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Point Sources Identification Problems for Heat Equations

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Abstract. We considered the point source identification problems for heat equations from noisy observation data taken at the minimum number of spatially fixed measurement points. We aim to identify the unknown number of sources and their locations along with their strengths. In our previous work, we proved that minimum measurement points needed under the noise-free setting. In this paper, we extend the proof to cover the noisy cases over a border class of source functions. We show that if the regularization parameter is chosen properly, the problem can be transformed into a poles identification problem. A reconstruction scheme is proposed on the basis of the developed theoretical results. Numerical demonstrations in 2D and 3D conclude the paper.

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1 Introduction

Inverse source identification problems are important in many branches of engineering sciences. For examples, an accurate estimation of a pollution source in a river [7], a determination of magnitude of groundwater pollution sources [15] are crucial to environmental protection. Other examples can be found in [20, 21] and the references therein. In general, a complete recovery of the unknown source is not attainable from practically restricted boundary measurements. The inverse source problem becomes solvable if certain *a priori* knowledge is assumed. Inverse problems are in nature *unstable* because the unknown solutions/ parameters have to be determined from indirect observable data

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which contain measurement errors. The major difficulty in establishing any numerical algorithm for approximating the solution is the severe ill-posedness of the problem and the ill-conditioning of the resultant discretized matrix.

The heat conduction process is irreducible in time, while the temperature profile becomes rapidly smoother in time. This means that the characteristic of the solution may not be affected by the observed data. To the knowledge of the authors, the mathematical analysis and efficient algorithms for inverse heat problems are still very limited. For instance, the uniqueness and conditional stability results for heat source identification problem can be found in [3, 4, 26]. Studies on stationary point source problem can be found in [2, 5, 16]. Some reconstruction schemes can be found in [22, 24, 28, 29].

In order to solve an inverse problem of any kind, it is well-known that more input data usually results in better estimation. In practise, it is not always possible to install a large amount of measuring instruments. Our interest is therefore investing on the minimum number of measurement points needed for the point sources identifications problems for heat equations. We first consider the problem of identifying, from data obtained by a single measurement point b

$$\|u(t,b)-u^{\delta}(t)\|_{L^2(0,T)}\leq\delta, \tag{1.1}$$

the source function of the following heat equation

$$\begin{cases} \partial_t u(x,t)=\Delta u(x,t)+f(x), & x\in\mathbb{R}^d,\ t>0, \\ u(x,0)=0, & x\in\mathbb{R}^d, \end{cases} \tag{1.2}$$

where the source function f is assumed to be a linear combination of dirac-delta function,

$$f(x)=\sum_{\ell=1}^{N}\sigma_\ell\delta(x-a_\ell). \tag{1.3}$$

In [17], we show that (1.2) is equivalent to the one with time dependent source function $h(t)f(x)$ using the Volterra equation of the second kind. Moreover, we analyzed the above problem without noise in two-dimension with the dirac-delta function $\delta(\cdot)$ in (1.3) approximated by some radially symmetric functions in the Schwartz space $\mathscr{S}(\mathbb{R}^2)$ of rapidly decreasing functions. Furthermore, we assumed all strengths are unitary, e.g. $\sigma_\ell=1$ for all ℓ in (1.3). We showed that one measurement point is sufficient to identify the number of sources and three measurement points are sufficient to determine all unknown source positions.

In this paper, the same results are shown to hold under a general setting: besides of the number of sources N and their locations a_ℓ, the strength σ_ℓ ($\ell=1,\cdots,N$) is now considered as considered as unknown. Moreover, the analysis of this paper takes noise into account. The work in this paper can be applied to the formulation in [17] when dirac-delta function $\delta(\cdot)$ is replaced by $\rho\in\mathscr{S}(\mathbb{R}^d)$.

Let b be a measurement point in $\mathbb{R}^d$. Our problem here is to identify the number of sources N, the source strengths σ_ℓ and the locations a_ℓ from the noisy data $u^{\delta}(t)$ at the

measurement point $b\in\mathbb{R}^d$ under some *a priori* assumptions of the source locations a_ℓ. In particular, since N is finite, we assume that the locations a_ℓ are all contained in a given bounded region Ω. We also assume that the measurement point b is located outside of Ω. For simplicity, we begin with only one measurement point. Then, we will see that multiple measurement points are needed to achieve our goal.

For the stationary problem

$$\triangle u(x)=\sum_{\ell=1}^{N}\sigma_\ell\delta(x-a_\ell)$$

in a bounded domain $\Omega\subset\mathbb{C}$ with Neumann condition $\partial_\nu u|_{\partial\Omega}=g$, the source term N and $\{\sigma_\ell,a_\ell\}$ can be determined by the relations

$$c_n=\sum_{\ell=1}^{N}\sigma_\ell a_\ell^n, \tag{1.4}$$

for each non-negative integer n where c_n can be computed from measurement data [10]. The equation (1.4) can be solved by some poles identification algorithms [19]. For other stationary inverse source problems, the readers are referred to [1,5,10,14,16].

We aim to derive a similar relation to (1.4) asymptotically for our nonstationary problem. Instead of the source locations, our equation will contain the measurement-to-source distances $r_\ell:=\|b-a_\ell\|$ between a spatially fixed measurement point b and the unknown source locations. In Section 2, we formulate our inverse source problem as a minimization problem and show the existence and uniqueness of its solution. Next, we show that the solution to the minimization problems will converge to the form of (1.4) in Section 3. In Section 4, the developed theories are transformed into numerical algorithm. Moreover, we re-derive the collocation method proposed in [17] in Section 5 with some necessary assumptions that we omitted previously. Finally, in Section 6, some numerical results are shown.

