag{3.4}$$

*where* dist *denotes the distance of a point to a set.*

The proof of Theorem 3.1 relies on the following three lemmas. The first two are familiar to the experts (see e.g., [4, Appendix 18.8] and [3, Lemma 2]).

**Lemma 3.1.** *Let $(v_n)$ be a bounded sequence in $BV(a,b)$ such that $v_n \to v$ in $L^1(a,b)$, $v_n(a) \to \alpha$, $v_n(b) \to \beta$ as $n \to \infty$. Then $v \in BV(a,b)$ and*

$$\liminf_{n \to \infty} \int_a^b |v_n'| \ge \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|, \tag{3.5}$$

*where we write for simplicity $v_n(a) = v_n(a+0)$, etc.*

*Proof.* Fix any function $h \in C_c^\infty(\mathbb{R})$ such that $h(a) = \alpha$ and $h(b) = \beta$. Consider the functions

$$w_n(t) := \begin{cases} h(t), & \text{if } t < a, \\ v_n(t), & \text{if } a \leq t \leq b, \\ h(t), & \text{if } t > b, \end{cases} \qquad w(t) := \begin{cases} h(t), & \text{if } t < a, \\ v(t), & \text{if } a \leq t \leq b, \\ h(t), & \text{if } t > b. \end{cases}$$

Clearly $w_n, w \in BV(\mathbb{R})$ and

$$\int_{\mathbb{R}} |w_n'| = \int_{-\infty}^{a} |h'| + \int_{a}^{b} |v_n'| + \int_{b}^{\infty} |h'| + |v_n(a) - \alpha| + |v_n(b) - \beta|, \tag{3.6a}$$

$$\int_{\mathbb{R}} |w'| = \int_{-\infty}^{a} |h'| + \int_{a}^{b} |v'| + \int_{b}^{\infty} |h'| + |v(0) - \alpha| + |v(b) - \beta|. \tag{3.6b}$$

Since $w_n \to w$ in $L^1(\mathbb{R})$ it is well-known that

$$\liminf_{n \to \infty} \int_{\mathbb{R}} |w_n'| \geq \int_{\mathbb{R}} |w'|.$$

Combining this with (3.6) yields (3.5). □

**Lemma 3.2.** *Given any $v \in BV(a, b)$ and constants $\alpha, \beta \in \mathbb{R}$, there exists a sequence $(v_n) \subset W^{1,1}(a, b)$ such that $v_n \to v$ in $L^2(a, b)$, $v_n(a) = \alpha$, $v_n(b) = \beta$, $\forall n$, and*

$$\lim_{n \to \infty} \int_{a}^{b} |v_n'| = \int_{a}^{b} |v'| + |v(a) - \alpha| + |v(b) - \beta|. \tag{3.7}$$

*Proof.* Set

$$w(t) := \begin{cases} \alpha, & \text{if } t < a, \\ v(t), & \text{if } a \leq t \leq b, \\ \beta, & \text{if } t > b. \end{cases}$$

Let $w_n = \rho_n * w$ where $(\rho_n)$ is a sequence of mollifiers. Clearly

$$\int_{\mathbb{R}} |w_n'| \leq \int_{\mathbb{R}} |w'| = \int_{a}^{b} |v'| + |v(a) - \alpha| + |v(b) - \beta|. \tag{3.8}$$

Moreover $w_n(t) = \alpha$ if $t < a - (1/n)$ and $w_n(t) = \beta$ if $t > b + (1/n)$. Rescaling the sequence $(w_n)$ by a change of variables we obtain a sequence $(v_n)$ of smooth functions such that $v_n \to v$ in $L^2(a, b)$, $v_n(a) = \alpha$, $v_n(b) = \beta$, $\forall n$, and

$$\limsup_{n \to \infty} \int_{a}^{b} |v_n'| \leq \int_{a}^{b} |v'| + |v(a) - \alpha| + |v(b) - \beta|.$$

Applying Lemma 3.1 we conclude that (3.7) holds. □

The third lemma relies on an elementary computation left to the reader.

**Lemma 3.3.** *Given any $\alpha, \beta, f \in \mathbb{R}$ we have*

$$\text{Inf}_{t \in \mathbb{R}} \left\{ |t - \alpha| + |t - \beta| + |t - f|^2 \right\} = |\alpha - \beta| + \varphi(\text{dist}(f, J)), \tag{3.9}$$

*where $J = [\min(\alpha, \beta), \max(\alpha, \beta)]$ and $\varphi$ has been defined in (3.1).*

*Proof* of Theorem 3.1. It consists of two steps.

**Step 1.** Given any $v \in BV(0,1)$ there exists a sequence $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ and

$$\lim_{n \to \infty} F(v_n) = F_r(v),$$

where $F_r(v)$ is defined by (3.4).

