Commun. Math. Res.
doi: 10.4208/cmr.2022-0015

Vol. **38**, No. 3, pp. 333-350
August 2022

Inverse Lax-Wendroff Boundary Treatment: a Survey

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Received 11 April 2022; Accepted 24 April 2022

Abstract. The inverse Lax-Wendroff (ILW) procedure is a numerical boundary treatment technique, which allows finite difference schemes and other schemes to achieve stability and high order accuracy when using cartesian meshes to solve boundary value problems defined on complex computational domain. In this short survey we summarize the main ingredients of the ILW procedure, discuss its applicability and stability properties, and provide possible directions of its future development.

AMS subject classifications: 65M06, 65M12

Key words: Inverse Lax-Wendroff procedure, boundary treatment, high order accuracy, stability, complex geometry.

1 Introduction

Finite difference methods are widely used to solve partial differential equations (PDEs). For example, to solve a hyperbolic equation

$$u_t + u_x = 0, \quad 0 \le x \le 1 \tag{1.1}$$

with the initial condition $u(x,0) = u^0(x)$ and the boundary condition

$$u(0,t) = g(t), \tag{1.2}$$

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a finite difference scheme approximates (1.1) on a spatial grid

$$0=x_0<x_1<\cdots<x_N=1 \tag{1.3}$$

and, for simplicity, we assume a uniform grid with the mesh size $\Delta x=x_{j+1}-x_j$. For example, we could use the following finite difference scheme:

$$u_j^{n+1}=au_{j-2}^n+bu_{j-1}^n+cu_j^n+du_{j+1}^n \tag{1.4}$$

with suitably chosen constants a,b,c and d (which depend on $\lambda=\frac{\Delta t}{\Delta x}$), approximating the PDE (1.1) to third order accuracy. Here and below, u_j^n is the numerical approximation at the grid point $x=x_j$ and at time $t=t^n$, and we assume, for simplicity, a constant time step size $\Delta t=t^{n+1}-t^n$.

The first difficulty associated with the boundary conditions of schemes such as (1.4) is the wide stencil, e.g., for the scheme (1.4) the stencil consists of four points $\{x_{j-2},x_{j-1},x_j,x_{j+1}\}$. Notice that the scheme (1.4) cannot be used to compute u_1^{n+1} and u_N^{n+1}, if we only have the information of u_j^n for $0\le j\le N$. We would need to either define the "ghost point" values u_{-1}^n and u_{N+1}^n and then use the scheme (1.4), or we could use a different scheme than (1.4) for the computation of u_1^{n+1} and u_N^{n+1}. In either case, we must analyze the stability and accuracy of the resulting approximations.

For a higher order finite difference scheme, the stencil is wider, and hence the number of such "abnormal" points near the boundary will be larger, causing more complications in either of the two approaches above.

The second difficulty associated with the boundary conditions of schemes such as (1.4) is the possibility that the boundary of the computational domain may not coincide with the grid points. For example, instead of the grid points defined in (1.3), we could imagine that the first grid point x_0 is not at $x=0$, for example $x_0=0.4\Delta x$. Such a configuration will make it difficult to apply the given boundary condition (1.2) which is defined at $x=0$. Of course, one might argue that such choice of grid points seems artificial. But even in one space dimension, such scenarios cannot be avoided if we are computing a moving (in time) domain with a fixed spatial grid. In two or higher dimensions, such scenarios will always happen if we are attempting to solve a PDE defined in a domain with complex geometry using cartesian meshes.

One of the major problems associated with the second difficulty is the possible appearance of small cells near the boundary, in the sense that the distance from the first grid point x_0 and the physical boundary $x=0$ is very small in comparison with the mesh size Δx. For many explicit schemes, the ratio of time step size over

the spatial mesh size must be bounded for stability, thus we might need a very small time step Δt to ensure stability due to the fact that $x_0 - 0 \ll \Delta x$. This is referred to as the "cut-cell problem" in the literature, which would need special techniques such as the h-box method in [4, 15] to fix.