2 Existence and uniqueness

Without loss of generality, we may assume that $r_\ell:=\|b-a_\ell\|\in[R_1,R_2]$ with

$$R_1=\inf_{x\in\Omega}\|b-x\|,\quad R_2=\sup_{x\in\Omega}\|b-x\|.$$

Define a partition $S_m:=\{s_k|\,k=1,\cdots,m\}$ of $[R_1,R_2]$ for $m\in\mathbb{N}$ such that $\text{gap}(S_m)\to 0$ as $m\to\infty$, where

$$\text{gap}(S_m)=\sup_{2\le k\le m}|s_k-s_{k-1}|.$$

A solution of (1.2) which decays at infinity is given by

$$u(x,t)=\sum_{\ell=1}^{N}\int_0^t\sigma_\ell\frac{1}{(4\pi\tau)^{\frac{d}{2}}}\exp\left(-\frac{\|x-a_\ell\|^2}{4\tau}\right)d\tau. \tag{2.1}$$

Motivated by (2.1), we define an operator $K_m:\mathbb{R}^m\to L^2(0,T)$ by

$$K_m(\lambda)(t)=\sum_{\{k|s_k\in S_m\}}\lambda_k\int_0^t\frac{1}{(4\pi\tau)^{\frac{d}{2}}}\exp\left(-\frac{s_k^2}{4\tau}\right)d\tau,\quad \lambda\in\mathbb{R}^m. \tag{2.2}$$

For any given positive number $\alpha>0$, we define a functional on $\mathbb{R}^m$ by

$$J(\lambda):=\|K_m(\lambda)(t)-u^\delta(t)\|^2_{L^2(0,T)}+\alpha\sum_{k=1}^m|\lambda_k|, \tag{2.3}$$

where $u^\delta(t)$ is the measurement data satisfying (1.1). We consider the minimization problem

$$\inf_{\lambda\in\mathbb{R}^m}J(\lambda). \tag{2.4}$$

We solve the minimization problem (2.4), which is a nonlinear optimization problem, with respect to λ. Let $\partial J(\lambda)$ denote the generalized Jacobian of J at $\lambda\in\mathbb{R}^m$, see [13,25].

Lemma 2.1. *For $\lambda\in\mathbb{R}^m$, the following statements are equivalent:*

(i) $0\in\partial J(\lambda)$.

(ii) *There exists a unique vector $\xi\in\mathbb{R}^m$ such that*

$$\begin{cases} K_m^*K_m(\lambda)-K_m^*u^\delta+\alpha\xi=0, \\ \xi_k=\dfrac{\lambda_k+\xi_k}{\max\{1,|\lambda_k+\xi_k|\}}, \qquad k=1,\cdots,m. \end{cases} \tag{2.5}$$

Proof. Denotes the set of points at which J is differentiable by D_J. Whenever the partial derivatives exist at point x, we write $\nabla J(x)$ for the usual $m\times m$ Jacobian of partial derivatives. From [9,13,25], we have a relation

$$\partial J(\lambda)=\mathrm{co}\left\{\lim_{x\to\lambda,x\in D_J}\nabla J(x)\right\}.$$

Here, $\mathrm{co}\{A\}$ is the convex hull of a set A. Then we have

$$\begin{aligned}\partial J(\lambda)=&\mathrm{co}\left\{K_m^*K_m(\lambda)-K_m^*u^\delta+\alpha\lim_{x\to\lambda,x\in D_J}\nabla|x|\right\}\\ =&K_m^*K_m(\lambda)-K_m^*u^\delta+\alpha\left\{(\xi_1,\xi_2,\cdots,\xi_m)^T\in\mathbb{R}^m:\lambda_k\xi_k=|\lambda_k|,|\xi_k|\le1\right\}.\end{aligned}$$

Similar approaches with slackness conditions have been successfully applied to other problems, see [11,12].

Using the fact that $\lambda_k\xi_k=|\lambda_k|$ and $|\xi_k|\leq 1$ are equivalent to

$$\xi_k=\frac{\lambda_k+\xi_k}{\max\{1,|\lambda_k+\xi_k|\}},$$

we can express $\partial J(\lambda)$ as

$$\partial J(\lambda)=K_m^*K_m(\lambda)-K_m^*u^\delta+\alpha\left\{(\xi_1,\xi_2,\cdots,\xi_m)^T\in\mathbb{R}^m:\xi_k=\frac{\lambda_k+\xi_k}{\max\{1,|\lambda_k+\xi_k|\}}\right\}.$$

The equivalence is now obvious. □

Since the functional J is strictly convex on $\mathbb{R}^m$, it contains the unique minimizer $\widehat{\lambda}=\lambda^{\alpha,m,\delta}\in\mathbb{R}^m$. In this case, the condition $0\in\partial J(\widehat{\lambda})$ becomes the necessary and sufficient condition that the $\widehat{\lambda}$ is the minimizer of J, see [9,13,25]. Hence, from Lemma 2.1, to seek the minimizer $\widehat{\lambda}$ of J becomes the task of finding the unique pair $(\widehat{\lambda},\widehat{\xi})\in\mathbb{R}^m\times\mathbb{R}^m$ that satisfies the nonlinear system (ii) in Lemma 2.1. This is the same as to find the unique zero point of a function $F:\mathbb{R}^{2m}\to\mathbb{R}^{2m}$ defined by

$$F(\lambda,\xi):=\begin{pmatrix} K_m^*K_m(\lambda)-K_m^*u^\delta+\alpha\xi \\ \xi_1-\dfrac{\lambda_1+\xi_1}{\max\{1,|\lambda_1+\xi_1|\}} \\ \vdots \\ \xi_m-\dfrac{\lambda_m+\xi_m}{\max\{1,|\lambda_m+\xi_m|\}} \end{pmatrix}.$$

A method to find the zero point of the function F is given in Section 4.

