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REVIEW ARTICLE

Review of Feynman's Path Integral in Quantum Statistics: from the Molecular Schrödinger Equation to Kleinert's Variational Perturbation Theory

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Abstract. Feynman's path integral reformulates the quantum Schrödinger differential equation to be an integral equation. It has been being widely used to compute internuclear quantum-statistical effects on many-body molecular systems. In this Review, the molecular Schrödinger equation will first be introduced, together with the Born-Oppenheimer approximation that decouples electronic and internuclear motions. Some effective semiclassical potentials, e.g., centroid potential, which are all formulated in terms of Feynman's path integral, will be discussed and compared. These semiclassical potentials can be used to directly calculate the quantum canonical partition function without individual Schrödinger's energy eigenvalues. As a result, path integrations are conventionally performed with Monte Carlo and molecular dynamics sampling techniques. To complement these techniques, we will examine how Kleinert's variational perturbation (KP) theory can provide a complete theoretical foundation for developing non-sampling/non-stochastic methods to systematically calculate centroid potential. To enable the powerful KP theory to be practical for many-body molecular systems, we have proposed a new path-integral method: automated integration-free path-integral (AIF-PI) method. Due to the integration-free and computationally inexpensive characteristics of our AIF-PI method, we have used it to perform *ab initio* path-integral calculations of kinetic isotope effects on proton-transfer and RNA-related phosphoryl-transfer chemical reactions. The computational procedure of using our AIF-PI method, along with the features of our new centroid path-integral theory at the minimum of the absolute-zero energy (AMAZE), are also highlighted in this review.

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Contents

1 Introduction

1.1 Molecular Schrödinger equation

Ever since quantum mechanics was constructed in the 1920s [1–24], solving the non-relativistic time-independent Schrödinger equation for a system of nuclei and electrons has become an essential step to understand every single detail of atomic or molecular properties [1]. The non-relativistic time-independent Schrödinger equation for a molecular system (hereafter we shorten it as the molecular Schrödinger equation) is [25–27]:

$$\hat{H}_{mole}\Psi_n = E_n\Psi_n, \tag{1.1}$$

where $\hat{H}_{mole}$ is the complete (non-relativistic) molecular Hamiltonian, Ψ_n and E_n are an energy eigenfunction (or wave function) and an energy eigenvalue at an eigenstate n, respectively. In contrast to the (intra)nuclear or nucleon Hamiltonian [28], the complete molecular Hamiltonian [4, 5, 25–27, 29] for N_n nuclei and N_e electrons can fortunately be written in an analytic closed form (thanks to the inverse square-distance proportionality in Coulomb's electrostatic force law):

$$\hat{H}_{mole} = \sum_{j}^{N_n} -\frac{1}{2M_j}\nabla_j^2 + \sum_{j<j'}^{N_n}\frac{Z_{j'}Z_j}{x_{jj'}} - \frac{1}{2}\sum_{i}^{N_e}\nabla_i^2 - \sum_{j}^{N_n}\sum_{i}^{N_e}\frac{Z_j}{r_{ij}} + \sum_{i<i'}^{N_e}\frac{1}{r_{ii'}}. \tag{1.2}$$

In Eq. (1.2), the units are atomic units [30], M_j is the mass ratio of nucleus j to an electron, and Z_j is the atomic number of nucleus j. The Laplacian operators ∇_j^2 and ∇_i^2 denote the second order differentiation with respect to the coordinates of the jth nucleus and the ith electron. The first term in Eq. (1.2) represents the kinetic energy operator for nuclei; the second term is the Coulomb repulsion between nuclei; the third term is the operator for the kinetic energy of electrons; the fourth and fifth terms indicate the Coulomb attraction between electrons and nuclei, and the repulsion between electrons, respectively. The distance between the jth and the j'th nuclei is $x_{jj'}$; the separation between the ith and the i'th electrons is $r_{ii'}$; the distance between the jth nucleus and the ith electrons is r_{ij}.

1.2 Central quantity in quantum thermodynamics: quantum partition function

Once the energy eigenvalues or the quantized energy spectrum in Eq. (1.1) are determined, it is straightforward to obtain a central physical quantity in thermodynamics, i.e., the quantum canonical partition function Q_{qm} [31], by the following summation of the Boltzmann energy distribution:

$$Q_{qm} = \sum_n \exp(-\beta E_n), \tag{1.3}$$

where $\beta = 1/k_B T$, k_B is Boltzmann's constant, and T is temperature. All standard thermodynamic quantities for a system of nuclei and electrons, e.g., free energy, internal energy, entropy, pressure, etc., can be derived from it [31–34]. In Eq. (1.3), the lowest energy level E_0, which is often called the ground state energy or zero-point energy (ZPE), is usually the dominant energy level contributing to the partition function. Further, by virtue of Heisenberg's uncertainty principle, the ZPE is always larger than the minimum value of potential energy because a particle can never be at rest anywhere in a given potential or a particle with a particular momentum can be everywhere in a given potential.

1.3 Origin of potential energy surface: Born-Oppenheimer approximation

Unfortunately, even though *all* physics and chemistry of a (time-independent) molecular system are essentially in the molecular Schrödinger equation [Eq. (1.1)] [6,7], the equation can be exactly solved only for the simplest one-electron atoms or ions. For other systems, approximations must be introduced to calculate numerical solutions with the aid of computers. The most common and perhaps the mildest approximation often made is the Born-Oppenheimer approximation [19,24–27,35–38]. It decouples internuclear motions from electrons so that nuclei effectively move on a potential energy surface (PES) obtained by solving the electronic part of Schrödinger equation.

This approximation is based on the fact that an electron is much lighter than any nucleus (e.g., a proton, the lightest nucleus, is about 1840 times heavier than an electron). Nuclei move, consequently, much slower. As a result, from the electronic perspective, for a given set of nuclear positions, electrons adjust their positions 'instantly' before nuclei have a chance to move. On the other hand, from the standpoint of nuclei, electrons are moving so fast that their effects on nuclei are averaged out over the electronic wave functions. Mathematically, to simplify the molecular Hamiltonian, we first solve the electronic

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part of the Schrödinger equation for a particular set of nuclear configurations $\{x_j\}$. The electronic part of the complete molecular Hamiltonian [Eq. (1.2)] is called the electronic Hamiltonian [4,5,25–27,29,39]:

$$\hat{H}_{elec} = -\frac{1}{2}\sum_{i}^{N_e}\nabla_i^2 - \sum_{j}^{N_n}\sum_{i}^{N_e}\frac{Z_j}{r_{ij}} + \sum_{i<i'}^{N_e}\frac{1}{r_{ii'}}. \tag{1.4}$$

With this electronic Hamiltonian, we can obtain the electronic energy E_{elec} from the corresponding electronic Schrödinger equation:

$$\hat{H}_{elec}\psi_{elec} = E_{elec}\left(\{x_j\}\right)\psi_{elec}, \tag{1.5}$$

where ψ_{elec} is the electronic wave function. Note that the electronic energy $E_{elec}\left(\{x_j\}\right)$ depends parametrically on the nuclear positions $\{x_j\}$. With this electronic energy, the molecular Hamiltonian in Eq. (1.2) can be simplified as follows:

$$\begin{aligned}\hat{H}_{mole} &\approx \sum_{j}^{N_n} -\frac{1}{2M_j}\nabla_j^2 + \sum_{j<j'}^{N_n}\frac{Z_{j'}Z_j}{x_{jj'}} + \left\langle -\frac{1}{2}\sum_{i}^{N_e}\nabla_i^2 - \sum_{j}^{N_n}\sum_{i}^{N_e}\frac{Z_j}{r_{ij}} + \sum_{i<i'}^{N_e}\frac{1}{r_{ii'}} \right\rangle \\ &= \sum_{j}^{N_n} -\frac{1}{2M_j}\nabla_j^2 + \left[\sum_{j<j'}^{N_n}\frac{Z_{j'}Z_j}{x_{jj'}} + E_{elec}\left(\{x_j\}\right)\right] \\ &= \sum_{j}^{N_n} -\frac{1}{2M_j}\nabla_j^2 + V\left(\{x_j\}\right),\end{aligned} \tag{1.6}$$

where $< \cdots >$ signifies the average over electronic wave functions or the expectation value. In Eq. (1.6), V is defined as the sum of the nuclear repulsion energy and electronic energy, which effectively turns out to be the internuclear potential energy function as a consequence of the Born-Oppenheimer approximation:

$$V\left(\{x_j\}\right) \equiv \sum_{j<j'}^{N_n}\frac{Z_{j'}Z_j}{x_{jj'}} + E_{elec}\left(\{x_j\}\right). \tag{1.7}$$

There are many systematic and rigorous theories in electronic structure calculations to derive the internuclear potential energy from first principles (i.e., besides the universal fundamental constants in physics, there is no other empirical parameter involved in the calculations), e.g., Hartree-Fock theory, configuration interaction method, Møller-Plesset perturbation theory, coupled cluster approach, and Kohn-Sham density functional theory. All these quantum mechanical (QM) approaches for electronic structure calculations are often known as *ab initio* methods [4,5,25–27,29,39–45,162].

In contrast, a complete empirical method to determine an internuclear potential energy surface is to parameterize an analytic function without treating electronic degrees of freedom. This type of approach is often referred to as molecular mechanical (MM) method and the empirical potential energy is called force-field energy. Comparing to *ab initio* approach, MM methods are computationally much less expensive and can be applied to describe equilibrium properties in macromolecular systems involving over tens of thousands of heavy atoms [46–50]. But for the process involving electronic redistributions (e.g., electronic transfer, chemical bond breaking or forming, etc.), MM force field often is unable to describe it [51,52].

For the rest of this review, all discussions are limited to the Born-Oppenheimer approximation, which adiabatically decouples nuclear and electronic degrees of freedom.

1.4 Classical free energy

In practice, quantum effects on internuclear motions are much smaller than those on the electronic part. In many applications, the internuclear quantum effects are insignificant and could even be neglected. Thus, the eigenenergy spectrum E_n in Eq. (1.1) would become continuous. Given an internuclear potential V, the quantum canonical partition function in Eq. (1.3) consequently reduces to the classical canonical partition function as [31,32,40–42,53]:

$$Q_{cl} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx^{3N_n} dp^{3N_n}}{h^{3N_n}} \exp\left\{ -\beta \left[\left(\sum_j^{3N_n} \frac{p_j^2}{2M_j} \right) + V\left(\{x_j\}\right) \right] \right\}, \tag{1.8}$$

where h is Planck's constant and p_j is an momentum associated with a nuclear coordinate x_j. Subsequently, the classical free energy G_{cl} of a molecular system can be expressed in terms of the classical partition function as follows:

$$G_{cl} = -k_B T \ln Q_{cl} = -k_B T \ln \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx^{3N_n} dp^{3N_n}}{h^{3N_n}} \exp\left\{ -\beta \left[\left(\sum_j^{3N_n} \frac{p_j^2}{2M_j} \right) + V\left(\{x_j\}\right) \right] \right\}. \tag{1.9}$$

1.5 Quantum free energy in terms of Feynman's path integral

The above discussions (Section 1.3) on internuclear thermodynamics are limited to classical mechanics (regardless of using QM, MM, hybrid QM/MM to construct potential energy). However, the real world is described by quantum mechanics, including nuclei. In some applications, such as hydrogen adsorption in carbon nanotechnology, the transport mechanism of hydrated hydroxide ions in aqueous solution, and kinetic isotope effects on a biochemical reaction, internuclear quantum-statistical effects (e.g., quantization of vibration and quantum tunneling) are not negligible. A popular and robust choice for incorporating such internuclear quantum-statistical effects in the conventional molecular dynamics (MD) or Monte Carlo (MC) simulations [54–59] is using Feynman's path integral [2,32–34,60–69].

This is largely because the essence of Feynman's path integrals is to transform the Schrödinger *differential* equation to become an *integral* equation [Eq. (2.3)]. As a result, the many-body path integrations can be carried out by the conventional MD or MC sampling techniques. In addition, the quantum canonical partition function can be directly obtained with no need to compute individual energy eigenvalues.