If a boundary problem has certain symmetry or anti-symmetry, that is, if we know the solution is an even or an odd function with respect to the boundary, then we can use reflecting or symmetric boundary conditions to define the numerical solution at ghost points. For example, if we know u is an even function with respect to $x = 0$, we can define the grid points symmetrically

$$\cdots, \quad x_{-2} = -\frac{3}{2}\Delta x, \quad x_{-1} = -\frac{1}{2}\Delta x, \quad x_0 = \frac{1}{2}\Delta x, \quad x_1 = \frac{3}{2}\Delta x, \quad \cdots$$

and then define the ghost point values as

$$\cdots, \quad u_{-2} = u_1, \quad u_{-1} = u_0, \quad \cdots.$$

Similarly, if we know u is an odd function with respect to $x = 0$, we can define the ghost point values as

$$\cdots, \quad u_{-2} = -u_1, \quad u_{-1} = -u_0, \quad \cdots.$$

However, in two or three spatial dimensions, such reflecting or symmetry boundary conditions can only be easily implemented on straight line (plane) boundaries. For curved boundaries, it is very difficult to implement such boundary conditions without using body-fitted meshes.

One commonly used numerical boundary condition treatment is through suitable extrapolation, namely build an interpolating polynomial using several grid point values inside the computational domain, and then read its values at the ghost points to serve as the ghost point values of the numerical solution. This approach has been analyzed for stability in e.g. [23–25, 40]. In general, suitable extrapolation leads to stable approximations at the outflow boundary for hyperbolic equations (e.g. near $x = 1$ for the PDE (1.1)), but care must be taken if it is used near the inflow boundary (e.g. near $x = 0$ for the PDE (1.1)) to ensure stability.

In this survey we will briefly describe and summarize a class of boundary condition treatments, termed the inverse Lax-Wendroff (ILW) procedure, initially developed in [18, 41], which can be applied to many boundary value problems resulting in stable and high order accurate approximations. For a handbook entry of the ILW procedure (development until 2017) we refer to [39].

As related earlier work, we would like to mention [12, 13], which initialized the idea of converting spatial derivative near the boundary to temporal derivatives for solving one-dimensional linear hyperbolic initial-boundary value problems. We would also like to mention the embedded boundary method for solving the wave equation with Dirichlet or Neumann boundary conditions by using finite difference methods on cartesian grids [1, 23–25, 35]. In [40] the authors applied this method to hyperbolic conservation laws and obtained a second order accurate scheme. Baeza *et al.* [2, 3] extended the approach from second order to fifth order using Lagrange extrapolation with a filter for the detection of discontinuities.

We should mention that a commonly used method of using finite difference schemes to solve problems in complex geometry is to generate a boundary fitted mesh which allows the given boundary conditions to be imposed directly. As a result, the governing equations are often transformed into a new differential form in a curvilinear coordinate system, e.g. [21]. If the domain is relatively simple, a smooth mapping can be found to transform the whole domain. However, in more complex cases, composite overlapping meshes are needed to fit the physical boundaries, while these meshes are connected via interpolation, see e.g. [9,16,17,37]. The drawback of this approach is the difficulty in generating the body-conforming grids, especially if the boundary changes with time. We will not discuss this approach in this survey paper.

2 The basic idea

The basic idea of the ILW procedure can be best described by the following simple example, namely solving (1.1) with the boundary condition (1.2). Our task is to define suitably a value u_{-1}^n at the ghost point x_{-1}, after which the scheme (1.4) can be used to compute u_1^{n+1}.

Before describing the basic ideas of the ILW procedure, let us first look at the tradition Lax-Wendroff (LW) procedure [26]. To solve the PDE (1.1), we start from a Taylor expansion in time

$$u_j^{n+1} = u_j + (u_t)_j \Delta t + \frac{1}{2}(u_{tt})_j \Delta t^2 + \cdots,$$

where we have omitted the time level n superscripts on the right-hand side. We then replace the time derivatives by the spatial derivatives through repeatedly using the PDE (1.1)

$$(u_t)_j = -(u_x)_j,$$

$$(u_{tt})_j = -\big((u_x)_t\big)_j = -\big((u_t)_x\big)_j = (u_{xx})_j$$

etc. We then obtain

$$u_j^{n+1} = u_j - (u_x)_j \Delta t + \frac{1}{2}(u_{xx})_j \Delta t^2 + \cdots \tag{2.1}$$

and each spatial derivative in (2.1) can be approximated by a suitable finite difference, e.g. on second order level we can use

$$(u_x)_j \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x}, \quad (u_{xx})_j \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2},$$

to obtain the classical second order (in space and time) LW scheme

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\big(u_{j+1}^n - u_{j-1}^n\big) + \frac{\lambda^2}{2}\big(u_{j+1}^n - u_j^n + u_{j-1}^n\big),$$

where $\lambda = \frac{\Delta t}{\Delta x}$.