*Proof.* Applying Lemma 3.3 with $\alpha = v(a_i - 0)$, $\beta = v(a_i + 0)$, and $f = f_i$, $1 \le i \le k$, we obtain some $t_i$ (a minimizer in (3.9)) such that

$$\begin{aligned} &|t_i - v(a_i - 0)| + |t_i - v(a_i + 0)| + |t_i - f_i|^2 \\ =\,&|v(a_i - 0) - v(a_i + 0)| + \varphi(\text{dist}\,(f_i, j(v)(a_i))). \end{aligned} \tag{3.10}$$

We next apply Lemma 3.2 successively on

$$(0, a_1), \quad (a_i, a_{i+1}), \quad 1 \le i \le k-1, \quad \text{and} \quad (a_k, 1).$$

First on $(0, a_1)$ with $\alpha = v(0+)$ and $\beta = t_1$. This yields a sequence $(v_n) \subset W^{1,1}(0, a_1)$ such that $v_n(0) = v(0+)$, $v_n(a_1) = t_1$, $\forall n$, $v_n \to v$ in $L^2(0, a_1)$, and

$$\int_0^{a_1} |v_n'| = \int_0^{a_1} |v'| + |v(a_1 - 0) - t_1| + o(1). \tag{3.11}$$

Next on $(a_i, a_{i+1})$, $1 \le i \le k-1$, with $\alpha = t_i$ and $\beta = t_{i+1}$; this yields a sequence $(v_n) \subset W^{1,1}(a_i, a_{i+1})$ such that $v_n(a_i) = t_i$, $v_n(a_{i+1}) = t_{i+1}$, $v_n \to v$ in $L^2(a_i, a_{i+1})$, and

$$\int_{a_i}^{a_{i+1}} |v_n'| = \int_{a_i}^{a_{i+1}} |v'| + |v(a_i + 0) - t_i| + |v(a_{i+1} - 0) - t_{i+1}| + o(1). \tag{3.12}$$

Finally on $(a_k, 1)$ with $\alpha = t_k$ and $\beta = v(1-)$; this yields a sequence $(v_n) \subset W^{1,1}(a_k, 1)$ such that $v_n(a_k) = t_k$, $v_n(1) = v(1-)$, $v_n \to v$ in $L^2(a_k, 1)$, and

$$\int_{a_k}^1 |v_n'| = \int_{a_k}^1 |v'| + |v(a_k + 0) - t_k| + o(1). \tag{3.13}$$

Glueing these functions we obtain a sequence $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$, and

$$\int_0^1 |v_n'| = \sum_{i=0}^k \int_{a_i}^{a_{i+1}} |v'| + \sum_{i=1}^k (|v(a_i - 0) - t_i| + |v(a_i + 0) - t_i|) + o(1), \tag{3.14}$$

with the convention that $a_0 = 0$ and $a_{k+1} = 1$.

Inserting (3.10) into (3.14) we see that

$$\int_0^1 |v'_n| = \int_0^1 |v'| - \sum_{i=1}^k |t_i - f_i|^2 + \sum_{i=1}^k \varphi(\text{dist } (f_i, j(v)(a_i))) + o(1). \qquad (3.15)$$

Since $v_n(a_i) = t_i, \forall n, \forall i$, we conclude that

$$F(v_n) = \int_0^1 |v'_n| + \sum_{i=1}^k |v_n(a_i) - f_i|^2 = F_r(v) + o(1), \qquad (3.16)$$

which completes the proof of Step 1.

**Step 2.** Let $(v_n)$ be a bounded sequence in $W^{1,1}(0,1)$ such that $v_n \to v$ in $L^1(0,1)$. Then $v \in BV(0,1)$ and

$$\liminf_{n \to \infty} F(v_n) \geq F_r(v). \qquad (3.17)$$

*Proof.* Passing to a subsequence we may always assume that, for every $i = 0, 1, \cdots, k+1$, there exists some $\ell_i$ such that

$$v_n(a_i) \to \ell_i \quad \text{as} \quad n \to \infty.$$

From Lemma 3.1 we know that for every $i = 0, 1, \cdots k$,

$$\int_{a_i}^{a_{i+1}} |v'_n| \geq \int_{a_i}^{a_{i+1}} |v'| + |v(a_i + 0) - \ell_i| + |v(a_{i+1} - 0) - \ell_{i+1}| + o(1).$$

Adding these inequalities yields

$$F(v_n) \geq \sum_{i=0}^k \int_{a_i}^{a_{i+1}} |v'| + \sum_{i=1}^k (|v(a_i + 0) - \ell_i| + |v(a_i - 0) - \ell_i| + |\ell_i - f_i|^2) + o(1).$$

Applying Lemma 3.3 we find that

$$F(v_n) \geq \sum_{i=0}^k \int_{a_i}^{a_{i+1}} |v'| + \sum_{i=1}^k |v(a_i + 0) - v(a_i - 0)| + \sum_{i=1}^k \varphi(\text{dist } (f_i, j(v)(a_i)) + o(1)$$
$$= F_r(v) + o(1),$$

which completes the proof of Step 2, and thereby the proof of Theorem 3.1. □

## 4  Some properties of $F_r$

We discuss in this section some properties of $F_r$. First a few straightforward facts. We have

$$F_r(v) \leq F(v), \qquad \forall v \in W^{1,1}(0,1), \qquad (4.1)$$

indeed it suffices to choose $v_n = v, \forall n$ in (3.1). It may happen that $F_r(v) < F(v)$ for some $v's$ in $W^{1,1}(0,1)$. In fact

$$[F_r(v) = F(v) \quad \text{for some} \quad v \in W^{1,1}(0,1)] \Leftrightarrow [|v(a_i) - f_i| \leq 1, \ \forall i = 1, \cdots, k], \quad (4.2)$$

this is an immediate consequence of (1.2), (3.4) and (3.3).