2 Feynman's path integral

2.1 Relation between Schrödinger's equation and Feynman's path integral

After being inspired by Dirac's earlier works on the action principle in quantum mechanics [68,69], Feynman invented his path-integral (PI) formulation [2,32–34,60–75] as a new space-time approach to non-relativistic quantum mechanics in his 1942 Ph.D. thesis [60–62]. In the PI formalism, the kernel describing a wave function to propagate from (x_1,t_1) to (x_2,t_2) can be calculated as follows:

$$K(x_2,t_2;x_1,t_1)=\int_{(x_1,t_1)}^{(x_2,t_2)} D[x(t)]\exp\left\{\frac{i}{\hbar}S[x(t)]\right\}, \tag{2.1}$$

where K is the propagator (a type of Green's function), t is time, $x(t)$ describes a trajectory or a path in space-time, $\int_{(x_1,t_1)}^{(x_2,t_2)} D[x(t)]$ denotes a summation over *all* possible paths from (x_1,t_1) to (x_2,t_2) (i.e., a *functional* integration), and $S[x(t)]$ is the action associated with the Lagrangian $L[x(t)]$ [76]:

$$S[x(t)]=\int_{t_1}^{t_2} dt L[x(t)]=\int_{t_1}^{t_2} dt\left\{\frac{M}{2}\dot{x}^2(t)-V[x(t)]\right\}. \tag{2.2}$$

M in Eq. (2.2) denotes the mass of a particle, whereas V is the potential energy acting on the particle. The propagator K governs how a wave function $\psi(x,t)$ evolves with time. If the wave function at an initial time t_1 is given as $\psi(x_1,t_1)$ and K is provided as well, then we can obtain the wave function at an arbitrary time t_2 by the following equation:

$$\psi(x_2,t_2)=\int_{-\infty}^{\infty} K(x_2,t_2;x_1,t_1)\psi(x_1,t_1)dx_1. \tag{2.3}$$

In fact, Eq. (2.3) is the *integral* equation of quantum mechanics, which are equivalent to the time-dependent Schrödinger *differential* equation [32,70], i.e.,

$$\begin{aligned}&\hat{H}\psi(x,t)=i\hbar\frac{\partial}{\partial t}\psi(x,t)\\ \Leftrightarrow\ &\psi(x,t)=\int_{-\infty}^{\infty} K(x,t;x_1,0)\psi(x_1,0)dx_1.\end{aligned} \tag{2.4}$$

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Eq. (2.4) is readily generalized to many-body systems [32].

The propagator can also be written in the basis formed by the eigenfunctions of the time-independent Schrödinger equation, $\phi_n(x)$:

$$K(x_2,t_2;x_1,t_1)=\sum_n \phi_n(x_2)\phi_n^*(x_1)e^{-(i/\hbar)E_n(t_2-t_1)}, \tag{2.5}$$

where $\hat{H}\phi_n = E_n\phi_n$ and E_n are the eigenenergies for the eigenstates n. As a result, we have the following identity connecting the Schrödinger eigenfunctions and eigenenergies to Feynman's path integral:

$$\sum_n \phi_n(x_2)\phi_n^*(x_1)e^{-(i/\hbar)E_n(t_2-t_1)} = \int_{(x_1,t_1)}^{(x_2,t_2)} D[x(t)]\exp\left\{\frac{i}{\hbar}S[x(t)]\right\}. \tag{2.6}$$

Kac realized that the above identity is still valid even at pure imaginary time $t=-i\tau$ (where τ is a real number) [63, 64]. Consequently, it can also be applied to statistical physics. In the imaginary time frame, the propagator K in quantum mechanics is equivalent to the density matrix $\rho(x_2,x_1)$ [31–34] in quantum statistics by putting $t=-i\tau$:

$$\begin{aligned}\sum_n \phi_n(x_2)\phi_n^*(x_1)e^{-(i/\hbar)E_n(t_2-t_1)} &= \sum_n \phi_n(x_2)\phi_n^*(x_1)e^{-(1/\hbar)E_n(\tau_2-\tau_1)} \\ &= \sum_n \phi_n(x_2)\phi_n^*(x_1)e^{-(1/k_BT)E_n} \\ &\equiv \rho(x_2,x_1),\end{aligned} \tag{2.7}$$

where $\tau_2-\tau_1=\hbar/k_BT$ (independent of the choices for τ_1 or τ_2), k_B is Boltzmann's constant, and T is temperature. Hence with the identity in Eq. (2.6), we have the following quantum-statistical path integrals for the density matrix:

$$\rho(x_2,x_1)=\int_{(x_1,0)}^{(x_2,\beta\hbar)} D[x(\tau)]\exp\left\{-\frac{1}{\hbar}A[x(\tau)]\right\}, \tag{2.8}$$

where $\beta=1/k_BT$ and $A[x(\tau)]$ is the quantum-statistical action†

$$A[x(\tau)]=\int_0^{\beta\hbar} d\tau\left\{\frac{M}{2}\dot{x}^2(\tau)+V[x(\tau)]\right\}. \tag{2.9}$$

[†] The quantum-statistical action is also named the Euclidean action. According to page 122 of Ref. [34], this name refers to the case that an N-dimensional (N-D) Euclidean space continued by the imaginary time axis $\tau=it$ has the identical geometric properties as an ($N+1$)-D Euclidean space. For example, a 4-vector in a Minkowski space-time has a square length $dx^2=-(cdt)^2+(d\mathbf{x})^2$, where c is the speed of light. Extended to imaginary time, this becomes $dx^2=(cd\tau)^2+(d\mathbf{x})^2$ which is the square distance in an Euclidean 4-D space with 4 vectors $(c\tau,\mathbf{x})$.

Note after the change of variable $t=-i\tau$ for the differential dt and for the kinetic energy [or $\dot{x}(t)$] in Eq. (2.2), the quantum-statistical action in Eq. (2.9) is effectively related to the *Hamiltonian* instead of the *Lagrangian* for the paths running along the imaginary time axis.

The trace of the density matrix in Eq. (2.7), which is the sum of the diagonal elements and is invariant with the representing basis functions for the matrix, provides us with a central physical quantity in quantum statistics, i.e., the quantum canonical partition function Q_{qm}:

$$\begin{aligned} Q_{qm} &= \mathrm{Trace}(\rho) = \int_{-\infty}^{\infty} dx \rho(x,x) \\ &= \sum_n e^{-(1/k_BT)E_n} \int_{-\infty}^{\infty} |\phi_n(x)|^2 dx = \sum_n e^{-\beta E_n}. \end{aligned} \tag{2.10}$$

The diagonal element $\rho(x,x)$, in which the eigenfunctions (in position space) are used to represent the matrix, is also known as the particle density $\rho(x)$:

$$\rho(x,x) = \rho(x) = \sum_n |\phi_n(x)|^2 e^{-\beta E_n}. \tag{2.11}$$

Combining Eq. (2.10) with Eq. (2.8), we have the following expression of the quantum partition function Q_{qm} in terms of Feynman's path integral:

$$\begin{aligned} Q_{qm} &= \int_{-\infty}^{\infty} dx \int_{(x,0)}^{(x,\beta\hbar)} D[x(\tau)] \exp\left\{-\frac{1}{\hbar} A[x(\tau)]\right\}, \\ &\equiv \oint D[x(\tau)] \exp\left\{-\frac{1}{\hbar} A[x(\tau)]\right\} \\ &\Rightarrow \oint D[x(\tau)] \exp\left\{-\frac{1}{\hbar} A[x(\tau)]\right\} = \sum_n e^{-\beta E_n}. \end{aligned} \tag{2.12}$$

In Eq. (2.12), $\oint D[x(\tau)]$ denotes the summation over all *closed* paths $x(\tau)$ in the time-interval $\beta\hbar$, i.e., a *functional* integration (an integration over a *function* instead of over a variable; see Appendix A for more details). The physical interpretation of Eq. (2.12) is simple and vivid, which can be reflected by the following quote in the Feynman-Hibbs textbook [32]:

> Consider all the possible paths, or 'motions', by which the system can travel between the initial and final configurations in the 'time' $\beta\hbar$. The density matrix ρ is a sum of contribution from each motion, the contribution from a particular motion being the 'time' integral of the 'energy' divided by $\hbar$ for the path in question. (See next page for the rest of the quote.)

> The partition function is derived by considering only those cases in which the final configuration is the same as the initial configuration, and we sum over all possible initial configurations.
>
> Feynman and Hibbs 1965

2.2 Effective semiclassical potentials

In contrast to quantum statistics, since the energy spectrum in classical mechanics is always continuous, the classical canonical partition function is written in terms of the energy density associated with momenta and positions (which is conceptually or intrinsically impossible in quantum mechanics to have a simultaneous probability function for momenta and positions by virtue of Heisenberg's uncertainty principle [15,77]):

$$\begin{aligned} Q_{cl} &= \sum_{\substack{\text{continuous}\\ \text{energy}E}} e^{-\beta E} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dxdp}{h} \exp\left\{-\beta\left[\frac{p^2}{2m}+V(x,p)\right]\right\}. \end{aligned} \tag{2.13}$$

The replacement of $\sum_{\substack{\text{continuous}\\ \text{energy}E}}$ with $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dxdp}{h}$ in Eq. (2.13) can be derived from the quantum partition function at the high temperature limit [31,32]. In practice, the potential energy V is usually independent of the momentum p. As a result, the classical partition function can be expressed in terms of V as a configuration integral:

$$\begin{aligned} Q_{cl} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dxdp}{h} \exp\left\{-\beta\left[\frac{p^2}{2m}+V(x)\right]\right\} \\ &= \frac{1}{h}\int_{-\infty}^{\infty} \exp\left(-\beta\frac{p^2}{2m}\right)dp \int_{-\infty}^{\infty} dx \exp\{-\beta[V(x)]\} = \sqrt{\frac{Mk_BT}{2\pi\hbar^2}}\int_{-\infty}^{\infty} e^{-\beta V(x)}dx. \end{aligned} \tag{2.14}$$

The inverse of the factor $\sqrt{Mk_BT/2\pi\hbar^2}$ in Eq. (2.14) is the thermal de Broglie wavelength.

It is of particular interest to approximate the quantum partition function in a classical fashion without explicitly solving the Schrödinger equation. There have been several studies on rewriting the partition function in terms of an effective semiclassical potential as a configuration integral, such as the Wigner-Kirkwood expansion [31,77–83] and the centroid density of path integrals [32,84–89].

2.2.1 Effective potentials based on particle density

The basic idea of the Wigner-Kirkwood expansion [31,77–83], which was developed in 1930s, is to express the particle density in Eq. (2.11) as a power series of $\hbar$:

$$\rho(x) = \rho_{cl}(x)\left[1+qC_1(x,\beta)+q^2C_2(x,\beta)+\cdots\right], \tag{2.15}$$

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where the classical particle density

$$\rho_{cl}(x)=\sqrt{\frac{Mk_BT}{2\pi\hbar^2}}e^{-\beta V(x)}, \tag{2.16}$$

and

$$q=\frac{\hbar^2}{2M}. \tag{2.17}$$

The $\hbar$-expansion parameter q is in units of [energy]×[length]^2. Here, we explicitly show the first two terms of the quantum correction functions $C_i(x,\beta)$:

$$C_1(x,\beta)=\frac{1}{12}\beta^3\left[V'(x)\right]^2-\frac{1}{6}\beta^2V''(x), \tag{2.18a}$$

$$\begin{aligned}C_2(x,\beta)=&\frac{1}{288}\beta^6\left[V'(x)\right]^4-\frac{11}{360}\beta^5\left[V'(x)\right]^2V''(x)\\&+\frac{1}{40}\beta^4\left[V''(x)\right]^2+\frac{1}{30}\beta^4V'(x)V'''(x)-\frac{1}{60}\beta^3V^{(4)}(x).\end{aligned} \tag{2.18b}$$

With the $\hbar$ expansion in Eq. (2.15), we may construct the corresponding Wigner-Kirkwood (WK) effective potential $V_{\text{WK}}^{\text{eff}}$ as follows:

$$\begin{aligned}&\sqrt{\frac{Mk_BT}{2\pi\hbar^2}}e^{-\beta V_{\text{WK}}^{\text{eff}}(x)}=\rho_{cl}(x)\left[1+qC_1(x,\beta)+q^2C_2(x,\beta)+\cdots\right]\\&\Rightarrow\ V_{\text{WK}}^{\text{eff}}(x)=-k_BT\ln\left\{e^{-\beta V(x)}\left[1+qC_1(x,\beta)+q^2C_2(x,\beta)+\cdots\right]\right\}.\end{aligned} \tag{2.19}$$

In order to include an infinite number of $\hbar$ terms in the expansion approximately, Fujiwara, Osborn, and Wilk [78, 81] (FOW) proposed a cumulant [90, 91] version of the Wigner-Kirkwood expansion:

$$\rho(x)=\rho_{cl}(x)\exp\left[q\varsigma_1(x,\beta)+q^2\varsigma_2(x,\beta)+\cdots\right], \tag{2.20}$$

where the cumulant ς are identified as:

$$\varsigma_1=C_1, \tag{2.21a}$$

$$\varsigma_2=C_2-\frac{1}{2}C_1^2. \tag{2.21b}$$

Thus the corresponding FOW effective potential $V_{\text{FOW},\hbar}^{\text{eff}}$ is:

$$\begin{aligned}&\sqrt{\frac{Mk_BT}{2\pi\hbar^2}}e^{-\beta V_{\text{FOW},\hbar}^{\text{eff}}}=\rho_{cl}(x)\exp\left[q\varsigma_1(x,\beta)+q^2\varsigma_2(x,\beta)+\cdots\right]\\&\Rightarrow\ V_{\text{FOW},\hbar}^{\text{eff}}(x)=V(x)-k_BT\left[q\varsigma_1(x,\beta)+q^2\varsigma_2(x,\beta)+\cdots\right].\end{aligned} \tag{2.22}$$