We now look at the basic idea of the ILW procedure, by switching the roles of x and t in the traditional LW procedure (hence the word "inverse"). Suppose we are solving (1.1) with the boundary condition (1.2), and suppose the boundary $x = 0$ is of distance $a\Delta x$ from the nearby grid point x_j (with a constant a, which may not be an integer, and can be either positive or negative depending upon whether x_j is inside or outside the computational domain). The ILW procedure to determine u_j is as follows: We perform a Taylor expansion in space

$$u_j = u(0,t) + u_x(0,t) a\Delta x + \frac{1}{2} u_{xx}(0,t)(a\Delta x)^2 + \frac{1}{6} u_{xxx}(0,t)(a\Delta x)^3 + \cdots. \tag{2.2}$$

We then replace the spatial derivatives by the time derivatives through repeatedly using the PDE (1.1)

$$\begin{aligned}
&u_x = -u_t, && u_x(0,t) = -u_t(0,t) = -g'(t),\\
&u_{xx} = (-u_t)_x = -(u_x)_t = u_{tt}, && u_{xx}(0,t) = u_{tt}(0,t) = g''(t),\\
&u_{xxx} = (u_{tt})_x = (u_x)_{tt} = -u_{ttt}, && u_{xxx}(0,t) = -u_{ttt}(0,t) = -g'''(t),
\end{aligned}$$

etc. Notice that, since the boundary condition $g(t)$ is given, we can obtain its derivatives $g'(t), g''(t), g'''(t)$ etc. either analytically (if $g(t)$ is given by a formula), or by a very accurate finite difference approximation (if $g(t)$ is only given at some or all values of t but not by a formula). Finally, by (2.2), we have the third order approximation of u_j as

$$u_j = g(t) - a\Delta x g'(t) + \frac{1}{2}(a\Delta x)^2 g''(t) - \frac{1}{6}(a\Delta x)^3 g'''(t) \tag{2.3}$$

and we can certainly go to any higher order of accuracy as we wish. Notice that we have used only the PDE (1.1) and the given boundary condition (1.2) repeatedly to obtain the approximation (2.3), hence it is very easy to prove, by the GKS (Gustafsson, Kreiss and Sundström) analysis [14], that the scheme (1.4) using the ghost values u^n_{-1} obtained through (2.3) with $j=-1$, $a=-1$ is a stable scheme under the same time step restriction as that for the internal scheme (1.4) with periodic boundary conditions.

3 Steady state Hamilton-Jacobi equations

If we are interested in obtaining steady state solution of the Hamilton-Jacobi equation

$$H(\phi_x,\phi_y)=f(x,y) \tag{3.1}$$

together with suitable boundary conditions, we can use a Runge-Kutta or other time-stepping methods to march in time for the time dependent PDE

$$\phi_t+H(\phi_x,\phi_y)=f(x,y) \tag{3.2}$$

until steady state is reached, but that is usually rather slow. One class of effective numerical methods to speed this up is the fast sweeping method [7, 50], which uses Gauss-Seidel sweeping with different directions alternatively to reach steady state faster than explicit time stepping. For high order finite difference fast sweeping methods [49], the first few points near an inflow boundary cannot be computed by the scheme. For example, if we are solving (3.1) over the domain

$$0\le x\le 1,\quad 0\le y\le 1$$

and assuming $H_u>0, H_v>0$ for the Hamiltonian $H(u,v)$, the suitable boundary condition to prescribe is

$$\phi(0,y)=g_1(y),\quad \phi(x,0)=g_2(x). \tag{3.3}$$

We use a uniform spatial grid

$$0=x_0<x_1<\cdots<x_N=1,\quad 0=x_y<y_1<\cdots<y_N=1$$

with uniform mesh sizes $\Delta x=x_{i+1}-x_i$ and $\Delta y=y_{j+1}-y_j$ for simplicity, and third order upwind biased approximations to the partial derivatives