**Lemma 4.1.** *The functional $F_r$ is convex on $BV(0,1)$ and it is lower semicontinuous in the sense that for every sequence $(v_n) \subset BV(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ as $n \to \infty$, we have*

$$\liminf_{n \to \infty} F_r(v_n) \geq F_r(v). \quad (4.3)$$

*Proof.* Given $v, w \in BV(0,1)$ there exist (by Step 1 above) sequences $(v_n), (w_n) \subset W^{1,1}(0,1)$ such that $v_n \to v, w_n \to w$ in $L^2(0,1)$ and $F(v_n) \to F_r(v), F(w_n) \to F_r(w)$. By convexity of $F$ we have

$$F(tv_n + (1-t)w_n) \leq tF(v_n) + (1-t)F(w_n), \qquad \forall t \in [0,1]. \quad (4.4)$$

Passing to the limit in (4.4) and using Step 2 we see that

$$F_r(tv + (1-t)w) \leq tF_r(v) + (1-t)F_r(w).$$

Next, the proof of (4.3). By Step 1 applied to $v_n$ with $n$ fixed we may find some $w_n \in W^{1,1}(0,1)$ such that

$$\|v_n - w_n\|_{L^2} < \frac{1}{n} \quad \text{and} \quad |F_r(v_n) - F(w_n)| < \frac{1}{n}. \quad (4.5)$$

Thus $w_n \to v$ in $L^2(0,1)$ and from the definition (3.1) we conclude that

$$F_r(v) \leq \liminf_{n \to \infty} F(w_n) = \liminf_{n \to \infty} F_r(v_n) \quad \text{by} \quad (4.5).$$

Thus, we complete the proof. □

We now discuss the minization of $F_r$ on $BV(0,1)$. Recall (see Theorem 2.1) that

$$m = \min_{v \in W^{1,1}} F(v), \quad (4.6)$$

where $m$ is defined by (1.5). Set

$$\mu := \inf_{v \in BV} F_r(v). \quad (4.7)$$

From Lemma 4.1 and the compactness of the embedding $BV(0,1) \subset L^2(0,1)$ we deduce that the Inf in (4.7) is achieved. Clearly, by (4.1),

$$\mu = \inf_{v \in BV} F_r(v) \leq \inf_{v \in W^{1,1}} F_r(v) \leq \inf_{v \in W^{1,1}} F(v) = m. \quad (4.8)$$

We claim that

$$\mu = m. \tag{4.9}$$

Indeed, by (4.6) we have

$$m \le F(v), \qquad \forall v \in W^{1,1}(0,1). \tag{4.10}$$

From Step 1 above and (4.10) we deduce that

$$m \le F_r(v), \qquad \forall v \in BV(0,1), \tag{4.11}$$

and thus

$$m \le \inf_{v \in BV} F_r(v) = \mu.$$

Combined with (4.8) this yields (4.9).

As a consequence, any minimizer for $F$ on $W^{1,1}(0,1)$ must be a minimizer for $F_r$ on $BV(0,1)$. Indeed if $F(u) = m$, then $\mu \le F_r(u) \le F(u) = m = \mu$ so that $F_r(u) = \mu$.

The next result provides a complete description of all minimizers of $F_r$ on $BV(0,1)$.

**Theorem 4.1.** *Assume that $u \in BV(0,1)$ satisfies the following three conditions:*

$$\begin{cases} u \text{ is monotone nondecreasing (resp. nonincreasing)} \\ \text{ on each interval } (a_i, a_{i+1}), \ i = 1, \cdots, k-1, \\ \text{ such that } U_i \le U_{i+1} (\text{ resp. } U_{i+1} \le U_i), \end{cases} \tag{4.12a}$$

$$\begin{cases} U_i \le u(a_i + 0) \ \text{ and } \ u(a_{i+1} - 0) \le U_{i+1} \ \text{ if} \\ U_i \le U_{i+1} (\text{resp. reverse inequalities if } U_{i+1} \le U_i), \end{cases} \tag{4.12b}$$

$$u(x) = U_1, \quad \forall x \in [0, a_1] \quad \text{and} \quad u(x) = U_k, \quad \forall x \in [a_k, 1], \tag{4.12c}$$

*then $u$ is a minimizer of $F_r$ on $BV(0,1)$. And conversely.*

**Remark 4.1.** We deduce from Theorem 4.1 that the relaxed functional $F_r$ admits *many more* minimizers than the original functional $F$, even when $F_r$ is restricted to $W^{1,1}(0,1)$, since they are not bound by the rigid constraint $u(a_i) = U_i, \forall i$.

The proof relies on the following monotone version of Lemma 3.2.