In addition, Fujiwara, Osborn, and Wilk also devised an alternative cumulant expansion version of the effective potential $V^{\text{eff}}_{\text{FOW},\hbar}$ as a power series of β [78, 81]:

$$V^{\text{eff}}_{\text{FOW},\beta}(x) = V(x) - \left[\frac{\beta}{2}\zeta_1(x,q) - \frac{\beta^2}{6}\zeta_2(x,q) + \cdots\right], \tag{2.23}$$

where

$$\zeta_1(x,q) = -\frac{q}{3}V''(x), \tag{2.24}$$

and

$$\zeta_2(x,q) = -\frac{q}{2}\left[V'(x)\right]^2 + \frac{q^2}{10}V^{(4)}(x). \tag{2.25}$$

2.2.2 Centroid effective potential

Based on the centroid density of path integrals [32, 70, 84–89, 92], Feynman and Hibbs suggested a completely different approach for constructing an effective semiclassical potential [32]. First, we recall that calculations of the quantum partition function requires a summation of all possible closed paths. Each path is indexed by its initial or final configuration [Eq. (2.12)]. The difference in the centroid path integrals is that instead of integrating over all possible initial/final positions of paths to obtain the partition function, we may group the closed paths in accordance with their time-average positions $\bar{x}$, which are also called centroids:

$$\bar{x} = \frac{1}{\beta\hbar}\int_0^{\beta\hbar} x(\tau)\,d\tau. \tag{2.26}$$

The quantum partition function is, then, obtained by carrying out the integrations over the centroid positions (see Appendix A for more details):

$$Q_{qm} = \oint D[x(\tau)] \int_{-\infty}^{\infty} \delta(\bar{x} - x_0)\,dx_0 \exp\left\{-\frac{1}{\hbar}A[x(\tau)]\right\}, \tag{2.27}$$

where $\delta(\bar{x} - x_0)$ is a delta function picking or constraining a closed path with the centroid position $\bar{x}$ at a value of x_0. Then the corresponding centroid effective potential W can be defined as follows (see Appendix A for more details):

$$\begin{aligned}
&W(x_0) = -k_BT\ln\left[\sqrt{\frac{2\pi\hbar^2}{Mk_BT}}\oint D[x(\tau)]\delta(\bar{x}-x_0)\exp\left\{-\frac{1}{\hbar}A[x(\tau)]\right\}\right] \\
&\Rightarrow \sqrt{\frac{Mk_BT}{2\pi\hbar^2}}e^{-\beta W(x_0)} = \rho_{centroid}(x_0) = \oint D[x(\tau)]\delta(\bar{x}-x_0)\exp\left\{-\frac{1}{\hbar}A[x(\tau)]\right\} \\
&\Rightarrow Q_{qm} = \int_{-\infty}^{\infty} \rho_{centroid}(x_0)\,dx_0.
\end{aligned} \tag{2.28}$$

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As Feynman and Hibbs pointed out [32], a major advantage of categorizing each closed path in terms of its centroid position over the initial/final position is that the first non-zero correction term of Eq. (2.27) after a Taylor series expansion for V about $\bar{x}$ is of second order:

$$
\begin{aligned}
Q_{qm} =& \int_{-\infty}^{\infty} dx_0 \oint D[x(\tau)]\delta(\bar{x}-x_0) \\
&\times \exp\left\{ -\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left[\frac{M}{2}\dot{x}^2 + V(\bar{x}) + (x-\bar{x})V'(\bar{x}) + \frac{1}{2}(x-\bar{x})^2 V''(\bar{x}) + \cdots \right] \right\} \\
=& \int_{-\infty}^{\infty} dx_0 \oint D[x(\tau)]\delta(\bar{x}-x_0) \\
&\times \exp\left\{ -\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left[\frac{M}{2}\dot{x}^2 + V(\bar{x}) + \frac{1}{2}(x-\bar{x})^2 V''(\bar{x}) + \cdots \right] \right\},
\end{aligned}
\tag{2.29}
$$

where the time integral associated with the first order derivative $V'(\bar{x})$ is zero because of Eq. (2.26), i.e.,

$$
-\frac{1}{\hbar}V'(\bar{x})\int_0^{\beta\hbar} d\tau (x-\bar{x}) = 0. \tag{2.30}
$$

Therefore, in comparison with the effective semiclassical potentials from the Wigner-Kirkwood or Fujiwara-Osborn-Wilk expansions, the effective semiclassical potential based on the centroid density of path integrals should be less sensitive to the curvature or the variation of the original 'classical' potential V. Consequently, the centroid effective potential W should be more classical-like.

2.2.3 Comparison of the two semiclassical potentials and path-integral quantum transition-state theory

To shed some light on the close analogy between the centroid potential W and the original 'classical' potential V, we consider a chemical reaction overcoming a barrier from the reactant state. The barrier is approximated as an un-truncated or infinite parabola (i.e., an inverted harmonic potential), while the reactant state is treated approximately as a harmonic potential well (Fig. 1). The quantum correction to the classical reaction rate constant due to the reactant state is the ratio of the quantum to classical partition functions [93–104]. Let the angular frequency of the harmonic vibration be ω. The classical canonical partition function for a linear harmonic oscillator, in which $V(x) = M\omega^2 x^2/2$, is given as follows:

$$
Q_{cl} = \sqrt{\frac{Mk_B T}{2\pi\hbar^2}} \int_{-\infty}^{\infty} \exp\left(-\beta \frac{1}{2} M\omega^2 x^2 \right) dx = \frac{1}{\beta\hbar\omega}. \tag{2.31}
$$

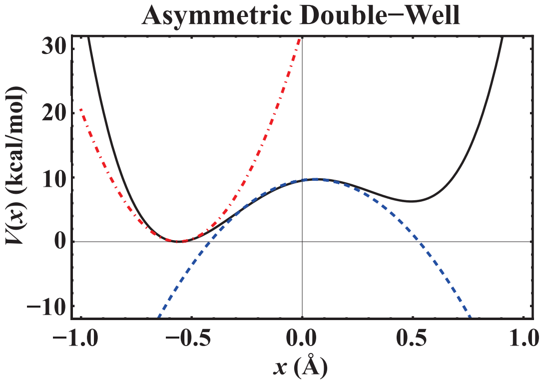

Figure 1: Schematic diagram for a chemical reaction, in which the reactant is approximated as a harmonic potential (red dashed-dotted line) and the transition state is approximated as an infinite parabola (blue dashed line).

The quantum canonical partition function for a linear harmonic oscillator can also be written in a closed-form expression:

$$Q_{qm}=\sum_{n=0}^{\infty}\exp\left[-\beta\left(n+\frac{1}{2}\right)\hbar\omega\right]=\frac{1}{2\sinh\left(\beta\hbar\omega/2\right)}. \tag{2.32}$$

Hence κ, the factor of the quantum correction to the classical reaction rate constant contributed from the reactant state, is:

$$\kappa_{qm/cl}(RS)=\frac{Q_{qm}}{Q_{cl}}=\frac{\beta\hbar\omega/2}{\sinh\left(\beta\hbar\omega/2\right)}. \tag{2.33}$$

For the transition state, the contribution to the quantum correction in this model reaction is only from quantum tunneling. Suppose the energy barrier from the reactant to transition states is V_0 and the imaginary angular frequency for the parabolic barrier is $i\nu$ (Fig. 1). The probability of tunneling transmission $\Upsilon(E)$ as a function of energy E (Ref. [105]) for an infinite parabolic barrier, in which $V(x)=V_0-M\nu^2x^2/2$, has been analytically solved by Bell [106, 107]:

$$\Upsilon(E)=\frac{1}{1+\exp\left(2\pi\frac{V_0-E}{\hbar\nu}\right)}. \tag{2.34}$$

With the transmission probability, we are in a position to derive the quantum tunneling correction to the classical reaction rate constant. According to the conventional transition-state theory [93–104], the classical rate constant is:

$$k_{cl}=\frac{k_BT}{h}\frac{1}{Q_{\mathrm{A}}}\exp(-\beta V_0), \tag{2.35}$$

where Q_{A} is the reactant partition function.

In the quantum world, we have to modify our way in computing the rate constant to account for the tunneling effects properly. For a one-dimensional barrier, as long as reactants move towards products from the beginning, they are *all* possible to overcome the barrier with *any* initial energy values (the energy is constant throughout the reaction). A factor governing the ability in passing through the barrier is the tunneling transmission probability. Therefore, instead of computing the reaction rate as the number of reactants passing through the transition state per unit time, we should consider the number of reactants per unit time in which their positions are at negative infinity but moving towards the product. As a result, we obtain the following quantum reaction rate constant expression for a one-dimensional barrier [108]:

$$k_{qm}=\frac{1}{h}\int_{-\infty}^{\infty}\exp(-\beta E)\Upsilon(E)\,dE\,\frac{1}{Q_{\mathrm{A}}}, \tag{2.36}$$

where Q_{A} is the reactant partition function. Dividing Eq. (2.36) by Eq. (2.35), we obtain the following quantum tunneling correction to the classical reaction rate constant for a one-dimensional barrier:

$$\kappa_{qm/cl}(TS)=\frac{\exp\left(V_0/k_BT\right)}{k_BT}\int_{-\infty}^{\infty}\Upsilon(E)\exp\left(-E/k_BT\right)dE. \tag{2.37}$$

Substituting Eq. (2.34) into Eq. (2.37), the analytical form of the tunneling correction for the infinite parabolic barrier can be written as:

$$\begin{aligned}\kappa_{qm/cl}(TS)&=\frac{\exp\left(V_0/k_BT\right)}{k_BT}\int_{-\infty}^{\infty}\frac{\exp\left(-E/k_BT\right)}{1+\exp\left(2\pi\frac{V_0-E}{\hbar\nu}\right)}dE\\&=\int_{0}^{\infty}\frac{du}{1+u^{\frac{\pi}{\beta\hbar\nu/2}}}=\frac{\beta\hbar\nu/2}{\sin\left(\beta\hbar\nu/2\right)}\qquad\text{iff }\ \beta\hbar\nu/2<\pi,\end{aligned} \tag{2.38}$$

where $u=\exp[(V_0-E)/k_BT]$. The integrals in Eq. (2.37) diverges when $\beta\hbar\nu/2>\pi$. This happens when the barrier is so narrow/sharp (i.e., the frequency ν is so high) or temperature is so low that the tunneling correction from the negative infinity energy weighted by the Boltzmann factor is not insignificant.

As a result, the total quantum correction to the classical reaction rate constant for a chemical reaction approximated as overcoming a parabolic barrier from a harmonic potential well [which is dividing Eq. (2.37) by Eq. (2.33)] is:

$$\begin{aligned}\kappa_{qm/cl}^{Tot}&=\kappa_{qm/cl}(TS)/\kappa_{qm/cl}(RS)\\&=\frac{\beta\hbar\nu/2}{\sin\left(\beta\hbar\nu/2\right)}\bigg/\frac{\beta\hbar\omega/2}{\sinh\left(\beta\hbar\omega/2\right)}.\end{aligned} \tag{2.39}$$

It is amazing to notice that the quantum corrections for the reactant [Eq. (2.33)] and transition [Eq. (2.37)] states, appealing to two different natures of quantum effects (i.e., quantization of vibration and quantum tunneling, respectively), are actually in the same mathematical closed form:

$$\kappa_{qm/cl} = \frac{\beta\hbar\Omega/2}{\sinh(\beta\hbar\Omega/2)}. \tag{2.40}$$

By putting $\Omega = \omega$ for the reactant state and $\Omega = i\nu$ for the transition state, Eq. (2.40) reduces to Eqs. (2.33) and (2.37), respectively. This correspondence has been described by Johnston as "an amusing coincidence" [109] and thought by Bell that it "may conceal some fundamental connection which is not yet fully understood" [107].

Interestingly, the truth behind the 'amusing coincidence' or the 'concealed fundamental connection' between the vibrational and tunneling effects could be answered at least in part by the centroid density of path integrals. The centroid effective potential W for the linear harmonic oscillator $V(x) = M\omega^2 x^2/2$ is [see Appendix A for more details, e.g., see Eq. (A.19)]:

$$W_{RS}(x) = \frac{1}{2}M\omega^2 x^2 - k_B T \ln\left[\frac{\beta\hbar\omega/2}{\sinh(\beta\hbar\omega/2)}\right]. \tag{2.41}$$

Note that the hyperbolic function $\frac{\beta\hbar\omega/2}{\sinh(\beta\hbar\omega/2)}$ is always *smaller* than *unity*. Consequently, W is basically identical to V but the bottom of the potential well is *raised* by the amount of energy equal to the free energy difference between the quantum and classical oscillators (Fig. 2). For the untruncated and infinite parabolic barrier $V(x) = V_0 - M\nu^2 x^2/2$, the centroid potential W is [see Appendix A for more details, e.g., see Eq. (A.19)]:

$$W_{TS}(x) = V_0 - \frac{1}{2}M\nu^2 x^2 - k_B T \ln\left[\frac{\beta\hbar\nu/2}{\sin(\beta\hbar\nu/2)}\right], \quad \text{for } \frac{\beta\hbar\nu}{2} < \pi. \tag{2.42}$$

This time, the trigonometric function $\frac{\beta\hbar\nu/2}{\sin(\beta\hbar\nu/2)}$ is always *greater* than *unity*. Thus W is nothing more than the infinite parabolic barrier, except that the original barrier V_0 is now *lowered* by $\left|-k_B T \ln\left[\frac{\beta\hbar\nu/2}{\sin(\beta\hbar\nu/2)}\right]\right|$. As a result, using the centroid potentials in Eqs. (2.41) and (2.42) to compute the reaction rate constant in a classical manner, we can obtain the exact quantum correction shown in Eq. (2.39).