3 Convergence

In this section, by applying some proper choices of the parameter $\alpha=\alpha(\delta)$ in the functional defined by (2.3) and the denseness of the partition (proportional to m) with respect to the given noise level $\delta>0$, we show the unknown coefficients $\hat{\lambda}_k$ in the operator K_m in (2.2) will lead to the desired moment equations

$$\sum_{k=1}^{m(\delta)}\widehat{\lambda}_k s_k^n\to\sum_{\ell=1}^{N}\sigma_\ell\|b-a_\ell\|^n \quad \text{as} \quad \delta\to 0, \tag{3.1}$$

for any real number n (instead of the desired $n\in\mathbb{N}$). Below states our main theorem.

Theorem 3.1. *If $\alpha=\alpha(\delta)$ and $m=m(\delta)$ are chosen so that $h_{m(\delta)}^2+\delta^2=\mathcal{O}(\alpha(\delta))$ where*

$$h_{m(\delta)}:=\max_{2\leq k\leq m(\delta)}\|v_k-v_{k-1}\|_{L^2(0,T)},$$

then we have

$$\lim_{\delta\to 0}\sum_{k=1}^{m(\delta)}\widehat{\lambda}_k g(s_k)=\sum_{\ell=1}^{N}\sigma_\ell g(r_\ell),$$

for all $g\in C^\infty(\mathbb{R}^+)$. *In particular, if we take* $g(r)=r^n$, *we have*

$$\lim_{\delta\to 0}\sum_{k=1}^{m(\delta)}\widehat{\lambda}_k s_k^n=\sum_{\ell=1}^{N}\sigma_\ell r_\ell^n,\quad \textit{for all } n\in\mathbb{R}.$$

To prove (3.1) and Theorem 3.1, we need the following notations and lemma.

Let $\lim_{\delta\to 0}\alpha(\delta)=0$ and $\lim_{\delta\to 0}m(\delta)=\infty$. Let $\widehat{\lambda}=\lambda^{\alpha,m,\delta}\in\mathbb{R}^m$ be the unique minimizer of (2.4). We define the functions

$$v_k(t):=\int_0^t\frac{1}{(4\pi\tau)^{\frac{d}{2}}}\exp\left(-\frac{s_k^2}{4\tau}\right)d\tau,$$

for $s_k\in S_m$, so that the operator in (2.2) can be written as $K_m(\lambda)(t)=\sum_{k=1}^m\lambda_k v_k(t)$. Moreover, for each $m\in\mathbb{N}$, we define a candidate to $\inf_{\lambda\in\mathbb{R}^m}J(\lambda)$ by using the coefficients $\eta^m\in\mathbb{R}^m$ defined as

$$\eta^m=(\eta_1^m,\cdots,\eta_m^m)=\sum_{\ell=1}^{N}\sigma_\ell e_{k(\ell)}\in\mathbb{R}^m,$$

where $e_1,\cdots,e_m$ are the standard vectors in $\mathbb{R}^m$ and $k(\ell)=\arg\min_{1\le k\le m}|s_k-r_\ell|$, $\ell=1,\cdots,N$. The corresponding approximation to $u(t,b)$ is of the form

$$u_m(t):=\sum_{k=1}^{m}\eta_k^m v_k(t).$$

Let us define a subspace of $C^\infty(\mathbb{R})$ by

$$\mathcal{K}=\overline{\text{Span}\left\{e^{-\frac{x^2}{4t}}:t\in(0,T)\right\}},$$

that is the closure under $L^\infty(\mathbb{R})$ of the space spanned by all Gaussian functions centering at the origin with different shape parameters t. The time interval for observation $[0,T]$ does not play any role in the theorems below. In fact, we can choose any observation time interval $[t_{\min},t_{\max}]$. Hereafter, we denote $(0,\infty)$ by $\mathbb{R}^+$.

Lemma 3.1. *Let* $\varphi\in C_0(\mathbb{R}^+)$, *then* $\varphi\in\mathcal{K}$. *In particular,* $\varphi(x^2)\in\mathcal{K}$.

Proof. We choose $\{\gamma_j\}_{j\in N}$ such that $\gamma_1<\gamma_2<\cdots$ with $0<\frac{1}{4T}<\gamma_1$ and $\sum_{j=1}^\infty\frac{1}{\gamma_j}=\infty$.