**Lemma 4.2.** *Given any nondecreasing function $v$ on $(a,b)$ and constants $\alpha \le v(a), \beta \ge v(b)$, there exists a sequence of nondecreasing functions $(v_n) \subset W^{1,1}(a,b)$ such that $v_n \to v$ in $L^2(a,b), v_n(a) = \alpha, v_n(b) = \beta \quad \forall n$, and*

$$\lim_{n \to \infty} \int_b^a |v_n'| = \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|. \tag{4.13}$$

The proof of Lemma 4.2 which is similar to the proof of Lemma 3.2 is left to the reader. We now turn to the

*Proof of Theorem 4.1.* Applying Lemma 4.2 on the interval $(a_i, a_{i+1})$ with $\alpha = U_i$ and $\beta = U_{i+1}$ we obtain a sequence $(v_n)$ of monotone functions in $W^{1,1}(a_i, a_{i+1})$ such that $v_n \to u$ in $L^2(a_i, a_{i+1})$,

$$v_n(a_i) = U_i \quad \text{and} \quad v_n(a_{i+1}) = U_{i+1}, \qquad \forall n, \tag{4.14a}$$

$$\lim_{n\to\infty} \int_{a_i}^{a_{i+1}} |v_n'| = \int_{a_i}^{a_{i+1}} |u'| + |u(a_i + 0) - U_i| + |u(a_{i+1} - 0) - U_{i+1}|. \tag{4.14b}$$

Next we set

$$v_n(x) = U_1, \quad \forall x \in [0, a_1] \quad \text{and} \quad v_n(x) = U_k, \quad \forall x \in [a_k, 1]. \tag{4.15}$$

Glueing the functions $v_n$ defined above we obtain a function still denoted $v_n \in W^{1,1}(0,1)$, satisfying all the requirements of Theorem 2.1. Thus $v_n$ is a minimizer for $F$ in $W^{1,1}(0,1)$ so that

$$F(v_n) = m, \qquad \forall n. \tag{4.16}$$

Since $v_n \to u$ in $L^2(0,1)$, we deduce (from the definition (3.1) of $F_r$) that

$$F_r(u) \le \lim_{n\to\infty} F(v_n) = m.$$

(Note that the full strength of Lemma 4.1 was not used). Invoking (4.9) we conclude that $u$ is a minimizer for $F_r$.

We now turn to the converse. Assume that $u$ is a minimizer for $F_r$ on $BV(0,1)$. Let $t_i, i = 1, \cdots, k$, be the unique minimizer in (3.9) corresponding to $\alpha = u(a_i - 0), \beta = u(a_i + 0)$ and $f = f_i$, so that

$$|t_i - u(a_i - 0)| + |t_i - u(a_i + 0)| + |t_i - f_i|^2$$
$$= |u(a_i - 0) - u(a_i + 0)| + \varphi(\text{dist}(f_i, j(u)(a_i))). \tag{4.17}$$

Next write, for $1 \le i \le k - 1$,

$$t_i - t_{i+1} = (t_i - u(a_i + 0)) + (u(a_i + 0) - u(a_{i+1} - 0)) + (u(a_{i+1} - 0) - t_{i+1}), \tag{4.18}$$

so that

$$|t_i - t_{i+1}| \le |t_i - u(a_i + 0)| + |u(a_i + 0) - u(a_{i+1} - 0)| + |u(a_{i+1} - 0) - t_{i+1}|. \tag{4.19}$$

We now compute, as in (1.4),

$$\Phi(\bar{t}) = \sum_{i=1}^{k-1} |t_i - t_{i+1}| + \sum_{i=1}^{k} |t_i - f_i|^2,$$

where $\bar{t}$ is defined by $\bar{t} := (t_1, \cdots, t_k)$. From (4.19) and (4.17) we have when $k \geq 3$ (if $k = 2$ go directly to (4.20))

$$
\begin{aligned}
\sum_{i=1}^{k-1} & |t_i - t_{i+1}| \\
\leq & \sum_{i=1}^{k-1} |t_i - u(a_i + 0)| + \sum_{i=2}^{k} |t_i - u(a_i - 0)| + \sum_{i=1}^{k-1} |u(a_i + 0) - u(a_{i+1} - 0)| \\
= & |t_1 - u(a_1 + 0)| + |t_k - u(a_k - 0)| + \sum_{i=2}^{k-1} (|t_i - u(a_i - 0)| + |t_i - u(a_i + 0)|) \\
& + \sum_{i=1}^{k-1} |u(a_i + 0) - u(a_{i+1} - 0)| \\
= & |t_1 - u(a_1 + 0)| + |t_k - u(a_k - 0)| + \sum_{i=2}^{k-1} |u(a_i + 0) - u(a_i - 0)| \\
& + \sum_{i=2}^{k-1} \varphi(\text{dist}(f_i, j(u)(a_i))) - \sum_{i=2}^{k-1} |t_i - f_i|^2 + \sum_{i=1}^{k-1} |u(a_i + 0) - u(a_{i+1} - 0)|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Phi(\bar{t}) = & \sum_{i=1}^{k-1} |t_i - t_{i+1}| + \sum_{i=1}^{k} |t_i - f_i|^2 \\
\leq & |t_1 - u(a_1 + 0)| + |t_k - u(a_k - 0)| + |t_1 - f_1|^2 + |t_k - f_k|^2 + \sum_{i=2}^{k-1} |u(a_i + 0) \\
& - u(a_i - 0)| + \sum_{i=1}^{k-1} |u(a_i + 0) - u(a_{i+1} - 0)| + \sum_{i=2}^{k-1} \varphi(\text{dist}(f_i, j(u)(a_i)) \\
\leq & |t_1 - u(a_1 + 0)| + |t_k - u(a_k - 0)| + |t_1 - f_1|^2 + |t_k - f_k|^2 \\
& + \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} |u'| + \sum_{i=2}^{k-1} |u(a_i + 0) - u(a_i - 0)| + \sum_{i=2}^{k-1} \varphi(\text{dist}\,(f_i, j(u)(a_i)) \\
= & |t_1 - u(a_1 + 0)| + |t_k - u(a_k - 0)| + |t_1 - f_1|^2 + |t_k - f_k|^2 \\
& + \int_0^1 |u'| - \int_0^{a_1} |u'| - \int_{a_k}^1 |u'| - |u(a_1 + 0) - u(a_1 - 0)| \\
& - |u(a_k + 0) - u(a_k - 0)| + \sum_{i=2}^{k-1} \varphi(\text{dist}(f_i, j(u)(a_i)).
\end{aligned}
$$