On the contrary, the above classical-like behavior in the centroid potential, which is basically shifting the 'classical' potential upward and downward without changing the frequency of the potential, cannot be observed for the effective potentials constructed by the particle density [Eq. (2.11)] or by the initial/final positions of closed paths. For the linear harmonic oscillator $V(x) = M\omega^2 x^2/2$, the closed form of the particle density is [32, 34]:

$$\rho_{RS}(x) = \sqrt{\frac{M\omega}{2\pi\hbar\sinh(\beta\hbar\omega)}} \exp\left\{-\frac{M\omega x^2}{\hbar\sinh(\beta\hbar\omega)}\left[\cosh(\beta\hbar\omega) - 1\right]\right\}. \tag{2.43}$$

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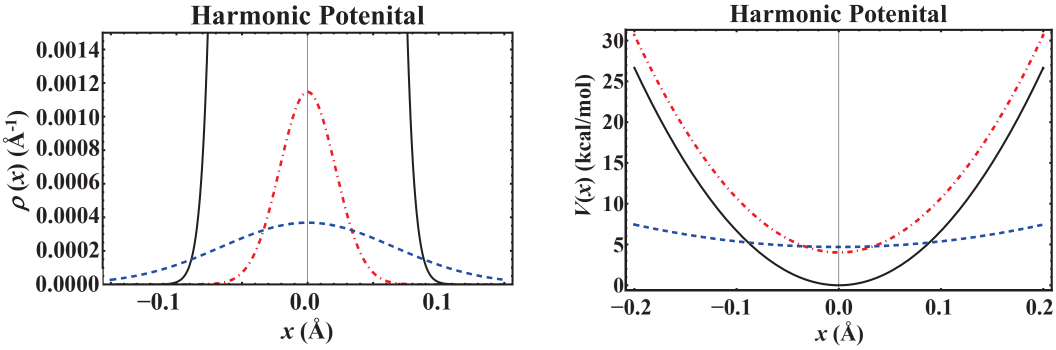

Figure 2: (Left) Particle density from the 'classical' harmonic potential [black solid line; Eq. (2.16)], quantum density matrix [blue dashed line; Eq. (2.11)], and centroid density of path integrals [red dashed-dotted line; Eq. (2.28) and Eq. (2.41)]. (Right) 'Classical' harmonic potential (black solid line), semiclassical potential from quantum particle density [blue dashed line; Eq. (2.45)], and centroid effective potential [red dashed-dotted line; Eq. (2.41)]. The mass M and the angular frequency ω of the classical harmonic potential $V(x)=M\omega^2x^2/2$ are: $M=1/(\frac{1}{19.0}+\frac{1}{1.01})$ AMU and $\omega=2.2112$ Å$^{-1}\sqrt{\frac{2\times 136.3\text{kcal/mol}}{M}}$.

For the untruncated or infinite parabolic barrier $V(x)=V_0-M\nu^2x^2/2$, the corresponding particle density is [113]:

$$\rho_{TS}(x)=e^{-\beta V_0}\sqrt{\frac{M\nu}{2\pi\hbar\sin(\beta\hbar\nu)}}\exp\left\{-\frac{M\nu x^2}{\hbar\sin(\beta\hbar\nu)}\left[\cos(\beta\hbar\nu)-1\right]\right\}. \tag{2.44}$$

As a consequence, the effective semiclassical potential V_ρ^{eff} constructed by the particle density, i.e.,

$$\sqrt{\frac{Mk_BT}{2\pi\hbar^2}}e^{-\beta V_\rho^{\text{eff}}}=\rho(x)\;\Rightarrow\;V_\rho^{\text{eff}}(x)=-k_BT\ln\left[\rho(x)\sqrt{\frac{2\pi\hbar^2}{Mk_BT}}\right] \tag{2.45}$$

behaves differently from the original potential V. For a direct comparison, Fig. 2 depicts these three types of harmonic potentials: V, V_ρ^{eff}, and W. The bottom of the potential V_ρ^{eff} is shifted upward the most, and its shape is the flattest too. We may recall that W essentially shifts up V by the free energy difference between quantum and classical harmonic oscillators. Therefore, in contrast to the case of using the centroid potential, if we replace the original potential V with V_ρ^{eff} to compute the reaction rate constant classically via the transition state theory, we would not be able to recover the exact quantum correction [Eq. (2.39)].

Through the above examples on a harmonic potential and an infinite parabola, we have demonstrated the ability in computing accurate quantum properties in classical formalisms by making use of the centroid density of path integrals. Indeed, this ability has been extensively applied in many theoretical or computational studies to accurately determine thermodynamic and quantum dynamic quantities [32, 34, 88, 89, 92, 114–120], including molecular spectroscopy of quantum fluids [121–124] and the rate constants of chemical and enzymatic reactions [57–59, 84–88, 110, 125–130]. The mass-dependent nature of W is also of particular interest because isotope effects can be obtained, and it has been applied to carbon nanotubes [54] in materials science and enzymatic reactions in life science [57–59, 126–128].

2.3 Path-integral Monte Carlo and molecular dynamics simulations

The success of path-integral calculations in various disciplines of science is partially owing to the emergence of Monte Carlo (MC) simulations, which started being widely used at about the same time around the birth of path integrals [87, 88, 131–145]. In practice, each path in space-time is conventionally either discretized into a set of virtual 'beads' on a sliced time axis or represented in Fourier space. The mathematical expression for discretized path integrations over all possible closed paths may be written as follows [32, 33, 60, 61]:

$$\oint D[x(\tau)] = \lim_{\substack{N\to\infty \\ \Delta\tau\to 0}} \int_{x_{N-1}=-\infty}^{\infty} \cdots \int_{x_2=-\infty}^{\infty} \int_{x_1=-\infty}^{\infty} \frac{dx_1}{\lambda_B^\tau} \frac{dx_2}{\lambda_B^\tau} \cdots \frac{dx_{N-1}}{\lambda_B^\tau}, \tag{2.46}$$

where N is the number of beads, $\lambda_B^\tau = \sqrt{\frac{2\pi\hbar\Delta\tau}{M}}$ is the thermal de Broglie wavelength between two adjacent beads, $\Delta\tau = \beta\hbar/N$ and $x_N = x_1$, for selecting closed paths. In Eq. (2.46), the path between two adjacent beads is 'approximated' as a free-particle path. The closed paths in discretized path integrals sometimes may be referred as polymer rings or necklaces. On the other hand, the Fourier path integrations over all closed paths, which are characterized by the centroid position $\bar{x}$, may be expressed as follows [34]:

$$\oint D[x(\tau)] = \sqrt{\frac{Mk_BT}{2\pi\hbar^2}} \int_{-\infty}^{\infty} d\bar{x} \prod_{m=1}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx_m^{\rm Im} dx_m^{\rm Re}}{\pi k_B T / M\omega_m^2} \right), \tag{2.47}$$

where $\omega_m = 2\pi m/\beta\hbar$, m is the index of Fourier coefficients, and the closed paths $x(\tau)$ are represented in terms of Fourier coefficients $x_m^{\rm Re}$ and $x_m^{\rm Im}$:

$$x(\tau) = \bar{x} + 2\sum_{m=1}^{\infty} \left(x_m^{\rm Re} \cos\omega_m\tau - x_m^{\rm Im} \sin\omega_m\tau \right). \tag{2.48}$$

More details about Fourier path integrals can be found in Appendix A.

By using Eq. (2.47) or (2.46), MC or molecular dynamics (MD) sampling techniques are subsequently performed to calculate the path integrations over a *finite* number of Fourier coefficients or beads, which are called PIMC [131–141] or PIMD simulations [87, 88, 142–145], respectively.

3 Kleinert's variational perturbation theory

Kleinert's variational perturbation (KP) theory [34] for the centroid density [32, 70, 84–89, 92] of Feynman path integrals [2, 32–34, 60–75] provides a complete theoretical foundation for developing non-stochastic/non-sampling methods to systematically incorporate internuclear quantum-statistical effects in condensed phase systems. Similar to the complementary interplay between the rapidly growing quantum Monte Carlo simulations [146–149] and the well-established *ab initio* or density-functional theories (DFT) for electronic structure calculations [4, 5, 25–27, 29], non-sampling/non-stochastic path-integral methods can complement the conventional Fourier or discretized path-integral Monte-Carlo (PIMC) [131, 136, 139–141] and molecular dynamics (PIMD) [87, 88] [e.g., Eqs. (2.46) and (2.47)] simulations which have been widely used in condensed phases.

To simplify the illustration of the essence of Kleinert's variational perturbation theory, we now consider a one-particle one-dimensional system. For a one-particle one-dimensional system, the classical canonical partition function in Eq. (1.8) reduces to:

$$Q_{cl}=\sqrt{\frac{Mk_BT}{2\pi\hbar^2}}\int_{-\infty}^{\infty}e^{-\beta V(x_0)}dx_0. \tag{3.1}$$

The traditional way to obtain the quantum canonical partition function, i.e., Eq. (1.3), is to solve the internuclear Schrödinger equation to get the individual energy eigenvalues. But in the path-integral (PI) formulation, we do not know the individual energy eigenvalues for obtaining the quantum partition function. This is because the PI representation of the quantum partition function can be written in terms of the centroid effective potential W as a classical configuration integral [32,34,84,85,88,89,92]:

$$Q_{qm}=\sum_n \exp(-\beta E_n)=\sqrt{\frac{Mk_BT}{2\pi\hbar^2}}\int_{-\infty}^{\infty}e^{-\beta W(x_0)}dx_0. \tag{3.2}$$

Given the centroid potential $W(x_0)$, thermodynamic and quantum dynamic quantities can be accurately determined, including molecular spectroscopy of quantum fluids and the rate constant of chemical and enzymatic reactions. The mass-dependent nature of $W(x_0)$ is also of particular interest because isotope effects can be obtained, and it has been applied to carbon nanotubes [54–56], proton-transfer reactions in solution [150], and biochemical reactions in protein [57–59] and RNA enzymes [151].

The centroid potential $W(x_0)$ in Eq. (3.2) is defined as follows [32,34,84,85,88,89,92]:

$$W(x_0)=-k_BT\ln\left[\sqrt{\frac{2\pi\hbar^2}{Mk_BT}}\oint D[x(\tau)]\delta(\bar{x}-x_0)\exp\{-A[x(\tau)]/\hbar\}\right], \tag{3.3}$$

where τ is a real number and represents the magnitude/component for pure imaginary time in path integral, $x(\tau)$ describes a path in space-time, $\oint Dx(\tau)\delta(\bar{x}-x_0)$ denotes a summation over *all* possible closed paths in which $\bar{x}$ is equal to x_0 (i.e., a functional integration), and $\bar{x}$ is the time-average position, called 'centroid'

$$\bar{x} \equiv \frac{1}{\beta\hbar} \int_0^{\beta\hbar} x(\tau)\, d\tau. \tag{3.4}$$

In Eq. (3.3), A is the quantum-statistical action:

$$A[x(\tau)] = \int_0^{\beta\hbar} d\tau \left\{ \frac{M}{2} \dot{x}^2(\tau) + V[x(\tau)] \right\}, \tag{3.5}$$

where $V(x)$ is the original potential energy of the system. More details can be found in Appendix A. Generalization of Eq. (3.3) to a multi-dimensional system is straightforward [32,34].

A number of non-stochastic approaches have been developed to estimate the centroid potential. For example, Feynman and Hibbs described a first-order cumulant expansion by introducing a Gaussian smearing function in a free-particle reference frame to yield an upper bound on the centroid potential [32]. This was subsequently modified by Doll and Myers (DM) by using a Gaussian width associated with the angular frequency at the minimum of the original potential [152]. Mielke and Truhlar employed a free-particle reference state and approximated the sum over paths by a minimal set of paths constrained for a harmonic oscillator. The action integral is obtained by using the three-point trapezoidal rule for the potential to yield the displaced-point path integral (DPPI) centroid potential [153].

A closely related theoretical approach to the KP theory is the variational method independently introduced by Giachetti and Tognetti [154], and by Feynman and Kleinert (hereafter labeled as GTFK) [155], which formally corresponds to the first order approximation in the KP theory, i.e., KP1. The GTFK approach is a variational method that adopts a harmonic reference state by variationally optimizing the angular frequency. This variational method has been applied to a variety of systems, including quantum dynamic processes in condensed phases (e.g., water and helium). Although the original GTFK approach is among the most accurate approximate methods for estimating the path-integral centroid potential in many applications [153], significant errors can exist in situations in which quantum effects are dominant, especially at low temperatures. Higher order perturbations of KP theory can significantly and systematically improve computational accuracy over the KP1 results [34,150,151,156,157].