By the Müntz theorem [18] (also see [8] for recent references), we know that $\overline{\text{Span}\{y^{\gamma_1},y^{\gamma_2},\cdots\}}$ is dense in $C[0,1]$. We are interested in $\phi\in C[0,1]$ and $\varphi\in C_0(\mathbb{R}^+)$ such

that $\phi(0)=0$ and $\phi(y)=\varphi\left(\log\frac{1}{y}\right)$ for $y\in[0,1]$. For any $\varepsilon>0$, there exists $M\in\mathbb{N}$ and a finite sequence $\alpha_j^\varepsilon\in\mathbb{R}$, depending on φ and ε, such that

$$\sup_{0<y\le 1}\left|\phi(y)-\sum_{j=1}^{M}\alpha_j^\varepsilon y^{\gamma_j}\right|=\sup_{0<y\le 1}\left|\varphi\left(\log\frac{1}{y}\right)-\sum_{j=1}^{M}\alpha_j^\varepsilon y^{\gamma_j}\right|<\epsilon.$$

For any $y\in(0,1]$, by taking $x=\log\frac{1}{y}\in\mathbb{R}^+$ we have

$$\sup_{x\in\mathbb{R}^+}\left|\varphi(x)-\sum_{j=1}^{M}\alpha_j^\varepsilon e^{-\gamma_j x}\right|<\varepsilon.$$

For all $\varphi\in C_0(\mathbb{R}^+)$, we have $\varphi(x)\in\mathcal{K}$. Putting $x^2=\log\frac{1}{y}$ results in the particular case. Note that the initial measurement time $t=0$ is not crucial in the proof above, i.e., one can take $t\in(t_1,t_2)$ as well. □

Proof of Theorem 3.1. It is sufficient to prove the first equality for arbitrary sequence $\{\delta_p\}$ such that $\lim_{p\to\infty}\delta_p=0$. We set $\widehat{\lambda}^p:=\lambda^{\alpha(\delta_p),m(\delta_p),\delta_p}$.

First, observe that

$$\begin{aligned}J(\eta^{m(\delta_p)})&=\|K_{m(\delta_p)}(\eta^{m(\delta_p)})-u^{\delta_p}\|^2_{L^2(0,T)}+\alpha(\delta_p)\sum_{k=1}^{m(\delta_p)}|\eta_k^{m(\delta_p)}|\\&\le\|K_{m(\delta_p)}(\eta^{m(\delta_p)})-u\|^2_{L^2(0,T)}+\|u-u^{\delta_p}\|^2_{L^2(0,T)}+\alpha(\delta_p)\sum_{k=1}^{m(\delta_p)}|\eta_k^{m(\delta_p)}|\\&\le h^2_{m(\delta_p)}\sum_{k=1}^{m(\delta_p)}|\sigma_k|+\delta_p^2+\alpha(\delta_p)\sum_{k=1}^{m(\delta_p)}|\eta_k^{m(\delta_p)}|.\end{aligned}$$

Since $J(\hat{\lambda}^p)\le J(\eta^{m(\delta_p)})$, we have

$$\begin{aligned}&\|K_{m(\delta_p)}(\widehat{\lambda}^p)(t)-u(b,t)\|_{L^2(0,T)}\\&\le\sqrt{J(\hat{\lambda}^p)}+\delta_p\le\sqrt{J(\eta^{m(\delta_p)})}+\delta_p\\&\le\left(h^2_{m(\delta_p)}\sum_{k=1}^{m(\delta_p)}|\sigma_k|+\delta_p^2+\alpha(\delta_p)\sum_{k=1}^{m(\delta_p)}|\eta_k^{m(\delta_p)}|\right)^{\frac{1}{2}}+\delta_p.\end{aligned}\tag{3.2}$$

Moreover, we have

$$\sum_{k=1}^{m(\delta_p)}|(\widehat{\lambda}^p)_k|\le\frac{J(\eta^{m(\delta_p)})}{\alpha(\delta)}\le\frac{h^2_{m(\delta_p)}(\sum_{k=1}^{m(\delta_p)}|\sigma_k|)^2+\delta_p^2}{\alpha(\delta)}+\sum_{k=1}^{m(\delta_p)}|\sigma_k|\le C,\tag{3.3}$$

where C indicates a generic constant hereafter.

For $p\in\mathbb{N}$, we define $\mu_p\in H^{-1}(\mathbb{R})$ by

$$\mu_p=\sum_{k=1}^{m(\delta_p)}(\widehat{\lambda}^p)_k\delta(x-s_k).$$

By (3.3), the norm of $\|\mu_p\|_{H^{-1}(\mathbb{R})}$ is bounded by

$$\begin{aligned}\|\mu_p\|^2_{H^{-1}(\mathbb{R})}&=\int_R|\mathcal{F}[\mu_p](\xi)|^2(1+\xi^2)^{-1}d\xi\\&=\int_R\Big|\sum_{k=1}^{m(\delta_p)}(\widehat{\lambda}^p)_ke^{-i\xi s_k}\Big|^2(1+\xi^2)^{-1}d\xi\\&\le C\left(\sum_{k=1}^{m(\delta_p)}|(\widehat{\lambda}^p)_k|\right)^2\le C.\end{aligned}$$

Hence, we can take a subsequence $\{\mu_{p_1}\}$ of $\{\mu_p\}$ that converges $*$-weakly in $H^{-1}(\mathbb{R})$ to an element μ^* whose support is contained in Ω. Therefore we have

$$\lim_{p_1\to\infty}\langle\mu_{p_1},\varphi\rangle=\langle\mu^*,\varphi\rangle\quad\text{for all }\varphi\in H^1(\mathbb{R}).\tag{3.4}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the duality bracket on $H^{-1}(\mathbb{R})\times H^1(\mathbb{R})$ defined by

$$\langle f,\varphi\rangle:=\int_{\mathbb{R}}\mathcal{F}(f)(\xi)\mathcal{F}(\varphi)(\xi)d\xi\quad\text{for }\ f\in H^{-1}(\mathbb{R}),\ \varphi\in H^1(\mathbb{R}),$$

where $\mathcal{F}(f)$ and $\mathcal{F}(\varphi)$ are the Fourier transforms of f and φ, respectively.

