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# On Some Properties of the Curl Operator and Their Consequences for the Navier-Stokes System

Nicolas Lerner[1] and François Vigneron[2,*]

[1] *Sorbonne Université, Institut de Mathématiques de Jussieu, UMR 7586, Campus Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris cedex 05, France.*
[2] *Université de Reims Champagne-Ardenne, Laboratoire de Mathématiques de Reims, UMR 9008, Moulin de la Housse, BP 1039, 51687 Reims cedex 2, France.*

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In honor of our friend Professor Chaojiang Xu, on the occasion of his 65th birthday.

**Abstract.** We investigate some geometric properties of the curl operator, based on its diagonalization and its expression as a non-local symmetry of the pseudo-derivative $(-\Delta)^{1/2}$ among divergence-free vector fields with finite energy. In this context, we introduce the notion of spin-definite fields, i.e. eigenvectors of $(-\Delta)^{-1/2}$curl. The two spin-definite components of a general 3D incompressible flow untangle the right-handed motion from the left-handed one. Having observed that the non-linearity of Navier-Stokes has the structure of a cross-product and its weak (distributional) form is a determinant that involves the vorticity, the velocity and a test function, we revisit the conservation of energy and the balance of helicity in a geometrical fashion. We show that in the case of a finite-time blow-up, both spin-definite components of the flow will explode simultaneously and with equal rates, i.e. singularities in 3D are the result of a conflict of spin, which is impossible in the poorer geometry of 2D flows. We investigate the role of the local and non-local determinants

$$\int_0^T \int_{\mathbb{R}^3} \det\big(\operatorname{curl} u, u, (-\Delta)^\theta u\big)$$

*Corresponding author. *Email address:* `francois.vigneron@univ-reims.fr` (F. Vigneron)

and their spin-definite counterparts, which drive the enstrophy and, more generally, are responsible for the regularity of the flow and the emergence of singularities or quasi-singularities. As such, they are at the core of turbulence phenomena.

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# 1 Introduction

The initial value problem for the Navier-Stokes system for incompressible fluids is usually written as

$$\begin{cases} \dfrac{\partial u}{\partial t}+(u\cdot\nabla)u-\nu\Delta u=-\nabla p, \quad \operatorname{div} u=0, \\ u_{|t=0}=u_0. \end{cases} \tag{1.1}$$

Here $u=u(t,x)$ is a time-dependent vector field on $\mathbb{R}^3$, the viscosity $\nu$ is a positive parameter (expressed in Stokes, i.e. $\mathtt{L}^2\mathtt{T}^{-1}$) and $u_0$ is a given divergence-free vector field.

In 1934, Leray [58] proved the existence of global weak solutions in $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$. In 3D, the question of their uniqueness remains elusive and is intimately connected to deciding whether the weak solutions enjoy a higher regularity. Well-posedness in various function spaces has been studied thoroughly and culminates in Koch and Tataru's result [52] if the data $u_0$ is small in the largest (i.e. less constraining) function space (called $\mathrm{BMO}^{-1}$) that is scale and translation invariant and on which the heat flow remains locally uniformly in $L^2_{t,x}$.

The set of singular times may or not be empty, but it is a compact subset of $\mathbb{R}_+$, whose Hausdorff measure of dimension $\frac{1}{2}$ is zero. The celebrated theorem of Caffarelli *et al.* [17] ensures that singularities form a subset of space-time whose parabolic Hausdorff measure of dimension 1 vanishes too (see also Arnold and Craig [2]).

Note that Eq. (1.1) corresponds to an Eulerian point of view, i.e. it describes the movement of the fluid in a fixed reference frame. The natural question of tracking individual fluid particles, i.e. the Lagrangian point of view, is equivalent to the existence of a flow $\xi:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d$

$$\frac{\partial\xi}{\partial t}=u\big(t,\xi(t,x)\big), \quad \xi(0,x)=x. \tag{1.2}$$

The volume preserving map $\xi(t,\cdot)$ tracks the deformations of the fluid [22,40].

For a comprehensive covering of most of the classical theory of Navier-Stokes, we refer the reader to, e.g., Lemarié's book [57] and the references therein. Berselli's recent book [9] offers an interesting complement that blends theoretical results on the energy fluxes with numerical methods and turbulence theory. Davidson's book [34] provides valuable physical insight on the latter subject.