$$\begin{aligned}(\phi_x)_{i,j}&\approx\frac{a_1\phi_{i-2,j}+b_1\phi_{i-1,j}+c_1\phi_{i,j}+d_1\phi_{i+1,j}}{\Delta x},\\ (\phi_y)_{i,j}&\approx\frac{a_2\phi_{i,j-2}+b_2\phi_{i,j-1}+c_2\phi_{i,j}+d_2\phi_{i,j+1}}{\Delta y},\end{aligned} \tag{3.4}$$

where the coefficients a_k, b_k, c_k and d_k are either constants or they could nonlinearly depend on ϕ for a weighted essentially non-oscillatory (WENO) approximation [19], then the scheme (3.4) cannot be used to compute $\phi_{1,j}$ and $\phi_{i,1}$ with only the given boundary condition (3.3)

$$\phi_{0,j} = g_1(y_j), \quad \phi_{i,0} = g_2(x_i).$$

We could either use a lower order scheme to compute $\phi_{1,j}$ and $\phi_{i,1}$, resulting in degeneration of accuracy throughout the computational domain (as errors will propagate into the computational domain through sweeping), or we could prescribe also $\phi_{1,j}$ and $\phi_{i,1}$ by the exact solution if this information is available (however, of course they are not in general available).

In [18], a Lax-Wendroff procedure is designed to overcome this difficulty, which is explained as follows. Consider Taylor expansion in x

$$\phi(x_1, y_j) = \phi(0, y_j) + \Delta x \phi_x(0, y_j) + \frac{\Delta x^2}{2} \phi_{xx}(0, y_j) + \mathcal{O}(\Delta x^3)$$

hence our desired approximation for the third order scheme is

$$\phi_{1,j} = \phi(0, y_j) + \Delta x \phi_x(0, y_j) + \frac{\Delta x^2}{2} \phi_{xx}(0, y_j).$$

We already have $\phi(0, y_j) = g(y_j)$. The PDE (3.1), evaluated at the point $(0, y_j)$, becomes

$$H\big(\phi_x(0, y_j), g'(y_j)\big) = f(0, y_j), \tag{3.5}$$

in which the only unknown quantity is $\phi_x(0, y_j)$. Solving this (usually nonlinear) equation should give us $\phi_x(0, y_j)$. Notice that the assumption $H_u > 0$ guarantees that the nonlinear algebraic equation (3.5) has a unique solution. This is the only nonlinear algebraic equation to solve in the Lax-Wendroff procedure, as we can see below, all higher order terms satisfy linear equations.

To obtain $\phi_{xx}(0, y_j)$, we first take the derivative with respect to y on the original PDE (3.1), and then evaluate it at the point $(0, y_j)$, which yields

$$H_u\big(\phi_x(0, y_j), g'(y_j)\big) \phi_{xy}(0, y_j) + H_v\big(\phi_x(0, y_j), g'(y_j)\big) g''(y_j) = f_y(0, y_j). \tag{3.6}$$

In this equation the only unknown quantity is $\phi_{xy}(0, y_j)$, hence we obtain easily its value since, by assumption, $H_u > 0$.

We then take the derivative with respect to x on the original PDE (3.1), and evaluate it at the point $(0, y_j)$ to obtain

$$H_u\big(\phi_x(0, y_j), g'(y_j)\big) \phi_{xx}(0, y_j) + H_v\big(\phi_x(0, y_j), g'(y_j)\big) \phi_{xy}(0, y_j) = f_x(0, y_j).$$

This time, the only unknown quantity is $\phi_{xx}(0,y_j)$, which we can obtain readily from this equality.

It is clear that this procedure can be carried out to any desired order of accuracy. Also, the inflow boundary can be any piece of a smooth curve and does not need to be aligned with the mesh points: we only need to change the x and y partial derivatives to normal and tangential derivatives with respect to the boundary. Then, the original PDE and its normal and tangential derivatives will give us equations relating a desired normal derivative in terms of the function and its tangential derivatives along the boundary, which are given by the boundary condition. We refer to [18,46] for more details and numerical experimental results.

This Lax-Wendroff procedure works extremely well and allows us to compute steady state solutions of the Hamilton-Jacobi equations defined in complex geometry (e.g. in a circle) using rectangular meshes not aligned with the computational domain boundary. This procedure has also been applied to fast sweeping discontinuous Galerkin methods for solving steady state solutions of Hamilton-Jacobi equations [27,48], with excellent convergence of only a few sweeps independent of the mesh sizes.