If we take $\varphi_t(x)=e^{-x^2/4t}(4\pi t)^{-d/2}$ for any $t>0$ in (3.4), we have

$$\lim_{p_1\to\infty}\langle\mu_{p_1},\varphi_t\rangle=\langle\mu^*,\varphi_t\rangle.\tag{3.5}$$

On the other hand, for all $t>0$, $|\langle\mu_{p_1},\varphi_t\rangle|$ is bounded by

$$\begin{aligned}|\langle\mu_{p_1},\varphi_t\rangle|&=\left|\sum_{k=1}^{m(\delta_{p_1})}(\widehat{\lambda}^{p_1})_k\frac{1}{(4\pi t)^{\frac{d}{2}}}\exp\left(-\frac{s_k^2}{4t}\right)\right|\\&\le\frac{e^{-s_1^2/4t}}{(4\pi t)^{\frac{d}{2}}}\sum_{k=1}^{m(\delta_{p_1})}|\widehat{\lambda}^{p_1}_k|\le C,\end{aligned}\tag{3.6}$$

with a constant $C>0$. Hence, by (3.5) and (3.6), we have

$$\lim_{p_1\to\infty}\int_0^t\langle\mu_{p_1},\varphi_s\rangle ds=\int_0^t\langle\mu^*,\varphi_s\rangle ds,\tag{3.7}$$

for all $t>0$.

On the other hand, from (3.2), by taking the subsequence $\{\mu_{p_2}\}_{p_2}$ of $\{\mu_{p_1}\}_{p_1}$, we have

$$\begin{aligned}\lim_{p_2\to\infty}\int_0^t \langle \mu_{p_2},\varphi_s\rangle\, ds &= \lim_{p_2\to\infty} K_{m(\delta_{p_2})}(\widehat{\lambda}^{p_2})(t)\\ &= u(b,t)=\int_0^t \langle f,\varphi_s\rangle\, ds,\end{aligned} \tag{3.8}$$

for almost every $t\in[0,T]$. From (3.7) and (3.8),we have

$$\int_0^t \langle \mu^*,\varphi_s\rangle\, ds=\int_0^t \langle f,\varphi_s\rangle\, ds, \tag{3.9}$$

for almost every $t\in[0,T]$. Since both functions are smooth with respect to $t>0$, by applying unique continuation theorem, we conclude that (3.9) holds for all $t>0$. Now, we can differentiate both side of (3.9) and we have

$$\langle \mu^*,\psi_t\rangle=\langle f,\psi_t\rangle, \quad \text{for all } t\in[0,T], \tag{3.10}$$

where $\psi_t(x)=\exp(-\frac{x^2}{4t})$.

Next, we show that μ^* can be extended uniquely to the functional $\widetilde{\mu}^*\in C(\mathbb{R})^*$. For any compact set $K\subset\mathbb{R}$ and $\varphi\in C_K^\infty(\mathbb{R})$, by (3.4), we have

$$\begin{aligned}\langle \mu^*,\varphi\rangle &= \lim_{p_2\to\infty}\langle \mu_{p_2},\varphi\rangle\\ &= \lim_{p_2\to\infty}\left|\sum_{k=1}^{m(\delta_{p_2})}(\widehat{\lambda}^{p_2})_k\varphi(y_k)\right|\\ &\le \lim_{p_2\to\infty}\sum_{k=1}^{m(\delta_{p_2})}\left|(\widehat{\lambda}^{p_2})_k\right|\|\varphi\|_\infty\le C\|\varphi\|_\infty.\end{aligned}$$

Hence, we can conclude that μ^* can be extended uniquely to the functional $\widetilde{\mu}^*\in C_0(\mathbb{R})^*$.

Furthermore, $\widetilde{\mu}^*$ has compact support because μ^* has compact support. We come to the conclusion that $\widetilde{\mu}^*\in C(\mathbb{R})^*$. Thus, from (3.10), we have

$$\widetilde{\mu}^*(\psi_t)=\langle \mu^*,\psi_t\rangle=\langle f,\psi_t\rangle.$$

On the other hand, we know that $f\in C(\mathbb{R})^*$ and $\langle f,\psi_t\rangle=f(\psi_t)$. Hence, we have

$$\widetilde{\mu}^*(\psi_t)=f(\psi_t). \tag{3.11}$$

Let $\theta(x)\in C_0^\infty(\mathbb{R}^+)$ be a continuous function such that $\theta=1$ on $[R_1,R_2]$ and $\theta=0$ on $[0,R_1/2]$. Let $g(r)\in C^\infty(\mathbb{R}^+)$. From Lemma 3.1, the function $\theta(|x|)g(|x|)$ belongs to $\mathcal{K}$;

hence, it can be approximated by a finite sum of the form $\sum a_j\psi_{t_j}$, $t_j\in[0,T]$ in $L^\infty(\mathbb{R})$. Therefore, from (3.11) and the fact that $\widetilde{\mu}^*,f\in C(\mathbb{R})^*$, we have

$$\widetilde{\mu}^*(\theta(|x|)g(|x|))=f(\theta(|x|)g(|x|)).$$

Noting that $\theta(|x|)g(|x|)\in C_0^\infty(\mathbb{R})$, we have

$$\langle\mu^*,\theta(|x|)g(|x|)\rangle=\langle f,\theta(|x|)g(|x|)\rangle.$$

By taking $\varphi(x)=\theta(|x|)g(|x|)$ in (3.4), we obtain

$$\begin{aligned}\lim_{p_2\to\infty}\sum_{k=1}^{m(\delta_{p_2})}(\widehat{\lambda}^{p_2})_k g(s_k)&=\lim_{p_2\to\infty}\langle\mu_{p_2},\varphi\rangle=\langle\mu^*,\varphi\rangle\\&=\langle f,\varphi\rangle=\sum_{\ell=1}^{N}\sigma_\ell g(r_\ell).\end{aligned}$$