In the next few lines, we will present a small subset of these classical results, not necessarily in chronological order, to provide some background on the arduous question of the regularity of the solutions. Then we will expose our own contribution, which is a new geometric approach based on the diagonalization of the curl operator.

## 1.1 Classical regularity theory near a singular event

The behavior of smooth solutions of the Navier-Stokes equation as they approach a (still conjectural) finite blow-up time has been studied very carefully.

For the $\dot{H}^1$ semi-norm, a precise rate has been known since Leray [58]: if the first time of singularity $T^*$ of a smooth solution is finite, then

$$\|\nabla u(t)\|_{L^2} \geq \frac{C}{(T^*-t)^{\frac{1}{4}}}. \tag{1.3}$$

This inequality is the immediate consequence of a bootstrap of the local well-posedness result for data in $H^1$, when one takes into account that if $u_0$ blows up at time $T^*$, then $u(t)$ will blow up at time $T^*-t$. Similarly, for any $0<\gamma<\frac{1}{2}$ and $p=\frac{3}{(1-2\gamma)}$

$$\|u(t)\|_{\dot{H}^{\frac{1}{2}+2\gamma}} \gtrsim \|u\|_{L^p} \geq \frac{C_\gamma}{(T^*-t)^\gamma}. \tag{1.4}$$

The endpoint $L^\infty$ is admissible with rate $\gamma=\frac{1}{2}$.

Thanks to the energy inequality and the Sobolev embedding, any Leray solution enjoys a uniform control in $L^\infty_t L^2_x \cap L^2_t L^6_x$, so in particular in $L^4_t L^3_x$ and $L^{2+2/3}_t L^4_x$. Various authors including Chemin[†], Foias *et al.* [39] and Cordoba *et al.* [33] independently observed that the amplitude of Leray solutions is controlled in $L^1_{\mathrm{loc}}(\mathbb{R}_+;L^\infty(\mathbb{R}^3))$ i.e.

$$\forall T>0, \quad \int_0^T \|u(t)\|_{L^\infty} dt < \infty. \tag{1.5}$$

This result is now known as the absence of squirt singularities (see e.g. [57, §11.6]).

[†]Personal communication (2004) and unpublished lecture notes.

In [74], Vasseur proved a family of estimates in various function spaces as long as the solution remains smooth, one of which reads

$$\int_0^{T^*} \sum_{|k|\leq 2} \left\|\nabla^k u(t)\right\|_{L^1} dt \leq C\left(1+\|u_0\|_{L^2}^4\right) \tag{1.6}$$

with a constant $C$ that does not depend on the solution $u$, nor on the blow-up time $T^*$. Interpolation between (1.5) and (1.6) ensures that

$$\forall p\in[1,\infty], \quad \int_0^{T^*} \|u(t)\|_{L^p} dt < \infty. \tag{1.7}$$

Using the energy inequality (1.7) can obviously be improved to $L^q([0,T^*);L^p)$ with

$$\begin{cases} q=1-\dfrac{3}{p}, & \text{if} \quad p\geq 6, \\ \dfrac{2}{q}+\dfrac{3}{p}=\dfrac{3}{2}, & \text{if} \quad 2\leq p\leq 6, \\ q=\dfrac{p}{2-p}, & \text{if} \quad 1\leq p\leq 2. \end{cases}$$

These universal qualitative upper bounds are in sharp contrast with the lower bounds, which generalize (1.3)-(1.4) in the case of a finite time blow-up. Regarding the supremum norm, (1.4) implies

$$\int_0^{T^*} \|u(t)\|_{L^\infty}^2 dt = +\infty. \tag{1.8}$$

In very rough terms and unless the amplitude oscillates wildly near $T^*$, the behavior depicted by (1.5) and (1.8) suggests that

$$\frac{C_\infty}{(T^*-t)^{\frac{1}{2}}} \leq \|u(t)\|_{L^\infty} \leq \frac{C(u)}{T^*-t}.$$