Since $u$ is a minimizer for $F_r$ we know by (4.7) and (4.9) that

$$
F_r(u) = \int_0^1 |u'| + \sum_{i=1}^{k} \varphi(\text{dist}(f_i, j(u)(a_i))) = m,
$$

so that

$$
\begin{aligned}
\Phi(\bar{t}) \leq & |t_1 - u(a_1 + 0)| + |t_k - u(a_k - 0)| + |t_1 - f_1|^2 + |t_k - f_k|^2 \\
& + m - \varphi(\text{dist}(f_1, j(u)(a_1))) - \varphi(\text{dist}(f_k, j(u)(a_k))) - \int_0^{a_1} |u'| \\
& - \int_{a_k}^1 |u'| - |u(a_1 + 0) - u(a_1 - 0)| - |u(a_k + 0) - u(a_k - 0)|.
\end{aligned} \tag{4.20}
$$

Finally we use (4.17) for $i = 1$ and $i = k$, and deduce from (4.20) that

$$
\Phi(\bar{t}) \leq -|t_1 - u(a_1 - 0)| - |t_k - u(a_k + 0)| - \int_0^{a_1} |u'| - \int_{a_k}^1 |u'| + m. \tag{4.21}
$$

Therefore

$$
\Phi(\bar{t}) \leq m,
$$

so that by (1.5), $\bar{t} = (t_1, \cdots, t_k)$ is a minimizer of $\Phi$ on $\mathbb{R}^k$. By uniqueness we have

$$
t_i = U_i, \qquad \forall i. \tag{4.22}
$$

Moreover from (4.21) we deduce that

$$
|t_1 - u(a_1 - 0)| = |t_k - u(a_k + 0)| = \int_0^{a_1} |u'| = \int_0^{a_k} |u'| = 0.
$$

Consequently (4.12c) holds. Returning to the above estimates we infer that all inequalities are equalities. In particular, $\forall i = 1, \cdots, k - 1$,

$$
\int_{a_i}^{a_{i+1}} |u'| = |u(a_i + 0) - u(a_{i+1} - 0)| \tag{4.23}
$$

and

$$
|t_i - t_{i+1}| = |t_i - u(a_i + 0)| + |u(a_i + 0) - u(a_{i+1} - 0)| + |u(a_{i+1} - 0) - t_{i+1}|. \tag{4.24}
$$

Equality (4.23) implies that $u$ is monotone on the interval $(a_i, a_{i+1})$, while equality (4.24) yields

$$
\begin{aligned}
\text{sign}(t_i - t_{i+1}) = & \text{sign}(t_i - u(a_i + 0)) = \text{sign}(u(a_i + 0) - u(a_{i+1} - 0)) \\
= & \text{sign}(u(a_{i+1} - 0) - t_{i+1}).
\end{aligned}
$$

In view of (4.22) we conclude easily that $u$ satisfies (4.12a) - (4.12b). $\qquad \square$

**Remark 4.2.** Theorem 4.1 is stated in T. Sznigir [6] though the proof in [6] is somewhat obscure.

**Remark 4.3.** We have

$$|U_i - f_i| \leq 1, \qquad \forall i = 1, \cdots, k. \tag{4.25}$$

Indeed consider the piecewise linear function $u_\ell$ defined in Remark 2.1. Then $u_\ell$ satisfies

$$m = F(u_\ell) = \mu = F_r(u_\ell).$$

In view of (4.2) this implies (4.25). Inequality (4.25) could also be deduced directly from the fact that $U = (U_1, \cdots, U_k)$ is a minimizer of $\Phi$ defined in (1.4). We have, using the theory of sub-differentials,

$$0 \in 2(U_i - f_i) + \text{Sign}\,(U_i - U_{i-1}) + \text{Sign}\,(U_i - U_{i+1}), \qquad \forall i = 2, \cdots, k-1, \tag{4.26}$$

where Sign denotes as usual the monotone graph defined by

$$\text{Sign}(s) := \begin{cases} +1, & \text{if } s > 0, \\ [-1, +1], & \text{if } s = 0, \\ -1, & \text{if } s < 0. \end{cases}$$