Thus, an arbitrary subsequence $\langle\mu_{p_1},\varphi\rangle$ of $\langle\mu_p,\varphi\rangle$ contains a subsequence $\langle\mu_{p_2},\varphi\rangle$ that is convergent to the unique limit $\sum_{\ell=1}^N\sigma_\ell g(r_\ell)$. Consequently, the original sequence $\langle\mu_p,\varphi\rangle$ itself converges to $\sum_{\ell=1}^N\sigma_\ell g(r_\ell)$. The proof is therefore completed. □

4 Numerical procedures

The numerical procedure contains three parts: (I) to solve the minimization problem (2.5) for the unknown coefficients, (II) to determine the total number of point sources, strength, and, measurement-to-source distances, and (III) to determine the source locations.

(I: for each measurement point) To derive a numerical procedure from the nonlinear system (2.5) and to obtain the minimizer $\hat{\lambda}$, we consider a nonsmooth version of Newton's method

$$\begin{pmatrix}\lambda^{n+1}\\ \xi^{n+1}\end{pmatrix}=\begin{pmatrix}\lambda^n\\ \xi^n\end{pmatrix}-{V_n}^{-1}F(\lambda^n,\xi^n),\tag{4.1}$$

where $V_n\in\partial F(\lambda^n,\xi^n)$ is given by

$$V_n=\begin{bmatrix}K_m^*K_m & \alpha I_{m\times m}\\ -\Theta & I_{m\times m}-\Theta\end{bmatrix},$$

and

$$\Theta=\mathrm{diag}(\theta_1,\cdots,\theta_m),\quad \theta_k:=\begin{cases}0 & \text{if } |\lambda_k+\xi_k|>1,\\ 0.5 & \text{if } |\lambda_k+\xi_k|=1,\\ 1 & \text{if } |\lambda_k+\xi_k|<1.\end{cases}$$

Since F is Lipschiz and semismooth [23] in $\mathbb{R}^m\times\mathbb{R}^m$ and any $V\in\partial F(\lambda,\xi),(\lambda,\xi)\in\mathbb{R}^m\times\mathbb{R}^m$ is nonsingular, the iteration (4.1) converges globally to the unique solution of $F(\lambda,\xi)=0$ [23, Theorem 3.3]. Therefore, we have proven the following:

Corollary 4.1. *Let* $(\hat{\lambda},\hat{\xi})$ *be the unique solution of* $F(\lambda,\xi)=0$. *Then* $\hat{\lambda}$ *is the unique minimizer of* J *defined by* (2.3).

In the next section, we show how this nonlinear procedure can be reduced to a linear collocation system, as we proposed in [17], so that one could avoid numerical integration and nonlinearity.

(II: for each measurement point) Once the system of nonlinear equations in **(I)** is solved, the constants c_n can be approximated by

$$\left(\sum_{k=1}^{m}\hat{\lambda}_j s_j^n\right)\to c_n:=\sum_{\ell=1}^{N}\sigma_\ell r_\ell^n,\quad n\in\mathbb{N},\text{ under the assumption of Theorem 3.1}\tag{4.2}$$

to which the pole identification algorithm [6] can be applied.

The pole identification algorithm has to be run for each measurement point. Moreover, the problem here is nonlinear and ill-posed in nature. The method is very sensitive to the error in measurements and may result in complex solutions when the noise level is large. In such cases, we conclude that our proposed algorithm fails to produce any estimations. As one may imagine, the total number of unknown sources cannot be large. In the next section, some numerical identifications are demonstrated.

(III: using all information) The final step is to locate the actual source locations a_j from the measurement-to-source distances. The number of required measurement points for unique determination of source locations depends on the dimension d. Each identification of measurement-to-source distance provides a sphere $S^{d-1}\in\mathbb{R}^d$. In the case of $\mathbb{R}^2$, at least three measurement points are needed [17]. In general, one can easily see that at least $d+1$ measurement points are required for solving problems in $\mathbb{R}^d$.

In $\mathbb{R}^2$, once we have determined the radius $r_{j,\ell}$ $(j=1,2,3,\ell=1,\cdots,N)$ between all three measurement points and the unknown source locations, we employ an algorithm by Vakulenko [27] to find the intersections of circles. Due to the presence of numerical errors, we do not expect the three circles intersect at exact one point.

If all radius are overestimated, there will be a nonempty intersection. If one or more radius are underestimated, there will be no intersection at all. In such cases, we gradually increase all radius until we obtain a nonempty intersection. The *estimated error* is the sum of

1. the radius of the smallest circle that contains the intersection, and
2. the change to $r_{j\ell}$ in order to get nonempty intersection.

In total, there are N^3 combinations. Under the condition that each $r_{j\ell}$ can only be used once, the N estimations with minimum estimated errors are chosen to be the estimated source locations to the N unknown source locations. The center of such smallest circle that contains the intersection is the corresponding estimation of source location.