To "thicken" the peaks of amplitude, one may look at uniform bounds for the heat flow, i.e. Besov norms of negative regularity index. The quantitative lower bound of Chemin & Gallagher [20]

$$\|u(t)\|_{\dot{B}^{-1+2\gamma}_{\infty,\infty}} = \sup_{\tau>0} \tau^{\frac{1}{2}-\gamma} \left\|e^{\tau\Delta}u(t)\right\|_{L^\infty} \geq \frac{C_\gamma}{(T^*-t)^\gamma}, \quad 0<\gamma<\frac{1}{2} \tag{1.9}$$

is coherent with the previous intuition when $\gamma=\frac{1}{2}$. The second endpoint ($\gamma=0$) requires special care because it is also the end of the chain of critical scale-invariant spaces

$$\dot{H}^{\frac{1}{2}} \subset L^3 \subset \mathrm{BMO}^{-1} \subset \dot{B}^{-1}_{\infty,\infty}$$

i.e. Galilean invariant spaces $X \subset \mathcal{S}'(\mathbb{R}^3)$ whose norm satisfies the additional relation $\|\lambda u(\lambda x)\|_X = \|u(x)\|_X$ (see Meyer [65] and also [20, Proposition 1.2]). Kato's [51] and Escauriaza *et al.* [47] theorems state that

$$\liminf_{t\to T^*} \|u(t)\|_{L^3} \geq c_0, \quad \limsup_{t\to T^*} \|u(t)\|_{L^3} = +\infty.$$

Later, Seregin [69] proved that there are no major fluctuations of the $L^3$ norm near the blow-up time, i.e.

$$\lim_{t\to T^*} \|u(t)\|_{L^3} = +\infty \tag{1.10}$$

and a quantitative polylogarithmic rate was obtained recently by Tao [71]

$$\limsup_{t\to T^*} \frac{\|u(t)\|_{L^3}}{\left(\log\log\log \frac{1}{T^*-t}\right)^c} = +\infty. \tag{1.11}$$

However, a simple scaling argument (see [6, §5.1]) forces the inferior limit (over all solutions) to be zero in (1.11) and in any similar estimate with a diverging rate, i.e. fluctuations of the $L^3$ norm will sometimes be visible at this time-scale. Soon afterwards, Barker & Prange [7] investigated the possibility of reducing the length of the polylogarithm.

The well known Ladyzhenskaya-Prodi-Serrin condition reads

$$\int_0^{T^*} \|u(t)\|_{L^p}^q dt = +\infty \quad \text{for} \quad \frac{2}{q} + \frac{3}{p} = 1, \quad p > 3. \tag{1.12}$$

Note that (1.10) corresponds to the endpoint $p = 3$, while (1.8) matches $p = \infty$. This second endpoint was investigated by Kozono & Taniuchi [53], who even generalized it to the (larger) BMO space.

The blow-up of scale-invariant Besov norms of negative regularity index was obtained by Gallagher *et al.* [41]. This lower bound implies that most supercritical norms i.e. Galilean invariant space-time function spaces $Y$ such that

$$\left\|\lambda u(\lambda^2 t, \lambda x)\right\|_Y \leq C\lambda^{1-\gamma} \|u(t,x)\|_Y$$

with $\gamma < 1$ will also blow up. For the subtle behavior at the endpoint among critical spaces, i.e. $\dot{B}^{-1}_{\infty,\infty}$, we refer to Cheskidov & Shvydkoy [25] and Ohkitani [66].

Concerning spaces of higher regularity, we have known since Kato that

$$\int_0^{T^*} \|\nabla u\|_{L^\infty} = +\infty. \tag{1.13}$$

The celebrated Beale-Kato-Majda criterion [8,68] reads

$$\int_0^{T^*} \|\mathrm{curl}\, u\|_{L^\infty} = +\infty \tag{1.14}$$

and various generalizations in more involved function spaces are possible (see e.g. [53,68]). We will briefly present Cheskidov & Shvydkoy's [26] variant of this criterion (see Eq. (4.28) below).

Alternatively, each of the identities (1.8), (1.10), (1.12)-(1.14) can also be stated as a regularity criterion. If the left-hand side integral is finite on some time interval $[0,T]$, then the corresponding solution remains regular up to and including at time $T$, i.e. one has $T^* > T$. Numerical investigations of quasi-singularities are still underway, e.g. by Sverak *et al.* [45,48], who possibly hint at the existence of actual singularities, or by Protas *et al.* [5,49,50], who seek flows that maximize the growth of various norms.