This implies (4.25). On the other hand, we have

$$0 \in 2(U_1 - f_1) + \text{Sign}\,(U_1 - U_2),$$

and

$$0 \in 2(U_k - f_k) + \text{Sign}\,(U_k - U_{k-1}),$$

which imply in fact that

$$|U_1 - f_1| \leq \frac{1}{2} \text{ and } |U_k - f_k| \leq \frac{1}{2}.$$

# 5   Where a mild pertubation can produce a big difference

In this section we consider a mild pertubation of the original functional $F$ defined by (1.2) and we show that the corresponding minimizing problems differ significantly from those associated with $F$.

Set

$$G(u) = F(u) + \int_0^1 |u|^2 = \int_0^1 |u'| + \sum_{i=1}^k |u(a_i) - f_i|^2 + \int_0^1 |u|^2, \tag{5.1}$$

where $u \in W^{1,1}(0,1)$. Our initial goal is to investigate the minimization problem

$$A = \underset{u \in W^{1,1}}{\text{Inf}}\, G(u). \tag{5.2}$$

It turns out that the infimum in (5.2) need *not* be achieved (see [6, 7] and Remark 5.2) and we will replace it by a relaxed problem defined on $BV(0,1)$ as we have done in Section 3. For every $v \in BV(0,1)$ set

$$G_r(v) = \text{Inf} \liminf_{n \to \infty} G(v_n), \qquad (5.3)$$

where the Inf in (5.3) is taken over all sequences $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$. It is easy to check that

$$G_r(v) = F_r(v) + \int_0^1 |v|^2, \qquad \forall v \in BV(0,1), \qquad (5.4)$$

so that $G_r$ is strictly convex on $BV(0,1)$ and it is lower semicontinuous in the sense that for every sequence $(v_n) \subset BV(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ as $n \to \infty$, we have

$$\liminf_{n \to \infty} G_r(v_n) \geq G_r(v).$$

Consequently

$$B = \text{Inf}_{v \in BV} G_r(v) \qquad (5.5)$$

is uniquely achieved, and we denote by $\bar{v} \in BV(0,1)$ its unique minimizer, i.e.,

$$B = G_r(\bar{v}). \qquad (5.6)$$

We claim that

$$A = B. \qquad (5.7)$$

From (4.1), we deduce that $G_r \leq G$ on $W^{1,1}(0,1)$, and thus

$$B = \text{Inf}_{v \in BV} G_r(v) \leq \text{Inf}_{v \in W^{1,1}} G_r(v) \leq \text{Inf}_{v \in W^{1,1}} G(v) = A. \qquad (5.8)$$

On the other hand we have by (5.2)

$$A \leq G(u) = F(u) + \int_0^1 |u|^2, \qquad \forall u \in W^{1,1}(0,1). \qquad (5.9)$$

From (5.9) and Step 1 in Section 3 we deduce that

$$A \leq F_r(v) + \int_0^1 |v|^2 = G_r(v), \qquad \forall v \in BV(0,1), \qquad (5.10)$$

and thus

$$A \leq \text{Inf}_{v \in BV} G_r(v) = B. \qquad (5.11)$$

Combining (5.11) with (5.8) yields $A = B$.

As a consequence, if Problem (5.2) admits a minimizer $v_0 \in W^{1,1}(0,1)$, then

$$B \leq G_r(v_0) \leq G(v_0) = A,$$

so that, by (5.7), $G_r(v_0) = B$, i.e., $v_0$ is a minimizer for Problem (5.5). By uniqueness $v_0 = \bar{v}$.

The bottom line is that we have replaced Problem (5.2) which need *not* have a solution by Problem (5.5) which *always* admits a unique solution $\bar{v}$. Therefore $\bar{v}$ may be viewed as *the generalized solution* of Problem (5.2).

**Remark 5.1.** This concept of generalized solution is quite robust. In particular if $(u_n) \subset W^{1,1}(0,1)$ is a minimizing sequence for (5.2), then $u_n \to \bar{v}$ as $n \to \infty$ in $L^2(0,1)$. Indeed, we have

$$G_r(u_n) \leq G(u_n) \leq A + o(1),$$

and a subsequence $(u_{n_k})$ converges in $L^2(0,1)$ to some $\bar{u} \in BV(0,1)$ satisfying

$$B \leq G_r(\bar{u}) = A,$$

so that $G_r(\bar{u}) = B$ and by uniqueness $\bar{u} = \bar{v}$. Similarly, if we consider as in Remark 2.1,

$$G_{1,\varepsilon}(u) = F_{1,\varepsilon}(u) + \int_0^1 |u|^2, \qquad G_{2,p}(u) = F_{2,p}(u) + \int_0^1 |u|^2,$$

$$G_{3,\varepsilon}(u) = F_{3,\varepsilon}(u) + \int_0^1 |u|^2,$$

their unique minimizers also converge in $L^2(0,1)$ to $\bar{v}$. This is an easy consequence of the fact that $C^\infty([0,1])$ is dense in $W^{1,1}(0,1)$.