4 Time dependent hyperbolic equations

The same idea we mentioned in Section 2 can be used to solve multi-dimensional hyperbolic conservation law systems, e.g.

$$u_t+f(u)_x+g(u)_y=0,$$

where, for systems, we assume hyperbolicity, namely $\xi_1 f'(u)+\xi_2 g'(u)$ is diagonalizable with real eigenvalues and a complete set of eigenvectors for any real coefficients (ξ_1,ξ_2). See [41]. The main ingredients can be summarized as follows:

1. Perform a Taylor expansion in the normal direction (relative to the boundary) to link the point at which the solution value is desired to a point on the computational boundary.

2. The inflow boundary is treated by the ILW procedure, via repeatedly using the PDE and its tangential and normal derivatives (relative to the boundary), to rewrite a desired normal derivative in terms of the values of the function as well as its tangential and temporal derivatives along the boundary, which are all given by the inflow boundary condition.

3. Suitable extrapolation can be used near the outflow boundary. Such extrapolation can be based either on standard polynomial extrapolation or on WENO extrapolation [20]. Suitable WENO extrapolation procedures for ILW have been discussed in, e.g., [34, 38].

4. For hyperbolic systems, the boundary can contain both inflow and outflow components. The ILW procedure for the incoming characteristic variables and the extrapolation for the outgoing characteristic variables are then combined to deal with the boundary condition.

One difficulty of the ILW procedure, especially for nonlinear systems in multiple-dimensions, is that the algebra becomes very heavy for higher order derivatives.

In [44], a simplified version of this inverse Lax-Wendroff procedure (SILW) is proposed. The standard ILW procedure is used only to compute the first order normal derivative, subsequent higher order normal derivatives are then obtained by standard extrapolation with suitable order of accuracy. This SILW procedure worked well for the computational examples in [44], in which the physical boundaries are aligned with the mesh points. For such cases and for the fifth order WENO schemes, this SILW procedure provides stable and accurate results for the very demanding detonation problems [44].

While stability is guaranteed for the original ILW procedure as can be easily analyzed by the GKS theory [14], analysis must be performed to find out how many normal derivatives must be performed in the SILW procedure to guarantee stability under the same time step restriction as that for the internal scheme, regardless of the relative location of the first grid point to the boundary (i.e. free from the "cut-cell" problem). In [45], a rigorous stability analysis using the GKS theory as well as a simplified eigenvalue analysis has been performed, for a class of central compact schemes developed in [32], Such analysis has also performed for upwind-biased finite difference schemes (prototypes of WENO schemes with linear weights) in [29]. We refer to [29, 45] for the detailed analysis and conclusions, which give very important guidance on how many terms to be retained by the ILW procedure in practical computation, to minimize the cost and yet to guarantee stability.

When solving nonlinear conservation laws

$$u_t + f(u)_x = 0,$$

if the boundary condition is given at the left boundary $x = 0$

$$u(0,t) = g(t),$$

then the ILW procedure is to obtain $u_x(0,t)$ through the PDE

$$u_x(0,t) = -\frac{u_t(0,t)}{f'(u(0,t))} = -\frac{g'(t)}{f'(g(t))}.$$

This works well if $f'(g(t)) > 0$ (to justify giving a boundary condition at the left boundary $x = 0$), but it causes problems if $f'(g(t))$ is very close to zero or is zero (close to or at the sonic points). This makes the computation using ILW or SILW procedures difficult for transonic boundaries, i.e. boundary where the inflow boundary becomes an outflow boundary, or vice versa, during the time interval of computation.

In [34], an alternative procedure is introduced to obtain the values of $f(u)$ (instead of u) at the ghost points by the ILW procedure. Then, we would need $f(u)_x$ at $x = 0$, which can be readily obtained as

$$\left(f(u)_x\right)|_{(0,t)} = -u_t(0,t) = -g'(t).$$

The remaining higher spatial derivatives of $f(u)$ are obtained by extrapolation. This alternative procedure works well at or near sonic points, allowing a smooth transition from inflow to outflow boundaries, especially for systems.

An important issue for conservation laws is numerical conservation. While this is straightforward for finite volume schemes, the very definition of conservation is not clear for finite difference schemes.