Let us close this first panorama by a word of caution. The heat equation is justly considered as the archetype of a well behaved parabolic regularizing model. Its solution is indeed obtained by convolution with a Gaussian kernel

$$e^{t\Delta}u_0 = \int_{\mathbb{R}^3} u_0(x-\sqrt{t}y)W(y)dy \quad \text{with} \quad W(y) = (4\pi)^{-\frac{3}{2}} e^{-\frac{y^2}{4}}. \tag{1.15}$$

However, as shown by Tychonov [38,72], this solution is not the only one if one fails to restrict the growth of $u$ at infinity to, e.g., $\mathcal{O}(e^{cx^2})$. For any $\alpha \in \mathbb{R}$, the following function is a smooth (but not tempered) solution of the heat equation that coincides with $u_0$ at $t=0$:

$$u(t,x) + \alpha \sum_{n=0}^{\infty} P_n\left(\frac{1}{t}\right) e^{-\frac{1}{t^2}} H(t) \frac{x^{2n}}{(2n)!}. \tag{1.16}$$

Here $u = e^{t\Delta}u_0$, $H(t)$ is the Heaviside function, $P_0 = 1$ and $P_{n+1}(z) = 2z^3 P_n(z) + P_n'(z)$ are the polynomials involved in the computation of the $n$-th derivative of $e^{-1/t^2}$. While this type of instability may seem far from the physical range of validity of hydrodynamical models, it remains instructive (see also [59]).

## 1.2 Geometric regularity theory near a singular event

All the criteria that we have mentioned up to now are obviously isotropic and do not rely on any geometric structure of the flow. There have been a few remarkable attempts to take into account the geometric nature of the Navier-Stokes equation and we shall now present them briefly.

A striking example of the importance of the geometry for hydrodynamics is turbulence, where radically anisotropic structures (vortex filaments and pancakes) play a central role [34]. This observation suggests that the most fundamental and universal behavior of fluids is a microlocal cascade. However, it is equally important (and more feasible) to describe the consequences of these interactions at intermediary scales, for example by estimating the growth of norms of geometric quantities.

An important step in this direction was achieved by Constantin & Fefferman [28,29], who studied the direction of the vorticity $\omega = \mathrm{curl}\, u$, i.e.

$$\xi(t,x) = \frac{\omega(t,x)}{|\omega(t,x)|} \in \mathbb{S}^2. \tag{1.17}$$

Using $\xi$ as a multiplier in the equation of vorticity (see (2.10) below), they established that

$$(\partial_t + u\cdot\nabla - \nu\Delta)(|\omega|) + \nu|\omega||\nabla\xi|^2 = \big\langle(\omega\cdot\nabla)u,\xi\big\rangle_{\mathbb{R}^3}. \tag{1.18}$$

Integrating over $[0,t]\times\Omega$ for $\Omega=\mathbb{R}^3$ or $\mathbb{T}^3$ leads (at first for smooth solutions, see [29, Eq. 20]) to the identity

$$\begin{aligned}&\int_\Omega |\omega(t,x)|dx + \nu\int_0^t\int_\Omega |\omega(t,x)||\nabla\xi(t',x)|^2 dxdt' \\ &= \int_\Omega |\omega(0,x)|dx + \int_0^t\int_\Omega \big\langle(\omega\cdot\nabla)u,\xi\big\rangle_{\mathbb{R}^3} dxdt'.\end{aligned}$$

The following geometric estimate follows immediately (using (3.2)):

$$\begin{aligned}&\int_\Omega |\omega(t,x)|dx + \nu\int_0^t\int_\Omega |\omega(t,x)||\nabla\xi(t',x)|^2 dxdt' \\ &\le \|\omega(0)\|_{L^1} + \frac{1}{2\nu}\Big(\|u(0)\|_{L^2}^2 - \|u(t)\|_{L^2}^2\Big).\end{aligned} \tag{1.19}$$

To the best of our knowledge, this global $L^\infty_t L^1_x$ estimate is the only known a priori bound on the vorticity that holds for any Leray solution, apart from the obvious $L^2_t L^2_x$ bound that follows from the energy inequality. This result illustrates how a local alignment in the direction of the vorticity can deplete the nonlinearity. An interpretation that connects this estimate to turbulence is given in [28].