It turns out that the minimizer $\bar{v}$ of (5.5) has a remarkable property:

**Theorem 5.1.** *The minimizer $\bar{v}$ of (5.5) is a constant $K_i$ on each interval $(a_i, a_{i+1})$, $i = 0, 1, \cdots, k$ with the convention that $a_0 = 0$ and $a_{k+1} = 1$.*

*Moreover*

$$|K_i| \leq 1/|a_{i+1} - a_i|, \qquad \forall i = 0, 1, \cdots, k. \tag{5.12}$$

The main ingredient in the proof of Theorem 5.1 is the following result taken from [3, Theorem 3] with roots in [6, Theorem 3.16].

**Lemma 5.1.** *Fix $\alpha, \beta, L \in \mathbb{R}$ and consider the minimization problem*

$$X = \mathrm{Inf} \left\{ \int_0^L |u'| + \int_0^L |u|^2; u \in BV(0,L), \ u(0) = \alpha \ \text{and} \ u(L) = \beta \right\}. \tag{5.13}$$

*A minimizer exists if and only if*

$$\alpha = \beta \quad \text{with} \quad |\alpha| = |\beta| \leq 1/L, \tag{5.14}$$

*and in this case the unique minimizer in (5.13) is the constant function $\alpha = \beta$.*

*Proof.* For the convenience of the reader we review briefly the argument from [3]. Set

$$H(u) = \int_0^L |u'| + \int_0^L |u|^2, \qquad u \in W^{1,1}(0,L), \qquad (5.15a)$$

$$u(0) = \alpha, \qquad\qquad u(L) = \beta, \qquad (5.15b)$$

and for $v \in BV(0,L)$,

$$H_r(v) = \text{Inf} \liminf_{n \to \infty} H(v_n), \qquad (5.16)$$

where the Inf in (5.16) is taken over all sequences $(v_n) \subset W^{1,1}(0,L)$ such that $v_n \to v$ in $L^2(0,L), v_n(0) = \alpha$ and $v_n(L) = \beta$.

From Lemmas 3.1 and 3.2 we know that

$$H_r(v) = \int_0^L |v'| + \int_0^L |v|^2 + |v(0) - \alpha| + |v(L) - \beta|, \qquad \forall v \in BV(0,L). \qquad (5.17)$$

Moreover,

$$X = \min_{v \in BV} H_r(v). \qquad (5.18)$$

Problem (5.13) usually admits *no* minimizer, while Problem (5.18) *always* admits a unique minimizer denoted $V \in BV(0,L)$. If (by chance!) Problem (5.13) admits a minimizer $U \in BV(0,L)$, then $U = V$. On the other hand, if we happen to know that the minimizer $V$ of (5.18) satisfies $V(0) = \alpha$ and $V(L) = \beta$ then $V$ is a minimizer for (5.13).

To summarize, the existence of a minimizer for (5.13) boils down to the question whether $V$ satisfies $V(0) = \alpha$ and $V(L) = \beta$. We are thus led to study the properties of $V$. It is convenient to distinguish two cases:

**Case 1**: $\alpha\beta \leq 0$.   **Case 2**: $\alpha\beta > 0$.

In Case 1 we have $V \equiv 0$ and we conclude that our original Problem (5.13) admits a solution only if $\alpha = \beta = 0$; in this case $U \equiv 0$ is the minimizer of (5.13).

In Case 2 we may assume, without loss of generality, that

$$0 < \alpha \leq \beta.$$

The heart of the matter is the surprising fact that $V$ is a constant function (see the proof of Lemma 5.1 in [3]). In order to identify the constant we compute $H_r$ given by (5.17) on the constant function $v \equiv t$; this yields

$$H_r(t) = Lt^2 + |t - \alpha| + |t - \beta|.$$

An easy inspection shows that $\min_{t \geq 0} H_r(t)$ is achieved at $t = 1/L$ if $\alpha > 1/L$ and at $t = \alpha$ if $\alpha \leq 1/L$. $\qquad \square$

*Proof of Theorem 5.1.* We apply Lemma 5.1 on each interval $(a_i, a_{i+1})$ with $L = a_{i+1} - a_i$, $\alpha = \bar{v}(a_i + 0)$ and $\beta = \bar{v}(a_{i+1} - 0)$. Clearly $\bar{v}$ restricted to $(a_i, a_{i+1})$ (and shifted) is a minimizer for (5.13). Otherwise we could find a function $w \in BV(a_i, a_{i+1})$ such that

$$\int_{a_i}^{a_{i+1}} |w'| + \int_{a_i}^{a_{i+1}} |w|^2 < \int_{a_i}^{a_{i+1}} |\bar{v}'| + \int_{a_i}^{a_{i+1}} |\bar{v}|^2,$$
$$w(a_i + 0) = \bar{v}(a_i + 0), \qquad w(a_{i+1} - 0) = \bar{v}(a_{i+1} - 0).$$