In essence, what Kleinert's variational perturbation (KP) theory does is to systematically builds up anharmonic corrections to the harmonic centroid potential calculated in a harmonic reference state characterized by a trial angular frequency Ω [34]. Given the reference, or trial harmonic action:

$$A_\Omega^{x_0}=\int_0^{\beta\hbar} d\tau\left\{\frac{M}{2}\dot{x}^2(\tau)+\frac{1}{2}M\Omega^2[x(\tau)-x_0]^2\right\}. \tag{3.6}$$

the centroid potential $W(x_0)$ in Eq. (3.3) can be expressed as a path integral of the harmonic action which is perturbed by the anharmonicity of the original potential:

$$\begin{aligned} e^{-\beta W(x_0)} &= \sqrt{\frac{2\pi\hbar^2}{Mk_BT}}\oint Dx(\tau)\delta(\bar{x}-x_0)e^{-A_\Omega^{x_0}/\hbar}e^{-\left(A-A_\Omega^{x_0}\right)/\hbar} \\ &= Q_\Omega^{x_0}\left\langle e^{-\left(A-A_\Omega^{x_0}\right)/\hbar}\right\rangle_\Omega^{x_0}, \end{aligned} \tag{3.7}$$

where $Q_\Omega^{x_0}$ is the local harmonic partition function given as follows:

$$Q_\Omega^{x_0}=\sqrt{\frac{2\pi\hbar^2}{Mk_BT}}\oint Dx(\tau)\delta(\bar{x}-x_0)e^{-A_\Omega^{x_0}/\hbar}=\frac{\beta\hbar\Omega/2}{\sinh(\beta\hbar\Omega/2)}, \tag{3.8}$$

and $\langle\cdots\rangle_\Omega^{x_0}$ is the expectation value over all closed paths of the action in Eq. (3.6):

$$\left\langle e^{-F[x(\tau)]/\hbar}\right\rangle_\Omega^{x_0}=\frac{1}{Q_\Omega^{x_0}}\sqrt{\frac{2\pi\hbar^2}{Mk_BT}}\oint Dx(\tau)\delta(\bar{x}-x_0)e^{-F[x(\tau)]/\hbar}e^{-A_\Omega^{x_0}/\hbar}. \tag{3.9}$$

See Appendix A for more details, e.g., see Eq. (A.19). In Eq. (3.9), $F[x(\tau)]$ denotes an arbitrary functional. It is of interest to note that Eq. (3.7) is the starting point of Zwanzig's free-energy perturbation, which has been extensively used in free-energy calculations through Monte Carlo and molecular dynamics simulations. In Appendix A, we briefly show how we can perform the *functional* integrations of Eq. (3.9) in Fourier space.

If we expand the exponential functional in Eq. (3.7) and sum up the prefactors into an exponential series of cumulants, then the nth-order approximation, $W_n^\Omega(x_0)$, to the centroid potential $W(x_0)$ can be written as follows [34]:

$$\begin{aligned} e^{-\beta W_n^\Omega(x_0)} =& Q_\Omega^{x_0}\exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar}d\tau\left\langle A_{\rm int}^{x_0}\right\rangle_{\Omega,c}^{x_0}+\frac{1}{2!\hbar^2}\int_0^{\beta\hbar}d\tau_1\int_0^{\beta\hbar}d\tau_2\left\langle A_{\rm int}^{x_0}[x(\tau_1)]A_{\rm int}^{x_0}[x(\tau_2)]\right\rangle_{\Omega,c}^{x_0}\right. \\ &\left.+\cdots+\left\{\prod_{j=1}^{n}\int_0^{\beta\hbar}d\tau_j\right\}\frac{(-1)^n}{n!\hbar^n}\left\langle\prod_{k=1}^{n}A_{\rm int}^{x_0}[x(\tau_k)]\right\rangle_{\Omega,c}^{x_0}\right\}, \end{aligned} \tag{3.10}$$

where $A_{\rm int}^{x_0}=A-A_\Omega^{x_0}$ is the so-called inter-*action*, representing the perturbation to the harmonic reference state, $\langle\cdots\rangle_{\Omega,c}^{x_0}$ is a cumulant which can be written in terms of expectation values $\langle\cdots\rangle_{\Omega}^{x_0}$ by the cumulant expansion [34,90,91], e.g.,

$$\left\langle A_{\rm int}^{x_0}[x(\tau)]\right\rangle_{\Omega,c}^{x_0}\equiv\left\langle A_{\rm int}^{x_0}[x(\tau)]\right\rangle_{\Omega}^{x_0}, \tag{3.11}$$

$$\left\langle A_{\rm int}^{x_0}[x(\tau_1)]\,A_{\rm int}^{x_0}[x(\tau_2)]\right\rangle_{\Omega,c}^{x_0}\equiv\left\langle A_{\rm int}^{x_0}[x(\tau_1)]\,A_{\rm int}^{x_0}[x(\tau_2)]\right\rangle_{\Omega}^{x_0}-\left\{\left\langle A_{\rm int}^{x_0}[x(\tau)]\right\rangle_{\Omega}^{x_0}\right\}^2, \tag{3.12}$$

$$\begin{aligned}&\left\langle A_{\rm int}^{x_0}[x(\tau_1)]\,A_{\rm int}^{x_0}[x(\tau_2)]\,A_{\rm int}^{x_0}[x(\tau_3)]\right\rangle_{\Omega,c}^{x_0}\\&\quad\equiv\left\langle A_{\rm int}^{x_0}[x(\tau_1)]\,A_{\rm int}^{x_0}[x(\tau_2)]\,A_{\rm int}^{x_0}[x(\tau_3)]\right\rangle_{\Omega}^{x_0}\\&\qquad-3\left\langle A_{\rm int}^{x_0}[x(\tau_1)]\,A_{\rm int}^{x_0}[x(\tau_2)]\right\rangle_{\Omega}^{x_0}\left\langle A_{\rm int}^{x_0}[x(\tau)]\right\rangle_{\Omega}^{x_0}+2\left\{\left\langle A_{\rm int}^{x_0}[x(\tau)]\right\rangle_{\Omega}^{x_0}\right\}^3,\end{aligned} \tag{3.13}$$

etc.

More importantly, Kleinert and co-workers derived a math equation for expressing the expectation value $\left\{\prod_{j=1}^{n}\int_0^{\beta\hbar}d\tau_j\right\}\langle\prod_{k=1}^{n}F_k[x(\tau_k)]\rangle_{\Omega}^{x_0}$ from the *functional-integral* form to be in terms of Gaussian smearing convolution integrals [34], which are ordinary integrals:

$$\begin{aligned}&\left\{\prod_{j=1}^{n}\int_0^{\beta\hbar}d\tau_j\right\}\left\langle\prod_{k=1}^{n}F_k[x(\tau_k)]\right\rangle_{\Omega}^{x_0}\\&=\left\{\prod_{j=1}^{n}\int_0^{\beta\hbar}d\tau_j\right\}\left\{\prod_{k=1}^{n}\int_{-\infty}^{\infty}dx_kF_k(x_k)\right\}\\&\quad\times\frac{1}{\sqrt{(2\pi)^n{\rm Det}\left[a^2_{\tau_k\tau_{k'}}(\Omega)\right]}}\exp\left\{-\frac{1}{2}\sum_{\substack{k=1\\k'=1}}^{n}(x_k-x_0)\,a^{-2}_{\tau_k\tau_{k'}}(\Omega)\,(x_{k'}-x_0)\right\},\end{aligned} \tag{3.14}$$

where ${\rm Det}\left[a^2_{\tau_k\tau_{k'}}(\Omega)\right]$ is the determinant of the $n\times n$-matrix consisting of the Gaussian width $a^2_{\tau_k\tau_{k'}}(\Omega)$, $a^{-2}_{\tau_k\tau_{k'}}(\Omega)$ is an element of the inverse matrix of $a^2_{\tau_k\tau_{k'}}(\Omega)$, and the Gaussian width is a function of the trial frequency Ω:

$$a^2_{\tau\tau'}(\Omega)=\frac{1}{\beta M\Omega^2}\left\{\frac{\beta\hbar\Omega}{2}\frac{\cosh[(|\tau-\tau'|-\beta\hbar/2)\Omega]}{\sinh(\beta\hbar\Omega/2)}-1\right\}. \tag{3.15}$$

In Appendix A, we briefly show a derivation of the Gaussian smearing convolution integral for the first order of $W_n^{\Omega}(x_0)$, i.e., $n=1$.

In contrast to the *functional*-integral expression shown in Eq. (3.10), we can now finally use the smearing potentials given in Eq. (3.14) to express the nth-order Kleinert

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variational perturbation (KPn) approximation, $W_n^{\Omega}(x_0)$ in terms of *ordinary* integrals [34]:

$$\begin{aligned} W_n^{\Omega}(x_0) = & -k_B T \ln Q_{\Omega}^{x_0} + \frac{k_B T}{\hbar} \int_0^{\beta\hbar} d\tau \left\langle V_{\text{int}}^{x_0}[x(\tau_1)] \right\rangle_{\Omega}^{x_0} \\ & - \frac{k_B T}{2!\hbar^2} \int_0^{\beta\hbar} d\tau_1 \int_0^{\beta\hbar} d\tau_2 \left\langle V_{\text{int}}^{x_0}[x(\tau_1)]\, V_{\text{int}}^{x_0}[x(\tau_2)] \right\rangle_{\Omega,c}^{x_0} \\ & + \cdots + k_B T \frac{(-1)^{n+1}}{n!\hbar^n} \left\{ \prod_{j=1}^{n} \int_0^{\beta\hbar} d\tau_j \right\} \left\langle \prod_{k=1}^{n} V_{\text{int}}^{x_0}[x(\tau_k)] \right\rangle_{\Omega,c}^{x_0}, \end{aligned} \tag{3.16}$$

where $V_{\text{int}}^{x_0}[x(\tau)] = V[x(\tau)] - \frac{1}{2} M\Omega^2 [x(\tau)-x_0]^2$ (the kinetic energy terms in Eq. (3.5) and Eq. (3.6) cancel out with one another).

As n tends to infinity, $W_n^{\Omega}(x_0)$ approaches the exact value of the centroid potential $W(x_0)$ in Eq. (3.3), which is independent of the trial Ω. But the truncated sum in Eq. (3.16) does depend on Ω, and the optimal choice of this trial frequency at a given order of KP expansion and at a particular centroid position x_0 is determined by the least-dependence of $W_n^{x_0}(\Omega)$ on Ω itself. This is the so-called frequency of least dependence, which provides a variational approach to determine the optimal value of Ω, i.e., $\Omega_{\text{opt},n}(x_0)$ [34].

Of particular interest is the special case when $n=1$, which turns out to be identical to the original GTFK variational approach. An important property of KP1 or the GTFK variational approach is that there is a definite upper bound for the computed $W_1^{\Omega}(x_0)$ by virtue of the Jensen-Peierls inequality, i.e., from Eqs. (3.7) and (3.10):

$$e^{-\beta W(x_0)} = Q_{\Omega}^{x_0} \left\langle \exp\left(-\frac{A - A_{\Omega}^{x_0}}{\hbar} \right) \right\rangle_{\Omega}^{x_0} \geq Q_{\Omega}^{x_0} \exp \left\langle -\frac{A - A_{\Omega}^{x_0}}{\hbar} \right\rangle_{\Omega}^{x_0} = e^{-\beta W_1^{\Omega}(x_0)}. \tag{3.17}$$

Note that by choosing $\Omega = 0$ (i.e., the reference state is for a free particle), KP1 or GTFK [154, 155] reduces to the Feynman-Hibbs approach [32]. For higher orders of n, unfortunately, it is not guaranteed that a minimum of $W_n^{x_0}(\Omega)$ actually exists as a function of Ω. In this case, the least dependent Ω is obtained from the condition that the next derivative of $W_n^{x_0}(\Omega)$ with respect to Ω is set to zero. Consequently, Ω is considered as a variational parameter in the Kleinert perturbation theory such that $W_n^{x_0}\left[\Omega_{\text{opt},n}(x_0)\right]$ is least-dependent on Ω.

This variational criterion relies on the uniformly and exponentially convergent property of the KP theory. Kleinert and coworkers proved that his theory exhibits this property in several strong anharmonic-coupling systems. More importantly, this remarkably fast convergent property can also be observed even for computing the *electronic* ground state energy of a hydrogen atom (3 degrees of freedom). The ground state energy was determined by calculating the electronic centroid potential at the zero-temperature limit. The accuracies of the first three orders of the KP theory for a hydrogen atom are 85%, 95%, and 98%, respectively [34].

In practice, for odd n, there is typically a minimum point in Ω, but due to the alternating sign of the cumulants in Eq. (3.16), there is usually no minimum in Ω for even n. Nevertheless, the frequency of least-dependence for an even order perturbation in n can be determined by locating the inflexion point, i.e., the zero-value of the second derivative of $W_n^{x_0}(\Omega)$ with respect to Ω. Since the KP expansion is uniformly and exponentially converged, Kleinert has demonstrated that the least-dependent plateau in $W_n^{x_0}(\Omega)$, which is characterized by a minimum point for odd n or by an inflexion point for even n (Fig. 3 [157]), grows larger and larger with increasing orders of n [34].