Another notable geometric result is the one from Vasseur [73] on the direction of the velocity field. This result is specific to the 3D case and can be stated as follows. If a solution blows-up at a finite time $T^*$, then

$$\int_0^{T^*} \left\|\mathrm{div}\left(\frac{u}{|u|}\right)\right\|_{L^p}^q dt = +\infty \quad \text{for} \quad \frac{2}{q}+\frac{3}{p}\le\frac{1}{2}, \quad q\ge 4, \quad p\ge 6. \tag{1.20}$$

Conversely, a control of the norm implies the regularity of the solution. This criterion is based on the incompressibility of the flow and the identity

$$|u|\operatorname{div}\frac{u}{|u|}=-\left(\frac{u}{|u|}\cdot\nabla\right)|u|.$$

It means that the growth of $|u|$ along the streamlines is linked to the divergence of the direction of $u$. In particular, the kinetic energy $|u|^2$ can only increase along the streamlines if they are bent and produce some divergence in the direction of the velocity.

Among anisotropic criteria, let us also mention a recent result by Chemin *et al.* [21] that investigates the possibility of detecting a singularity through one component only. If $u$ is a smooth solution that presents a blow-up at a finite time $T^*$, then

$$\inf_{\sigma\in\mathbb{S}^2}\int_0^{T^*}\|u(t)\cdot\sigma\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}dt=+\infty \quad \text{for} \quad p\geq 2, \tag{1.21}$$

which means that all components will be affected. They also show that

$$\inf_{\sigma\in\mathbb{S}^2}\sup_{t'>t}\|u(t')\cdot\sigma\|_{\dot{H}^{\frac{1}{2}}}\geq C\log^{-\frac{1}{2}}\left(e+\frac{\|u(t)\|^4_{L^2}}{T^*-t}\right). \tag{1.22}$$

The fact that the right-hand side vanishes is coherent with the remark that follows (1.11).

## 1.3 Structure of this article and summary of our results

Our article is structured as follows. In Section 2, we expose some geometric properties of the curl operator. The non-local diagonalization of the curl (see Lemma 2.2 and Remark 2.3) establishes a geometrical link with the pseudo-derivative, i.e. $|D|=(-\Delta)^{1/2}$: both operators are images of one another by a certain symmetry of the subset of $L^2$ formed by the divergence-free vector fields. This property leads us to introduce (Definition 2.1) the notion of spin-definite vector field, i.e. divergence-free fields such that

$$\operatorname{curl} u=\pm|D|u.$$

The end of Section 2 is dedicated to the study of such fields. In layman's terms, fields with positive spin display an exclusive right-handedness motion at all scales, while fields with negative spin are their chiral image in a mirror. Spin-definite fields are build as superpositions of planar Beltrami waves (2.19) with independent directions of propagation and various frequencies. Any divergence-free field can be decomposed in a unique way as the sum of two fields with respectively a positive and a negative spin. A few numerical simulations illustrate the importance of this notion for the description of vortex filaments.

The next key observation is that the non-linearity of Navier-Stokes has a cross-product structure and its weak (i.e. distributional) form is a determinant (see Eqs. (2.7) and (2.8) below). In Section 3, we use this geometric approach to revisit the two classical conservation laws for Navier-Stokes, i.e. the balance of energy and the balance of helicity. While the former is constitutional of the definition of the Leray space $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$, we expose the latter (see Eqs. (3.4) and (3.8)) as a conservation law in

$$L^\infty_t \dot{H}^{\frac{1}{2}}_x \cap L^2_t \dot{H}^{\frac{3}{2}}_x$$

for the spin-definite components of the flow. Our main Theorem 3.1, can be restated as follows.

**Theorem 1.1.** *In the case of a finite-time blow-up of Navier-Stokes, both of the spin-definite components of the flow will explode simultaneously and with equal rates.*

In simple terms, this means that singularities can only appear as the result of an unresolved conflict of spin that escalated out of control. Proposition 3.2, Theorems 3.2 and 3.3 quantify how an imbalance between the two spins actually prevents singularities. In Subsection 3.6, we explain how the poorer geometry of 2D flows either enforces the victory of one direction of rotation over the other or lets the viscosity dissolve the attempted conflict, while the richer 3D geometry allows for the possibility of an escalation of conflicting spins. In Subsection 3.4, we also briefly discuss the recent developments regarding Onsager's conjecture for the balance of energy and its counterpart for the balance of helicity.