Then the function $\bar{w} \in BV(0,1)$ defined by

$$\bar{w} := \begin{cases} w & \text{on } (a_i, a_{i+1}), \\ \bar{v} & \text{on } (0,1) \setminus (a_i, a_{i+1}), \end{cases}$$

would satisfy $G_r(\bar{w}) < G_r(\bar{v})$, which is imposible since $\bar{v}$ is a minimizer for $G_r$ on $BV(0,1)$. We deduce from Lemma 5.1 that $\bar{v} = K_i$ on $(a_i, a_{i+1})$, for some constant $K_i$ satisfying (5.12). $\qquad \square$

As an immediate consequence of Theorem 5.1 we have now an explicit *finite-dimensional* convex minimization problem which governs Problem (5.5):

**Corollary 5.1.** *The unique minimizer $\bar{v}$ of* (5.5) *is given by*

$$\bar{v} = \sum_{i=0}^{k} \bar{K}_i \mathbb{1}_{(a_i, a_{i+1})} \tag{5.19}$$

*and the constants $\bar{K}_i$ are obtained by minimizing*

$$\Psi(K) = \sum_{i=0}^{k-1} |K_{i+1} - K_i| + \sum_{i=1}^{k} \varphi(\text{dist }(f_i, J_i) + \sum_{i=0}^{k} K_i^2 (a_{i+1} - a_i) \tag{5.20}$$

*over $K = (K_0, \cdots, K_k) \in \mathbb{R}^{k+1}$, where $J_i$ denotes the interval $[\min(K_{i-1}, K_i), \max(K_{i-1}, K_i)]$.*

Finally we return to Problem (5.1) and derive some *necessary conditions* for the existence of a minimizer.

**Corollary 5.2.** *Assume that Problem* (5.1) *admits a minimizer $\bar{u} \in W^{1,1}(0,1)$, then necessarily*

$$\bar{u} \equiv \bar{K} \equiv \frac{1}{k+1} \sum_{i=1}^{k} f_i. \tag{5.21}$$

*Moreover we must have*

$$|f_i - \bar{K}| \leq 1, \qquad\qquad \forall i = 1, \cdots, k, \tag{5.22a}$$
$$(a_{i+1} - a_i)|\bar{K}| \leq 1, \qquad\qquad \forall i = 0, 1, \cdots, k, \tag{5.22b}$$

*so that in particular*

$$|\bar{K}| \leq k + 1. \tag{5.23}$$

*Proof of Corollary 5.2.* Since $\bar{u}$ is also a minimizer for (5.5) we know by Theorem 5.1 that $\bar{u}$ is constant on each interval $(a_i, a_{i+1})$. On the other hand $\bar{u} \in W^{1,1}(0,1)$, and thus

$$\bar{u} \equiv \bar{K} \quad \text{on} \quad (0,1),$$

for some constant $\bar{K}$. To identify $\bar{K}$ we write that

$$G(\bar{K}) \leq G(t), \qquad \forall t \in \mathbb{R},$$

i.e.,

$$\sum_{i=1}^{k} |\bar{K} - f_i|^2 + |\bar{K}|^2 \leq \sum_{i=1}^{k} |t - f_i|^2 + t^2, \qquad \forall t \in \mathbb{R},$$

which implies (5.21). Next we recall that, by (5.7),

$$G(\bar{u}) = G_r(\bar{u}).$$

Going back to (5.1) and (5.4) we see that

$$F(\bar{u}) = F_r(\bar{u}),$$

which implies (5.22a) by (4.2). Finally (5.22b) comes from (5.12).          □

**Remark 5.2.** In view of Corollary 5.2 it is easy to construct examples where Problem (5.2) admits no minimizer. Take for example $k = 2$ and $f_1, f_2$ such that $|2f_1 - f_2| > 3$.

## 6 Further directions of research

6.1) Try to adapt results from the previous sections to the following situations:
6.1.a) Let $\mu$ be a probability measure on $[0,1]$ and let

$$F(u) = \int_0^1 |u'| + \int_0^1 |u - f|^2 d\mu, \qquad u \in W^{1,1}(0,1),$$

where $f$ is a given (smooth) function on $[0,1]$.
6.1.b) Let

$$F(u) = \int_0^1 (1 + |u'|^2)^{1/2} + \sum_{i=1}^{k} |u(a_i) - f_i|^2, \qquad u \in W^{1,1}(0,1),$$

where $(a_i)$ and $(f_i)$ are as in Section 1.
6.2) Investigate the following minimization problem:

$$\text{Inf} \left\{ \sum_{i=1}^{k} |u(a_i) - f_i|^2; \; u \in W^{1,1}(0,1), \int_0^1 |u'| \leq A, \left( \text{resp.} \int_0^1 (|u'| + |u|^2) \leq A \right) \right\},$$

where $A > 0$ is given.

6.3) Let $\Gamma$ be a smooth curve in a domain $\Omega \subset \mathbb{R}^2$ and let

$$F(u) = \int_\Omega |\nabla u| + \int_\Gamma |u - f| d\sigma, \qquad u \in W^{1,1}(\Omega),$$

where $f$ is a given (smooth) function on $\Gamma$. Study the minimization of $F$.

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