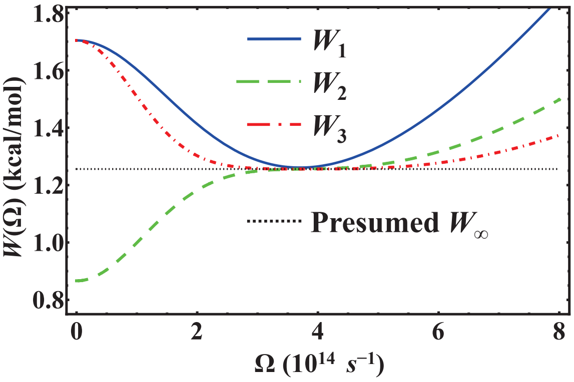


Figure 3: The minimum and inflexion points of the centroid potential calculated from the first three orders of Kleinert's variational perturbation theory. The first three order centroid potentials are denoted as W_1, W_2, W_3, respectively, which are plotted as a function of the variational angular frequency Ω [157]. W_∞ is the exact value of W.

4 Automated integration-free path-integral (AIF-PI) method

An especially attractive feature of Eq. (3.16) is that the if the real system potential is expressed as a series of polynomials or Gaussians, analytic expressions of Eq. (3.16) can be obtained, making the computation extremely efficient because the time-demanding Monte Carlo samplings for multi-dimensional numerical integrations could be avoided. Hereafter, the level of calculations up to nth order KP expansion for an mth-order-polynomial potential is denoted as KPn/Pm. For other potentials, KPn theory still involves elaborate n-dimensional space-time ($2n$ degrees of freedom) smearing integrals in Eq. (3.14). The intricacy of the smearing integrals increases tremendously for multi-dimensional potentials, where Ω becomes a $3N\times 3N$ matrix Ω_{ij} for N nuclei. This complexity is a major factor limiting applications of the KP theory beyond KP1, the original FK approach.

To render the KP theory feasible for many-body systems with N particles, we decouple the instantaneous normal mode (INM) coordinates $\{q^{x_0}\}^{3N}$ for a given configuration $\{x_0\}^{3N}$ [150, 151, 156–158]. Hence the multidimensional V effectively reduces to $3N$ one-dimensional potentials along each normal mode coordinate. Note that INM are naturally decoupled through the 2nd order Taylor expansion. The approximation of decoupling the INM coordinates has also been used elsewhere [159,160]. This approximation is particularly suited for the KP theory because of the exponential decaying property of the Gaussian convolution integrals in Eq. (3.14). In the decoupling INM approximation, the total effective centroid potential for N nuclei can be simplified as:

$$W_n^{\Omega}\left(\{x_0\}^{3N}\right) \approx V\left(\{x_0\}^{3N}\right) + \sum_{i=1}^{3N} w_{i,n}^{\Omega}\left(q_i^{x_0}\right), \tag{4.1}$$

where $w_{i,n}^{\Omega}\left(q_i^{x_0}\right)$ is the centroid potential for normal mode i. Although the INM approximation sacrifices some accuracy, in exchange, it allows analyses of quantum mechanical vibration and tunneling, and their separate contributions to the W. Positive and negative values of w_i raise (vibration) and lower (tunneling) the original potential, respectively. In practice, real frequencies from the INM analysis often yields positive w_i's in Eq. (4.1) with dominant contributions from zero-point-energy effects. By contrast, for imaginary frequencies in the INM, the values of w_i are often negative, due to tunneling contributions.

To obtain analytical expressions for the expectation values in Eq. (3.16), we use an mth order polynomial (Pm) to approximate or interpolate the potential along q_i. Hereafter, an mth order polynomial representation of the original potential energy function obtained with an interpolating step size q Å both in the forward and backward directions along the normal mode coordinate at x_0 is denoted as Pm-qA. Note that analytical results for P4 have been used by Kleinert for a quadratic-quartic anharmonic potential and a double-well potential [34]; however, higher order polynomials are needed to achieve the desired accuracy in real systems. We have thus derived the analytical closed forms of Eq. (3.16) up to P20 [150,151,156–158]. Consequently, the W as a function of an arbitrary Ω can be promptly obtained. This provides a convenient way to determine the least dependent Ω value without computing the complicated smearing integrals [Eq. (3.14)] iteratively for different trial values of Ω by Monte Carlo multi-dimensional numerical integrations. In fact, after the interpolating potential along each instantaneous normal-mode coordinate is determined, there is little computational cost for obtaining the W. Thereby, high level *ab initio* or density-functional (DFT) methods can be used to evaluate the potential energy function for *ab initio* path-integral calculations [150,151].

The computational procedure for obtaining the first and second order KP approximations to the centroid potential using our automated integration-free path-integral (AIF-PI) method is summarized below [150,151,156–158]:

1. For each $\{x_0\}^{3N}$, the mass-scaled Hessian matrix is diagonalized to obtain $\{q^{x_0}\}^{3N}$.
2. The original potential V is scanned from the configuration $\{x_0\}^{3N}$ along each $q_i^{x_0}$ for 10 points respectively in the forward and backward directions to interpolate V as P20-0.1A. We found that a step size of 0.1 Å is usually a reasonable choice to yield W within a few percent of the exact.

3. After the P20-0.1A interpolations, each $w_{i,n}^{\Omega}\left(q_i^{x_0}\right)$ as a function of Ω is readily obtained using the analytical expressions of KP1/P20 or KP2/P20. Note that the path integrals for these polynomials have been analytically integrated.

4. The values of $w_{i,n}^{\Omega}\left(q_i^{x_0}\right)$ are determined by numerically locating the least dependence of $w_{i,n}^{\Omega}\left(q_i^{x_0}\right)$ on Ω, i.e., zeroing the lowest order derivative of $w_{i,n}^{\Omega}\left(q_i^{x_0}\right)$ w.r.t. Ω (1st derivative for KP1 and usually 2nd derivative for KP2).

The procedure presented above is integration-free and essentially automated [150, 151, 156–158]. We hope it could be used by non-path-integral experts or experimentalists as a "black-box" for any given system. We are currently developing a formalism to systematically couple instantaneous normal-mode coordinates.

Due to the integration-free feature, our AIF-PI method is computationally efficient such that the potential energy can be evaluated using *ab initio* or density-functional theory (DFT) for performing the so-called *ab initio* path-integral calculations. Consequently, we used the hybrid density-functional B3LYP to construct the internuclear potential energy function for computing kinetic isotope effects (KIE) on several series of proton-transfer chemical reactions in water using the AIF-PI method. These proton-transfer chemical reactions are highly relevant to the bacterial squalene-to-hopene polycyclization. And the computed KIE results at the KP2 level are in good agreement with experiment (Fig. 4) [150]. Recently, we also employed the same computational technique to perform *ab initio* path-integral calculations of KIE on some RNA-model chemical reactions. Again, the calculated KIE values are in good agreement with experiments, which is a cover-image paper shown in Fig. 5 [151].

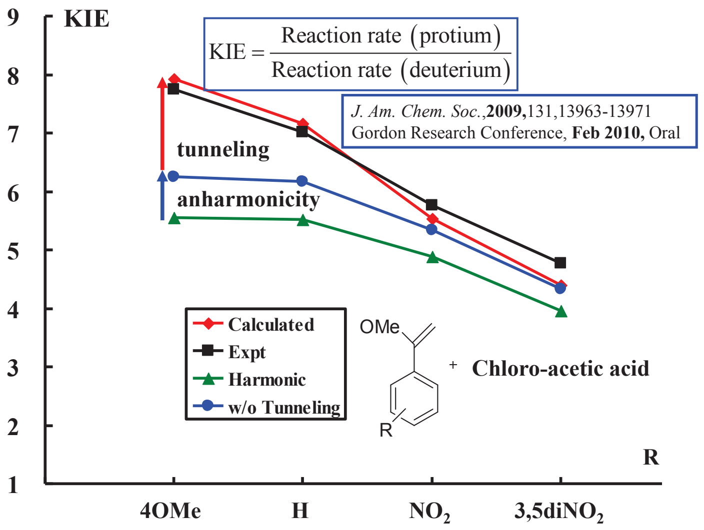
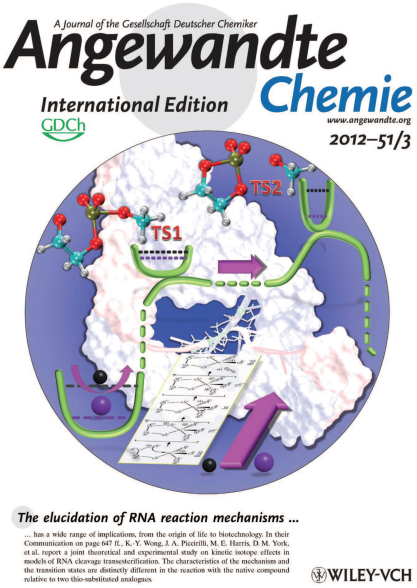


Figure 4: Our kinetic isotope effects (KIE) calculations on a series of proton-transfer reactions published in *J. Am. Chem. Soc.* (Impact Factor: 9.9) [150].



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The elucidation of RNA reaction mechanisms …

… has a wide range of implications, from the origin of life to biotechnology. In their Communication on page 647 ff., K.-Y. Wong, J. A. Piccirilli, M. E. Harris, D. M. York, et al. report a joint theoretical and experimental study on kinetic isotope effects in models of RNA cleavage transesterification. The characteristics of the mechanism and the transition states are distinctly different in the reaction with the native compound relative to two thio-substituted analogues.

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Figure 5: Cover image for our kinetic isotope effects (KIE) calculations on RNA transphosphorylation models published in *Angew. Chem. Int. Ed.* (Impact Factor: 13.5) [151].

5 Concluding remarks and outlook

In this review, we discuss how Feynman's path integral has been being used in the computations of internuclear quantum-statistical effects (e.g., tunneling, zero-point motion, and isotope effects) on many-body molecular systems.

Even though every single detail of molecular properties is essentially given in the molecular Schrödinger equation [Eq. (1.2)], the equation can be exactly solved only for the simplest one-electron atoms or ions. The most common and perhaps the mildest approximation often made to the molecular Schrödinger equation is the Born-Oppenheimer approximation, which adiabatically decouples nuclear and electronic degrees of freedom.

Although quantum effects on internuclear motions are much smaller than that on the electronic part, internuclear quantum-statistical effects actually could be important for many applications. A robust and popular choice for including such internuclear quantum effects is using Feynman's path integral [Eq. (2.12)]. This is largely because the essence of Feynman's path integrals is to transform the Schrödinger *differential* equation to become an *integral* equation [Eq. (2.3)]. As a result, the quantum canonical partition function can be directly obtained with some effective semiclassical potential [Eqs. (2.19), (2.22), (2.23), and (2.28); Fig. 2], e.g., centroid potential [Eq. (2.28)], without computing individual energy eigenvalues. In addition, the many-body path integrations can be carried out by the conventional molecular dynamics (MD) or Monte Carlo (MC) sampling techniques [Eqs. (2.46) and (2.47)].

To complement the conventional MD/MC sampling approach, Kleinert's variational perturbation (KP) theory provides a complete theoretical foundation for developing non-sampling/non-stochastic methods to systematically compute the centroid potential. In essence, what KP theory does is to systematically builds up anharmonic corrections to the harmonic centroid potential that is calculated in a harmonic reference state, which is characterized by a variational angular frequency [Eq. (3.16)]. Nevertheless, for multidimensional potentials, the complexity of the smearing integrals in the KP theory is increased terrifically. This intricacy is a major factor limiting applications of the KP theory beyond KP1, the 1st order approximation.

To make the powerful KP theory efficient beyond KP1 and applicable to realistic systems, we proposed our automated integration-free path-integral (AIF-PI) method (Section 4). In our AIF-PI method, a major achievement is to use a polynomial interpolation of the potential on each instantaneous normal mode (INM) coordinate for many-body systems to derive analytic expression for the path integrals. The implementation is sufficiently general for any systems described by smooth internuclear potential energy functions. Although the decoupled INM coordinate approximation in the AIF-PI method neglects correlations between instantaneous normal modes, it provides further insights into quantum contributions from vibration and tunneling.

In addition, the integration-free feature of our AIF-PI method enables us to perform *ab initio* path-integration calculations of kinetic isotope effects (KIE) on both proton-transfer and phosphoryl-transfer reactions. The calculated KIE results are in good agreement with experiment that help us to elucidate the enzymatic mechanisms underlying the bacterial squalene-to-hopene polycyclization and the RNA 2′-*O*-transphosphorylation (Figs. 4 and 5).

Nonetheless, to go beyond the INM approximation, we are developing a formalism to systematically couple the INM. And, we are also working on a new centroid path-integral theory at the minimum of the absolute-zero energy (AMAZE) [161]. Using this new AMAZE theory, along with other centroid properties reported in the literature, it is possible that we can accurately calculate many-body quantum free energies, tunneling splittings, and molecular (anharmonic) spectroscopy simply by minimization. Further, the long-existing problem in free-energy simulations, which is caused by multiple conformers orthogonal to the reaction coordinate, could also potentially be solved by our new AMAZE theory [161].

Acknowledgments

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A Path integrals in Fourier space and first order Gaussian smearing convolution integral

To derive the smearing formula for Kleinert's variational perturbation theory, we first determine the universal normalization constant, say C_m, for the quantum-statistical centroid-path-integrals (*functional* integration) at imaginary time τ in Fourier space.