5 Collocation method

To derive the direct collocation method proposed in [17] from the nonlinear system (2.5), we need to apply two extra approximations. First, the nonlinear system (2.5) has to be linearized. Next, the Gram matrix in (2.5) is approximated by the normal equation of some overdetermined collocation systems.

For a large number of partition points m and data with small noise level $\delta \ll 1$, we may assume that the parameter $\alpha \ll 1$. Hence, the nonlinear system (2.5) corresponding to *each* measurement point b can be linearized to obtain

$$K_m^* K_m(\widehat{\lambda}) = K_m^* u^\delta. \tag{5.1}$$

Because of these assumptions, the linear algorithm proposed in [17] works very well with noise-free data. Let

$$Y_k(\cdot,t) = \int_0^t \int_{\mathbb{R}^2} \frac{1}{4\pi\tau} \exp\left(-\frac{\|\cdot - z\|^2}{4\tau}\right) \rho(z-y_k)\, dz d\tau,$$

and the kj-th component of the $m \times m$ square matrix M^*M is given by

$$[K_m^* K_m]_{kj} = \int_0^T Y_k(b,t) Y_j(b,t)\, dt, \tag{5.2}$$

whereas the k-th component of M^*u is given by

$$[K_m^* u]_k = \int_0^T Y_k(b,t) u(b,t)\, dt. \tag{5.3}$$

Solving (5.1) for the minimizer $\widehat{\lambda}$ of (2.4), we obtain in a set of moment equations as in (3.1) for a single measurement point.

Further simplification is possible. Suppose we partition the time interval $[0,T]$ equally by Q points

$$t_{min} = t_0 < \cdots < t_q = t_{max},$$

where $\Delta t := t_j - t_{j-1}$ for $j = 1, \cdots, q$. If we approximate (5.2) and (5.3) by some numerical integration schemes, then the linear system (5.1) can be approximated by

$$A^T W A \widehat{\lambda} = A^T W u, \tag{5.4}$$

where A is a $q \times m$ matrix with entries

$$[A]_{jk} = Y_k(b,t_j) \quad j = 1, \cdots, q, \;\; k = 1, \cdots, m,$$

and the components of u are given by

$$u_j = u(b,t_j) \quad j = 1, \cdots, q.$$

Here, $W=\text{diag}(w_j)$ is to be determined by the choice of numerical integration scheme. It is well-known that the solution to (5.4) is equivalent to the least squares solution to

$$W^{1/2}A\widehat{\lambda}=W^{1/2}u. \tag{5.5}$$

Although different numerical schemes result in different non-singular square diagonal weight matrices W, the system (5.5) is equivalent to

$$A\widehat{\lambda}=u \tag{5.6}$$

up to some constant factors. The overdetermined system (5.6) is the same linear system derived in [17] for obtaining unknown coefficients under the noise-free setting.

6 Numerical demonstrations

Here, we give an example in two-dimension and three-dimension respectively. Our methodology reduces the problem in $\mathbb{R}^d$ to one-dimension with respect to the measurement-to-source distances. The only complication arise is in procedure (III) in which one has to investigate the intersections of spheres $S^{d-1}\subset\mathbb{R}^d$.

6.1 Example in two-dimension

We take our source points to be $\{a_\ell\}_{\ell=1}^3=\{[.4,0],[-.26,.25],[.13,-.65]\}\subset\mathbb{R}^2$ with unit strength, i.e, $\sigma_\ell=1$ for $\ell=1,2,3$. For the sake of Procedure (I), we distribute a 250-point partition into Ω that is the unit circle. Three measurement points are chosen. The maximum errors in measurement-to-source distances and maximum errors in source strength are reported in Table 1.

Table 1: Maximum errors of estimated measurement-to-source distances and source strength.

	b_1	b_2	b_3
Distance	0.0001	0.0015	0.0045
Strength	0.0004	0.0035	1.0004

In Fig. 1, the resulting circles are shown. The exact source locations are identified as the intersection of three circles; they are indicated by ◯. For the first two measurement points $b_1=[1,0]$ and $b_2=[-\sqrt{2},\sqrt{2}]$, three distinct circles are associated with each measurement points.

For the measurement point at $b_3=[-1,0]$, there are only two associated circles. This is due to the fact that two source points are showing very similar distances to the measurement point b_3. In Table 1, the corresponding error in source strength is 1.0004 indicating the same fact. In fact, our algorithm identifies that the strength associated with that particular measurement-to-source distance is 2.