Section 4 pursues the geometric investigation of the weak form of the non-linearity, i.e. critical determinants. Applying this point of view to study the enstrophy produces a proof of the regularity of 2D flows based on the identity

$$\int_{\mathbb{R}^3} \det(\operatorname{curl} u, u, -\Delta u) = 0,$$

which is valid if $u$ is a 2D divergence-free field embedded in 3D space; note that the integrand is not identically zero and that the cancellation is the result of a space average. More generally, we investigate (Propositions 4.1 and 4.2 and the identity (4.12)) how the sign of

$$\int_0^t \int_{\mathbb{R}^3} \det\big(\operatorname{curl} u, u, |D|^{2\theta} u\big)\, dx dt'$$

relates to the growth of the Sobolev norm $\dot{H}^\theta$ of the spin-definite components of the flow. This analysis suggests that even though the definition of regularity is obviously local, its control in the case of Navier-Stokes flows will likely involve non-local estimates. We also obtain a stability estimate among Leray solutions that has a geometric form

$$\|u_1(t)-u_2(t)\|_{L^2}^2 \leq \|u_1(0)-u_2(0)\|_{L^2}^2 \exp\left(\int_0^T \frac{\|u_1\times u_2\|_{L^2}^2}{\|u_1-u_2\|_{L^2}^2}\right)$$

and a variant of the Beale-Kato-Majda criterion.

For the convenience of the reader, Appendix A recalls the geometric proof of some vector calculus identities whose direct computational proofs in coordinates would be non-trivial.

# 2 Geometric properties of the curl operator

In this section, we collect some classical facts related to vorticity and we introduce notations, in particular the signed decomposition of curl in Subsection 2.3 and the associated notion of spin of a 3D divergence-free field, that will be used throughout the article.

We use the following definition for the Fourier transform on $\mathbb{R}^n$:

$$\hat{u}(\xi)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx, \quad u(x)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{u}(\xi)d\xi. \tag{2.1}$$

This definition provides a unitary transformation in $L^2(\mathbb{R}^n)$. The operator $D=-i\nabla$ satisfies

$$\widehat{D^\alpha u}(\xi)=\xi^\alpha\hat{u}(\xi).$$

The operator $|D|=(-\Delta)^{1/2}$ has symbol $|\xi|$. We focus exclusively on the three dimensional case, i.e. $n=3$, except in the brief Subsection 3.6.

## 2.1 The curl operator

We use the notation $\mathbf{C}=\mathrm{curl}$. It is a Fourier multiplier of symbol

$$\mathbf{C}(\xi)=\begin{pmatrix} 0 & -i\xi_3 & i\xi_2 \\ i\xi_3 & 0 & -i\xi_1 \\ -i\xi_2 & i\xi_1 & 0 \end{pmatrix}, \tag{2.2}$$

which is the matrix of $\eta \mapsto i\xi\times\eta$ seen as an endomorphism of the Hermitian space $\mathbb{C}^3$. Obviously

$$\mathbf{C}(\xi)^* = \overline{{}^t\,\mathbf{C}(\xi)} = \mathbf{C}(\xi), \tag{2.3}$$

which implies that the curl is (formally) self-adjoint over $L^2(\mathbb{R}^3)$. One has

$$\begin{aligned}\mathbf{C}(\xi)^2 &= -\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} |\xi|^2-\xi_1^2 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ -\xi_1\xi_2 & |\xi|^2-\xi_2^2 & -\xi_2\xi_3 \\ -\xi_1\xi_3 & -\xi_3\xi_2 & |\xi|^2-\xi_3^2 \end{pmatrix},\end{aligned}$$

so that

$$\mathbf{C}(\xi)^2 = |\xi|^2\mathrm{Id} - \xi\otimes\xi \quad \text{i.e.} \quad \mathbf{C}^2 = \nabla\,\mathrm{div} - \Delta.$$

The columns of $\mathbf{C}(\xi)$ are clearly orthogonal to $\xi$, which reflects the classical fact that $\mathrm{div}\circ\mathrm{curl}=0$. The operator $|D|^{-1}\mathbf{C}$ is obviously bounded on $L^2$ and is even of Calderón-Zygmund type by Mikhlin's multiplier theorem. The Leray projection onto divergence-free vector fields can be expressed in terms of $\mathbf{C}$

$$\mathbb{P} = |D|^{-2}\mathbf{C}^2 = \mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}. \tag{2.4}$$

The operator $\mathbb{P}$ is an orthogonal projection since $\mathbb{P}=\mathbb{P}^*$ and $\mathbb{P}^2=\mathbb{P}$. Similarly, $\mathrm{Id}-\mathbb{P}$ is an orthogonal projection onto gradient fields[‡], i.e. the nullspace of $\mathbf{C}$. Note also that $\mathbb{P}$ and $\mathbf{C}$ commute.