Any closed-path $x(\tau)$ in space-time with the time-average or the 'centroid' position $\bar{x}=\frac{1}{\beta\hbar}\int_0^{\beta\hbar}x(\tau)d\tau$ can be written as follows:

$$\begin{aligned} x(\tau)&=\bar{x}+\sum_{m=1}^{\infty}\left(x_m e^{i\omega_m\tau}+x_m^* e^{-i\omega_m\tau}\right)\\ &=\bar{x}+2\sum_{m=1}^{\infty}\left(x_m^{\mathrm{Re}}\cos\omega_m\tau-x_m^{\mathrm{Im}}\sin\omega_m\tau\right), \end{aligned} \tag{A.1}$$

where

$$\omega_m\equiv\frac{2\pi m}{\beta\hbar}. \tag{A.2}$$

After defining the notation $\oint D[x(\tau)]$ as the summation over all possible closed paths $x(\tau)$ in space-time (which is dimensionless):

$$\oint D[x(\tau)]\equiv\sqrt{\frac{Mk_BT}{2\pi\hbar^2}}\int_{-\infty}^{\infty}d\bar{x}\prod_{m=1}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_m^{\mathrm{Im}}dx_m^{\mathrm{Re}}}{C_m}\right), \tag{A.3}$$

the quantum canonical partition function Q_{qm} can be expressed in terms of Feynman's path integrals:

$$Q_{qm}=\oint D[x(\tau)]e^{-\frac{1}{\hbar}A[x(\tau)]}, \tag{A.4}$$

where $A[x(\tau)]$ is the quantum-statistical action:

$$A[x(\tau)]=\int_0^{\beta\hbar}d\tau\left\{\frac{M}{2}\dot{x}^2(\tau)+V[x(\tau)]\right\}. \tag{A.5}$$

The physical meaning of Eq. (A.4) is that the quantum partition function is the summation over all closed paths weighted by the exponential factor $e^{-\frac{1}{\hbar}A[x(\tau)]}$.

We determine the normalization constant C_m in Eq. (A.3) by considering a linear harmonic oscillator, in which the potential is $V(x)=\frac{1}{2}M\Omega^2x^2$. The action associated with this harmonic potential is:

$$\begin{aligned}A[x(\tau)]=\int_0^{\beta\hbar}d\tau\Bigg\{&\frac{M}{2}\left[2\sum_{m=1}^{\infty}\left(x_m^{\mathrm{Re}}\omega_m(-\sin\omega_m\tau)-x_m^{\mathrm{Im}}\omega_m(\cos\omega_m\tau)\right)\right]^2\\&+\frac{M}{2}\Omega^2\Bigg[\bar{x}^2+2\bar{x}\sum_{m=1}^{\infty}\left(x_m^{\mathrm{Re}}\cos\omega_m\tau-x_m^{\mathrm{Im}}\sin\omega_m\tau\right)\\&+\left(2\sum_{m=1}^{\infty}\left(x_m^{\mathrm{Re}}\cos\omega_m\tau-x_m^{\mathrm{Im}}\sin\omega_m\tau\right)\right)^2\Bigg]\Bigg\}.\end{aligned}\tag{A.6}$$

Since only the terms $\int_0^{\beta\hbar}d\tau\sin^2\omega_m\tau$ and $\int_0^{\beta\hbar}d\tau\cos^2\omega_m\tau$ are non-zero and all the other cross terms are at zero value, the harmonic quantum-statistical action can be simplified as follows:

$$\begin{aligned}A[x(\tau)]=\int_0^{\beta\hbar}d\tau\Bigg\{&\frac{M}{2}\left[4\sum_{m=1}^{\infty}\left(\left(x_m^{\mathrm{Re}}\right)^2\omega_m^2\sin^2\omega_m\tau+\left(x_m^{\mathrm{Im}}\right)^2\omega_m^2\cos^2\omega_m\tau\right)\right]\\&+\frac{M}{2}\Omega^2\left[\bar{x}^2+4\sum_{m=1}^{\infty}\left(\left(x_m^{\mathrm{Re}}\right)^2\cos^2\omega_m\tau+\left(x_m^{\mathrm{Im}}\right)^2\sin^2\omega_m\tau\right)\right]\Bigg\}\\ \Rightarrow A[x(\tau)]=\beta\hbar\Bigg\{&\frac{M\Omega^2\bar{x}^2}{2}+M\sum_{m=1}^{\infty}\left[\Omega^2+\omega_m^2\right]\left[\left(x_m^{\mathrm{Re}}\right)^2+\left(x_m^{\mathrm{Im}}\right)^2\right]\Bigg\}.\end{aligned}\tag{A.7}$$

Substituting Eq. (A.7) into Eq. (A.4), the harmonic quantum partition function is:

$$\begin{aligned}&\oint D[x(\tau)]e^{-\frac{1}{\hbar}A[x(\tau)]}\\&=\left(\int_{-\infty}^{\infty}d\bar{x}e^{-\beta\frac{1}{2}M\Omega^2\bar{x}^2}\sqrt{\frac{Mk_BT}{2\pi\hbar^2}}\right)\\&\quad\times\prod_{m=1}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_m^{\mathrm{Im}}dx_m^{\mathrm{Re}}}{C_m}\right)\times\exp\left\{-\beta M\sum_{m=1}^{\infty}\left[\Omega^2+\omega_m^2\right]\left[\left(x_m^{\mathrm{Re}}\right)^2+\left(x_m^{\mathrm{Im}}\right)^2\right]\right\}\\&=\left(\frac{1}{\beta\hbar\Omega}\right)\prod_{m=1}^{\infty}\frac{1}{C_m}\left[\sqrt{\frac{\pi}{\beta M(\Omega^2+\omega_m^2)}}\right]^2\\&=\frac{1}{\beta\hbar\Omega}\prod_{m=1}^{\infty}\frac{1}{C_m}\left[\frac{\pi}{\beta M(\Omega^2+\omega_m^2)}\right].\end{aligned}\tag{A.8}$$

For a free particle, we put $\Omega=0$ before integrating $\bar{x}$ in Eq. (A.8) to obtain the following partition function:

$$Q_{qm}(\text{free particle}) = \sqrt{\frac{Mk_BT}{2\pi\hbar^2}}\,Volume\prod_{m=1}^{\infty}\frac{1}{C_m}\left[\frac{\pi}{\beta M\omega_m^2}\right]$$
$$= Q_{cl}(\text{free particle})\prod_{m=1}^{\infty}\frac{1}{C_m}\left[\frac{\pi}{\beta M\omega_m^2}\right]. \tag{A.9}$$

Since the classical and quantum partition functions for a free particle are the same (because the eigen-energy spectrum is continuous), this leads the value of C_m to be

$$C_m = \pi k_B T/M\omega_m^2. \tag{A.10}$$

On the other hand, by putting $C_m = \pi k_B T/M\omega_m^2$, the quantum canonical partition function for a linear harmonic oscillator can be re-written as follows:

$$\begin{aligned}
Q_{qm}(\text{harmonic}) &= \oint D[x(\tau)]e^{-\frac{1}{\hbar}A[x(\tau)]}\\
&= \frac{1}{\beta\hbar\Omega}\prod_{m=1}^{\infty}\frac{\omega_m^2}{\Omega^2+\omega_m^2} = \frac{1}{\beta\hbar\Omega}\prod_{m=1}^{\infty}\frac{\frac{4\pi^2m^2}{\beta^2\hbar^2}}{\frac{4\pi^2m^2}{\beta^2\hbar^2}+\Omega^2}\\
&= \frac{1}{\beta\hbar\Omega}\prod_{m=1}^{\infty}\frac{m^2\pi^2}{m^2\pi^2+\left(\frac{\beta\hbar\Omega}{2}\right)^2} = \frac{1}{\beta\hbar\Omega}\frac{\beta\hbar\Omega/2}{\sinh(\beta\hbar\Omega/2)}\\
&= \frac{1}{2\sinh(\beta\hbar\Omega/2)} = \sum_{n=0}^{\infty}e^{-\beta\hbar\Omega\left(n+\frac{1}{2}\right)} \quad (\text{for all real } \Omega).
\end{aligned} \tag{A.11}$$

The proof of

$$\prod_{m=1}^{\infty}\frac{m^2\pi^2}{m^2\pi^2+\left(\frac{\beta\hbar\Omega}{2}\right)^2} = \frac{\beta\hbar\Omega/2}{\sinh(\beta\hbar\Omega/2)} \tag{A.12}$$

can be found in Refs. [34,70].

Now using (A.10) and (A.3), we can define the notation $\oint D[x(\tau)]\delta(\bar{x}-x_0)e^{-\frac{1}{\hbar}A[x(\tau)]}$ as follows:

$$\oint D[x(\tau)]\delta(\bar{x}-x_0)e^{-\frac{1}{\hbar}A[x(\tau)]} \equiv \sqrt{\frac{Mk_BT}{2\pi\hbar^2}}\prod_{m=1}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_m^{\mathrm{Im}}dx_m^{\mathrm{Re}}}{\pi k_BT/M\omega_m^2}e^{-\frac{1}{\hbar}A[\bar{x}_0(\tau)]}\right), \tag{A.13}$$

where $\bar{x}_0(\tau)$ denotes a closed path in which its average position is equal to x_0:

$$\begin{aligned}
&\bar{x}_0(\tau) = x_0 + 2\sum_{m=1}^{\infty}\left(x_m^{\mathrm{Re}}\cos\omega_m\tau - x_m^{\mathrm{Im}}\sin\omega_m\tau\right)\\
\Rightarrow\ &\frac{1}{\beta\hbar}\int_0^{\beta\hbar}\bar{x}_0(\tau)d\tau = x_0.
\end{aligned} \tag{A.14}$$

In other words, we also can use Eq. (A.13) to re-express Eq. (A.4) as follows:

$$\begin{aligned} Q_{qm} &= \oint D[x(\tau)] e^{-\frac{1}{\hbar}A[x(\tau)]} \\ &= \oint D[x(\tau)] \int_{-\infty}^{\infty} \delta(\bar{x}-x_0)\, dx_0 e^{-\frac{1}{\hbar}A[\bar{x}_0(\tau)]} \\ &= \sqrt{\frac{Mk_BT}{2\pi\hbar^2}} \int_{-\infty}^{\infty} dx_0 \prod_{m=1}^{\infty} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx_m^{\mathrm{Im}} dx_m^{\mathrm{Re}}}{\pi k_B T/M\omega_m^2} e^{-\frac{1}{\hbar}A[\bar{x}_0(\tau)]} \right). \end{aligned} \tag{A.15}$$

If we define the centroid effective potential W as follows:

$$\begin{aligned} e^{-\beta W(x_0)} &= \prod_{m=1}^{\infty} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx_m^{\mathrm{Im}} dx_m^{\mathrm{Re}}}{\pi k_B T/M\omega_m^2} e^{-\frac{1}{\hbar}A[\bar{x}_0(\tau)]} \right) \\ &= \sqrt{\frac{2\pi\hbar^2}{Mk_BT}} \oint D[x(\tau)] \delta(\bar{x}-x_0) e^{-\frac{1}{\hbar}A[x(\tau)]}, \end{aligned} \tag{A.16}$$

then finally we can express the quantum canonical partition function in a classical fashion or in terms of classical configuration integral:

$$Q_{qm} = \oint D[x(\tau)] e^{-\frac{1}{\hbar}A[x(\tau)]} \Rightarrow Q_{qm} = \sqrt{\frac{Mk_BT}{2\pi\hbar^2}} \int_{-\infty}^{\infty} dx_0 e^{-\beta W(x_0)}. \tag{A.17}$$

Next, we define a local harmonic action:

$$\begin{aligned} A_{\Omega}^{x_0}[\bar{x}_0(\tau)] \equiv A_{\Omega}^{x_0} &\equiv \int_0^{\beta\hbar} d\tau \left\{ \frac{M}{2}\dot{\bar{x}}_0^2(\tau) + \frac{1}{2}M\Omega^2[\bar{x}_0(\tau)-x_0]^2 \right\} \\ &= \beta\hbar \left\{ M \sum_{m=1}^{\infty} \left[\Omega^2+\omega_m^2\right] \left[\left(x_m^{\mathrm{Re}}\right)^2 + \left(x_m^{\mathrm{Im}}\right)^2 \right] \right\} \end{aligned} \tag{A.18}$$

in which we have used Eqs. (A.14) and (A.7) for the simplification of the final expression. From Eqs. (A.8), (A.10), (A.11), and (A.18), the corresponding local harmonic partition function Q and the harmonic centroid effective potential W can be written as:

$$\begin{aligned} Q_{\Omega}^{x_0} = e^{-\beta W_{\Omega}(x_0)} &= \prod_{m=1}^{\infty} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx_m^{\mathrm{Im}} dx_m^{\mathrm{Re}}}{\pi k_B T/M\omega_m^2} e^{-\frac{1}{\hbar}A_{\Omega}^{x_0}} \right) \\ &= \prod_{m=1}^{\infty} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx_m^{\mathrm{Im}} dx_m^{\mathrm{Re}}}{\pi k_B T/M\omega_m^2} \exp\left\{ -\beta M \sum_{m=1}^{\infty} \left[\Omega^2+\omega_m^2\right] \left[\left(x_m^{\mathrm{Re}}\right)^2 + \left(x_m^{\mathrm{Im}}\right)^2 \right] \right\} \right) \\ &= \prod_{m=1}^{\infty} \frac{\omega_m^2}{\Omega^2+\omega_m^2} = \frac{\beta\hbar\Omega/2}{\sinh(\beta\hbar\Omega/2)}. \end{aligned} \tag{A.19}$$