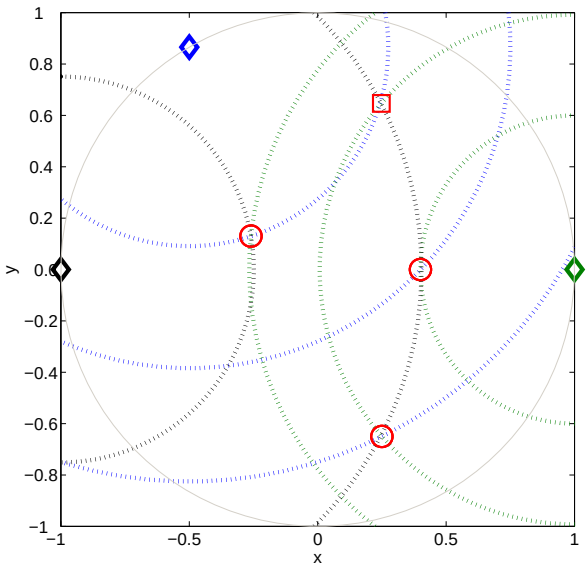


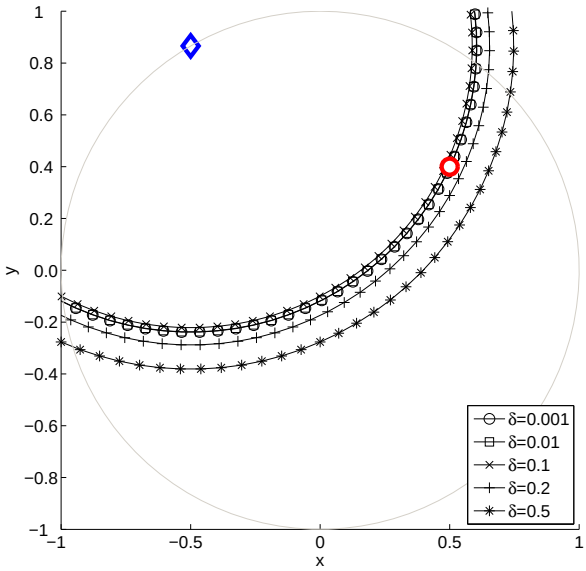
Figure 1: Demonstration of source location identification– ◯ indicates the correct identification, whereas □ indicates a false positive.

By the construction of this example, we see a false positive source location in Fig. 1 indicated by □. This is a coincidence because of the geometry of the source points and measurement points. One could, of course, add another measurement point to eliminate this false identification of source location. Other methods, like our previously proposed iterative scheme [16], can be applied to improve the accuracy of source location and strength estimation (if one can provide some initial guesses on the source locations). Furthermore, such method can be used to identify false positive location–the estimated strength will be zero or insignificant compared with the others.

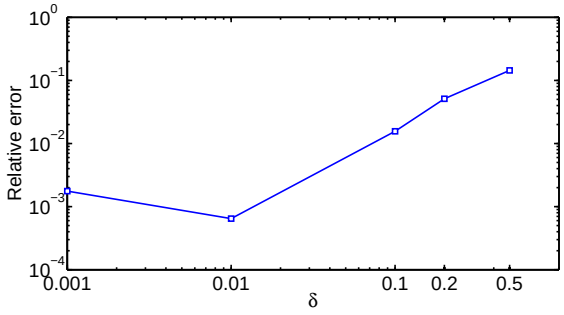
In Fig. 2, we show the estimations of the source-to-measurement distance under different levels of additive random noise. Due to the ill-posed nature and nonlinearity of the problem, the algorithm is rather sensitive to noise. If the error in the moment equation becomes an obstacle in Procedure (I), the pole identification algorithm may fail to identify the unknowns. In particular, for large noise level δ, the pole identification algorithm returns complex numbers as outputs. Since the distance r_ℓ and strength σ_ℓ must be real numbers, the method fails in due course. Improving the numerical stability of these procedures is left for future study.

6.2 Example in three-dimension

Both our theory and numerical results suggest that dimension does not have a strong effect on the moment equations in Theorem 3.1. The measurement points are place at $\pm e_j$ for $j=1,2,3$ where e_j is the standard vector. The one- and two-source cases, respectively,



(a) Absolute error



(b) Relative error

Figure 2: Demonstration of numerical accuracy to the approximation of measurement-to-source distances under the presence of noise.

with source locations

$$\{a_\ell\}_{\ell=1} = \{[0.5,\, 0.4,\, 0.1]\},$$

and

$$\{a_\ell\}_{\ell=1}^{2} = \{[0.4,\, 0,\, 0],\, [-0.26,\, 0.13,\, 0.13]\},$$

are tested. The results are shown in Table 2. It is clear that the number of unknown source points can affect the accuracy. Comparatively, the geometry of source points and measurement points have some minor influence on the source locations identifying problem.

Table 2: 3D Example: Maximum errors of estimated measurement-to-source distances and source strength.

$b=$	$+\hat{i}$	$-\hat{i}$	$+\hat{j}$	$-\hat{j}$	$+\hat{k}$	$-\hat{k}$
	One Source					
Distance	2.4e-8	1.6e-9	3.2e-12	2.3e-9	2.3e-9	1.9e-10
Strength	1.0e-10	1.6e-11	8.4e-15	2.3e-11	1.7e-11	1.5e-12
	Two Sources					
Distance	1.8e-6	1.8e-4	5.0e-6	1.3e-5	5.2e-4	5.3e-5
Strength	3.4e-6	3.9e-4	5.7e-5	2.7e-4	6.0e-3	1.1e-3

7 Conclusion

In this paper, we consider the identification of unknown point source in heat equations. With noisy data, we prove that the distances between the measurement points and all the unknown source locations satisfy a sequence of moment equations as the noise level tends to zero. Moreover, we build the connection of the work in this paper to our previously proposed linear scheme based on the general settings in this work which explains why the previous method does not work under noisy date. The theoretical results are transformed into a completed numerical scheme, which consists of three separated algorithms. Numerical demonstrations suggest that the proposed method in this paper is capable of identifying the unknown source function based on the information from as few as three measurement points. Due to the nonlinearity and complexity of the method, the accuracy may not be acceptable in certain cases. However, our method does successfully provide the initial guess and extra information to some efficient linear solvers for accuracy refinement. We aim to use these results as numerical verifications to the theory on the minimum measurement points needed for the point heat sources identification problems.

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