For a conservative finite difference scheme

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}\right), \tag{4.1}$$

where u_j is an approximation to the point value of the solution $u(x,t)$ at $x = x_j$, the locally conserved variable appears to be $u_j \Delta x$ (in the sense that its change over time is purely due to the net inflow and outflow through the cell boundaries $x = x_{j-1/2}$ and $x = x_{j+1/2}$), and the conserved total "mass" appears to be

$$\tilde{S} = \sum_{j=0}^{N} u_j \Delta x, \tag{4.2}$$

in the sense that

$$\tilde{S}^{n+1} = \tilde{S}^n - \Delta t\left(\hat{f}_{N+\frac{1}{2}} - \hat{f}_{-\frac{1}{2}}\right)$$

and, with periodic or compactly supported boundary conditions, $\hat{f}_{N+1/2} = \hat{f}_{-1/2}$ and we have total "mass" conservation

$$\tilde{S}^{n+1} = \tilde{S}^n.$$

In fact, the local "mass" is related to the true local mass by

$$u_j\Delta x=\bar{u}_j\Delta x+\mathcal{O}(\Delta x^3),$$

where

$$\bar{u}_j=\frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}u(x,t)dx$$

is the cell average. It would appear that the local "mass" is only a third order approximation to the true local mass $\bar{u}_j\Delta x$ in smooth regions, however the total "mass" $\tilde{S}$, as defined in (4.2), is equal to the true total mass S

$$S=\int_a^b u(x,t)dx \tag{4.3}$$

for any N-th degree trigonometric polynomial (assuming N is even for convenience)

$$u(x)=\sum_{k=-\frac{N}{2}}^{\frac{N}{2}}a_k e^{ikx}. \tag{4.4}$$

That is,

$$\tilde{S}=\sum_{j=0}^{N}u_j\Delta x=\sum_{j=0}^{N}\bar{u}_j\Delta x=S,$$

if the point values u_j and the cell averages $\bar{u}_j$ are both from an N-th degree trigonometric polynomial (4.4). Therefore, a conservative finite difference scheme (4.1) conserves the total "mass" $\tilde{S}$ as defined in (4.2), which is a spectrally accurate approximation to the true total mass S as defined in (4.3), for any smooth periodic or compactly supported solutions.

However, if the solution is not periodic or compactly supported, then we only have

$$\tilde{S}=\sum_{j=0}^{N}u_j\Delta x=\sum_{j=0}^{N}\bar{u}_j\Delta x+\mathcal{O}(\Delta x^2)=S+\mathcal{O}(\Delta x^2). \tag{4.5}$$

That is, the total "mass" $\tilde{S}$, as defined in (4.2), is only a second order approximation to the true total mass S for non-periodic functions. Therefore, if a finite difference scheme conserves the total "mass" $\tilde{S}$ (subject to net inflow and outflow at the domain boundaries), it can only be second order accurate. This problem exists already for regular finite difference schemes, but is compounded by the ILW procedure at the numerical boundaries.

In [10], conservative finite difference schemes were obtained using the ILW procedure, with the following ingredients:

- Using the numerical quadrature formula

$$\int_0^\infty h(x)dx = \Delta x \sum_{j=0}^{+\infty} \omega_j h(x_j) + \mathcal{O}\left((\Delta x)^\nu\right), \tag{4.6}$$

where the weights ω_j depend on ν, but $\omega_j = 1$ for $j \geq \nu$, we define the numerical total mass consistent with high order accuracy to the true total mass as

$$\tilde{S} = \sum_{j=0}^{N} \omega_j u_j \Delta x. \tag{4.7}$$

- We modify the numerical fluxes near the boundary (the modification is local and is a high order accuracy perturbation from the original inverse Lax-Wendroff scheme), so that the resulting scheme is conservative (subject to inflow and outflow) with respect to the numerical total mass (4.7).

- The conservative inverse Lax-Wendroff scheme works equally well as the original inverse Lax-Wendroff scheme in accuracy and non-oscillatory performance, through extensive numerical tests, and it shows an advantage in shock location resolution for long time simulation.

In [51], an ILW procedure is designed for the boundary treatment of implicit-explicit (IMEX) Runge-Kutta method for hyperbolic systems with stiff source terms. The ILW procedure has been applied to boundary conditions of the Boltzmann type models in [11] and of the simulations of relativistic stars in [47], among others. Engineering applications of the ILW and SILW procedures can be found in, e.g., [6, 22].