The final expression in Eq. (A.19) is true for all real Ω; if Ω is pure imaginary, we need $\left|\frac{\beta\hbar\Omega}{2}\right| < \pi$. By using the local harmonic action in Eq. (A.18) as the reference frame, we now derive Kleinert's variational perturbation theory as follows. First, we re-write the expression of the centroid potential W in Eq. (2.28) as Zwanzig's free-energy perturbation in the harmonic reference frame:

$$\begin{aligned} e^{-\beta W(x_0)} &= \prod_{m=1}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_m^{\rm Im}dx_m^{\rm Re}}{\pi k_B T/M\omega_m^2}e^{-\frac{1}{\hbar}\left(A^{x_0}-A_\Omega^{x_0}\right)}e^{-\frac{1}{\hbar}A_\Omega^{x_0}}\right)\\ &= Q_\Omega^{x_0}\frac{\prod\limits_{m=1}^{\infty}\left(\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{dx_m^{\rm Im}dx_m^{\rm Re}}{\pi k_B T/M\omega_m^2}e^{-\frac{1}{\hbar}\left(A^{x_0}-A_\Omega^{x_0}\right)}e^{-\frac{1}{\hbar}A_\Omega^{x_0}}\right)}{\prod\limits_{m=1}^{\infty}\left(\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{dx_m^{\rm Im}dx_m^{\rm Re}}{\pi k_B T/M\omega_m^2}e^{-\frac{1}{\hbar}A_\Omega^{x_0}}\right)}\\ &= Q_\Omega^{x_0}\left\langle e^{-\frac{1}{\hbar}\left(A^{x_0}-A_\Omega^{x_0}\right)}\right\rangle_\Omega^{x_0} = Q_\Omega^{x_0}\left\langle e^{-\frac{1}{\hbar}A_{\rm int}^{x_0}}\right\rangle_\Omega^{x_0}, \end{aligned} \tag{A.20}$$

where $\langle\cdots\rangle_\Omega^{x_0}$ is the *functional* average or expectation value over all closed paths in the harmonic reference frame described by the harmonic action in Eq. (A.18) and $A_{\rm int}^{x_0}=A^{x_0}-A_\Omega^{x_0}$ is the so-called *inter*-action. We expand the functional average in Eq. (A.20) by the functional cumulant expansion [91]:

$$\begin{aligned} e^{-\beta W(x_0)} &= Q_\Omega^{x_0}\left\langle e^{-\frac{1}{\hbar}A_{\rm int}^{x_0}}\right\rangle_\Omega^{x_0}\\ &= Q_\Omega^{x_0}\left\langle \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar}d\tau\left\{V[\bar{x}_0(\tau)]-\frac{1}{2}M\Omega^2[\bar{x}_0(\tau)-x_0]^2\right\}\right]\right\rangle_\Omega^{x_0}\\ &= Q_\Omega^{x_0}\exp\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n!\hbar^n}\int_0^{\beta\hbar}d\tau_1\cdots\int_0^{\beta\hbar}d\tau_n\left\langle V_{\rm int}^{x_0}[\bar{x}_0(\tau_1)]\cdots V_{\rm int}^{x_0}[\bar{x}_0(\tau_n)]\right\rangle_{\Omega,c}^{x_0}\right\}, \end{aligned} \tag{A.21}$$

where $\langle\cdots\rangle_{\Omega,c}^{x_0}$ is the cumulant average and $V_{\rm int}^{x_0}[\bar{x}_0(\tau)]=V[\bar{x}_0(\tau)]-\frac{1}{2}M\Omega^2[\bar{x}_0(\tau)-x_0]^2$. The cumulant average can be expressed in terms of the following *functional* average:

$$\begin{aligned} &\left\langle V_{\rm int}^{x_0}[\bar{x}_0(\tau_1)]\cdots V_{\rm int}^{x_0}[\bar{x}_0(\tau_n)]\right\rangle_\Omega^{x_0}\\ =&\frac{1}{Q_\Omega^{x_0}}\prod_{m=1}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_m^{\rm Im}dx_m^{\rm Re}}{\pi k_B T/M\omega_m^2}\left(V_{\rm int}^{x_0}[\bar{x}_0(\tau_1)]\cdots V_{\rm int}^{x_0}[\bar{x}_0(\tau_n)]\right)e^{-\frac{1}{\hbar}A_\Omega^{x_0}}\right\}. \end{aligned} \tag{A.22}$$

For the rest of this Appendix A, we are going to derive the smearing formula for the *functional* average in Eq. (A.22) when $n=1$, so that the path-integrations can be calculated in terms of *ordinary* integrations (i.e., no more *functional* integrations).

We first replace $V_{\rm int}^{x_0}[\bar{x}_0(\tau_n)]$ with an arbitrary function F_n in Eq. (A.22) for $n=1$, i.e.,

$$\begin{aligned}
&\langle F_1[\bar{x}_0(\tau)]\rangle_\Omega^{x_0}\\
=&\frac{1}{Q_\Omega^{x_0}}\prod_{m=1}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_m^{\rm Im}dx_m^{\rm Re}}{\pi k_BT/M\omega_m^2}F_1[\bar{x}_0(\tau)]e^{-\frac{1}{\hbar}A_\Omega^{x_0}}\right\}\\
=&\frac{1}{Q_\Omega^{x_0}}\prod_{m=1}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_m^{\rm Im}dx_m^{\rm Re}}{\pi k_BT/M\omega_m^2}F_1\left[x_0+2\sum_{m=1}^{\infty}\left(x_m^{\rm Re}\cos\omega_m\tau-x_m^{\rm Im}\sin\omega_m\tau\right)\right]\right.\\
&\left.\times\exp\left\{-\beta M\sum_{m=1}^{\infty}\left[\Omega^2+\omega_m^2\right]\left[\left(x_m^{\rm Re}\right)^2+\left(x_m^{\rm Im}\right)^2\right]\right\}\right\}. \qquad (A.23)
\end{aligned}$$

Now we Fourier transform $F_1\left[\bar{x}_0(\tau)\right]$ in k space:

$$F_1[\bar{x}_0(\tau)]=\int_{-\infty}^{\infty}\frac{dk_1}{\sqrt{2\pi}}e^{ik_1\bar{x}_0(\tau)}\tilde{F}_1(k_1). \qquad (A.24)$$

Then Eq. (A.23) becomes

$$\begin{aligned}
&\langle F_1[\bar{x}_0(\tau)]\rangle_\Omega^{x_0}\\
=&\frac{1}{Q_\Omega^{x_0}}\times\prod_{m=1}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_m^{\rm Im}dx_m^{\rm Re}}{\pi k_BT/M\omega_m^2}\int_{-\infty}^{\infty}\frac{dk_1}{\sqrt{2\pi}}\tilde{F}_1(k_1)\exp\left\{ik_1x_0-\beta M\sum_{m=1}^{\infty}\left[\Omega^2+\omega_m^2\right]\right.\right.\\
&\left.\left.\times\left[\left(x_m^{\rm Re}\right)^2+\left(x_m^{\rm Im}\right)^2-\frac{2ik_1}{\beta M(\Omega^2+\omega_m^2)}\left(x_m^{\rm Re}\cos\omega_m\tau-x_m^{\rm Im}\sin\omega_m\tau\right)\right]\right\}\right). \qquad (A.25)
\end{aligned}$$

Doing the completing the square of $x_m^{\rm Re}$ and $x_m^{\rm Im}$, Eq. (A.25) becomes

$$\begin{aligned}
&\langle F_1[\bar{x}_0(\tau)]\rangle_\Omega^{x_0}\\
=&\frac{1}{Q_\Omega^{x_0}}\prod_{m=1}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_m^{\rm Im}dx_m^{\rm Re}}{\pi k_BT/M\omega_m^2}\int_{-\infty}^{\infty}\frac{dk_1}{\sqrt{2\pi}}\tilde{F}_1(k_1)\right.\\
&\times\exp\left\{ik_1x_0-\beta M\sum_{m=1}^{\infty}\left[\Omega^2+\omega_m^2\right]\left[\left(x_m^{\rm Re}-\frac{ik_1\cos\omega_m\tau}{\beta M(\Omega^2+\omega_m^2)}\right)^2\right.\right.\\
&\left.\left.\left.+\left(x_m^{\rm Im}+\frac{ik_1\sin\omega_m\tau}{\beta M(\Omega^2+\omega_m^2)}\right)^2+\frac{k_1^2}{\beta^2M^2(\Omega^2+\omega_m^2)^2}\right]\right\}\right). \qquad (A.26)
\end{aligned}$$

Since

$$\prod_{m=1}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx_m^{\rm Im}dx_m^{\rm Re}}{\pi k_B T/M\omega_m^2}\exp\left\{-\beta M\sum_{m=1}^{\infty}\left[\Omega^2+\omega_m^2\right]\left[\left(x_m^{\rm Re}\right)^2+\left(x_m^{\rm Im}\right)^2\right]\right\}\right)$$
$$=\frac{\beta\hbar\Omega/2}{\sinh\left(\beta\hbar\Omega/2\right)}=Q_\Omega^{x_0} \tag{A.27}$$

[see Eq. (A.19)], after integrating $x_m^{\rm Im}$ and $x_m^{\rm Re}$, the denominator $Q_\Omega^{x_0}$ in Eq. (A.26) is cancelled out by the integrations. The remainders are:

$$\langle F_1\left[\bar{x}_0(\tau)\right]\rangle_\Omega^{x_0}=\int_{-\infty}^{\infty}\frac{dk_1}{\sqrt{2\pi}}\tilde{F}_1(k_1)\exp\left\{ik_1x_0-\frac{k_1^2}{\beta M}\sum_{m=1}^{\infty}\frac{1}{\Omega^2+\omega_m^2}\right\}. \tag{A.28}$$

Now let time-*independent* smearing Gaussian width $a(\Omega)$ be:

$$\begin{aligned}a^2(\Omega)&=\frac{2}{\beta M}\sum_{m=1}^{\infty}\frac{1}{\Omega^2+\omega_m^2}\\&=\frac{1}{\beta M\Omega^2}\left[\frac{\beta\hbar\Omega}{2}\coth\left(\frac{\beta\hbar\Omega}{2}\right)-1\right].\end{aligned} \tag{A.29}$$

The proof of

$$\frac{2}{\beta M}\sum_{m=1}^{\infty}\frac{1}{\Omega^2+\frac{4\pi^2m^2}{\beta^2\hbar^2}}=\frac{1}{\beta M\Omega^2}\left[\frac{\beta\hbar\Omega}{2}\coth\left(\frac{\beta\hbar\Omega}{2}\right)-1\right] \tag{A.30}$$

can be found in Refs. [34,70]. Then Eq. (A.28) can be simplified as follows:

$$\langle F_1\left[\bar{x}_0(\tau)\right]\rangle_\Omega^{x_0}=\int_{-\infty}^{\infty}\frac{dk_1}{\sqrt{2\pi}}\tilde{F}_1(k_1)\exp\left\{ik_1x_0-\frac{a^2(\Omega)k_1^2}{2}\right\}. \tag{A.31}$$

We Fourier transform $\tilde{F}_1(k_1)$ [Eq. (A.24)] back to x space, Eq. (A.31) becomes

$$\langle F_1\left[\bar{x}_0(\tau)\right]\rangle_\Omega^{x_0}=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_1\int_{-\infty}^{\infty}dx_1F_1(x_1)\exp\left\{ik_1(x_0-x_1)-\frac{a^2(\Omega)k_1^2}{2}\right\}. \tag{A.32}$$

After integrating k_1, we obtain the smearing formula for the first order Kleinert's variational perturbation theory:

$$\begin{aligned}
&\langle F_1[\bar{x}_0(\tau)]\rangle_\Omega^{x_0}\\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty} dx_1 F_1(x_1)\int_{-\infty}^{\infty} dk_1 \exp\left\{-\frac{a^2(\Omega)}{2}\left[k_1-\frac{i(x_0-x_1)}{a^2(\Omega)}\right]^2-\left[\frac{a^2(\Omega)}{2}\right]\frac{(x_0-x_1)^2}{a^4(\Omega)}\right\}\\
=&\frac{1}{2\pi}\sqrt{\frac{2\pi}{a^2(\Omega)}}\int_{-\infty}^{\infty} F_1(x_1)\exp\left[-\frac{(x_0-x_1)^2}{2a^2(\Omega)}\right]dx_1\\
\Rightarrow&\ \langle F_1[\bar{x}_0(\tau)]\rangle_\Omega^{x_0}=\frac{1}{\sqrt{2\pi a^2(\Omega)}}\int_{-\infty}^{\infty} F_1(x_1)\exp\left[-\frac{(x_0-x_1)^2}{2a^2(\Omega)}\right]dx_1. \qquad \text{(A.33)}
\end{aligned}$$

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