5 The ILW procedure for problems involving complex moving geometries

We can easily extend the ILW and SILW procedures, defined on fixed geometry, to deal with problems of solutions involving complex moving geometries.

For problems in such geometries, it is difficult to use body-fitted meshes which conform to the moving geometry. Instead, methods based on fixed cartesian meshes have been successfully developed. For example, the immersed boundary (IB) method introduced by Peskin [36] is widely used. One of the challenges of the IB method is the representation of the moving objects which cut through

the grid lines in an arbitrary fashion. To solve compressible inviscid flows in complex moving geometries, most methods in the literature are based on finite volume schemes. The challenge mainly comes from the so-called "small-cell" problem. Namely, one obtains irregular cut cells near the boundary, which may be orders of magnitude smaller than the regular grid cells, leading to a severe time step restriction. Also, in terms of accuracy, most finite volume schemes in the literature are at most second order. In particular, the errors at the boundaries sometimes often fall short of second order.

Our ILW procedure can be extended to such situations with moving geometries. The only change is to obtain relationships between the temporal and spatial derivatives via the PDE in the moving Lagrangian framework. That is, in replacing the usual time derivatives by the material derivatives (measured along the moving boundary) in the ILW procedure. We refer to [42] for the details. See also [43].

6 Convection-diffusion equations

The extension of the ILW and SILW procedures to convection-diffusion equations is non-trivial, since totally different boundary treatments are needed for the diffusion-dominated and the convection-dominated regimes.

Let us first look at the simple example of the heat equation

$$u_t = u_{xx}, \quad 0 < x < \infty$$

with the boundary condition

$$u(0,t) = g(t).$$

If we perform a Taylor expansion at $x = 0$

$$u_j = u(0,t) + u_x(0,t)x_j + \frac{1}{2}u_{xx}(0,t)x_j^2 + \cdots,$$

then the ILW procedure can only determine the even order derivatives

$$u(0,t) = g(t), \quad u_{xx}(0,t) = g'(t), \quad \cdots$$

and the odd derivatives must be obtained by extrapolation.

Stability of such ILW procedure, when the relative location of the boundary and the closest grid point is arbitrary, and for both Dirichlet and Neumann boundary conditions, is systematically analyzed in [28, 30].

Now, suppose we have a convection-diffusion equation

$$u_t + a u_x = \varepsilon u_{xx}, \quad 0 < x < \infty$$

with the boundary condition

$$u(0,t) = g(t).$$

There are two ways of applying the ILW procedure to obtain the first two spatial derivatives:

1. The first approach is to use extrapolation to obtain the second derivative $u_{xx}(0,t) = u_{xx}^{ext}$, and then use the ILW procedure to obtain the first derivative

$$u_x(x,0) = -\frac{1}{a}\left(g'(t) - \varepsilon u_{xx}^{ext}\right).$$

   The remaining derivatives can be obtained similarly. This is the approach for the purely convection equation ($\varepsilon = 0$), hence it is expected to work well for convection-dominated situation.

2. The second approach is to use extrapolation to obtain the first derivative $u_x(0,t) = u_x^{ext}$, and then use the ILW procedure to obtain the second derivative

$$u_{xx}(x,0) = \frac{1}{\varepsilon}\left(g'(t) + a u_x^{ext}\right).$$

   The remaining derivatives can be obtained similarly. This is the approach for the purely diffusion equation ($a = 0$), hence it is expected to work well for diffusion-dominated situation.

In [33], a carefully designed combination of the boundary treatments for the two regimes has been proposed and a stable and accurate boundary condition for general convection-diffusion equations has been obtained, which worked well for various test cases including compressible Navier-Stokes equations. Engineering applications can be found in, e.g., [5,8,31].

7 Concluding remarks

In this short survey, we have demonstrated an inverse Lax-Wendroff (ILW) procedure for boundary treatment, which yields stable discretization with the same CFL number as the inner scheme and allows us to compute problems on arbitrary domains using cartesian meshes. The technique can be applied to inviscid

and viscous flows with complex moving geometries, yielding stable and high order accurate solutions. The simplified ILW (SILW) procedure further improves the efficiency and effectiveness of such boundary treatments.

Current and future extensions of the ILW procedure would involve a generalization of this technique to other schemes such as the discontinuous Galerkin method, and to problems with deformable structures and interface problems.

Acknowledgments

The author is supported by NSF (Grant DMS-2010107).

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