## 2.2 The Navier-Stokes velocity equation in curl form

Let us briefly present some alternative expressions of the Navier-Stokes equation involving the operator $\mathbf{C}$. For now, we are not directly interested in the standard equation of vorticity but rather in expressing the linear and non-linear terms as curls.

When $u$ is a divergence-free vector field, the identity (2.4) implies that $\mathbf{C}^2u = -\Delta u$. The Navier-Stokes equation (1.1) can thus also be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathbb{P}\big((u\cdot\nabla)u\big) + \nu\mathbf{C}^2u = 0, \\ u_{t=0} = u_0, \ \ \mathrm{div}\,u_0 = 0. \end{cases} \tag{2.5}$$

[‡]As we plant our discussion exclusively within the $L^2$ framework, there are no potential flows like $\nabla(x^2-y^2+x^3-3xz^2)$, which is both a gradient and a divergence-free field on $\mathbb{R}^3$. Such a field is formally in the range of $\mathbb{P}$.

Applying $\mathbb{P}$ to Eq. (1.1) leads directly to (2.5). Conversely, (2.5) implies that both $\frac{\partial u}{\partial t}$ and $u(t=0)$ are divergence-free, proving that $\operatorname{div} u=0$. Applying $\mathrm{Id}=\mathbb{P}-\nabla(-\Delta)^{-1}\operatorname{div}$ to $u\cdot\nabla u$, the pressure is immediately reconstructed with the identity $\nabla p=\nabla(-\Delta)^{-1}\operatorname{div}(u\cdot\nabla)u$.

**Remark 2.1.** A common observation is that

$$(u\cdot\nabla)u=\sum_j u_j\partial_j u=\sum_j \partial_j(u_j u)=\operatorname{div}(u\otimes u)$$

because $u$ is a divergence-free vector field, which gives a meaning in the distributional sense to the non-linear term as soon as $u(t,\cdot)$ belongs to any space embedded in $L^2_{\mathrm{loc}}$.

Let us now recall a well known identity of vector calculus. For any vector field $u$, one has

$$(\mathbf{C}u)\times u=(u\cdot\nabla)u-\frac{1}{2}\nabla|u|^2. \tag{2.6}$$

The first coordinate of $\mathbf{C}u\times u$ is indeed

$$\begin{aligned}&(\partial_3 u_1-\partial_1 u_3)u_3-(\partial_1 u_2-\partial_2 u_1)u_2\\ &=(u_3\partial_3+u_2\partial_2)(u_1)-\frac{1}{2}\partial_1\left(u_3^2+u_2^2\right)\\ &=(u\cdot\nabla)u_1-\frac{1}{2}\partial_1\left(|u|^2\right).\end{aligned}$$

The identity then follows by circular permutation among indices. There is also a profound geometric reason for the above identity (known sometimes as the dot product rule), as it is elemental in the definition of Riemannian connections (see Appendix A, Eq. (A.11)).

A direct consequence of (2.6) for the non-linear term of Navier-Stokes is that

$$\mathbb{P}\big((u\cdot\nabla)u\big)=\mathbb{P}\big((\mathbf{C}u)\times u\big). \tag{2.7}$$

We may therefore rewrite the Navier-Stokes equation in (2.5) as follows:

$$\frac{\partial u}{\partial t}+\mathbb{P}\big((\mathbf{C}u)\times u\big)+\nu\mathbf{C}^2 u=0. \tag{2.8}$$

This particular form of the equation will be of central importance in what follows. It suggests a new form of cancellations based on the following identity:

$$\big\langle(u\cdot\nabla)u,w\big\rangle_{L^2(\mathbb{R}^3)}=\langle\mathbf{C}u\times u,w\rangle_{L^2(\mathbb{R}^3)}=\int_{\mathbb{R}^3}\det(\mathbf{C}u,u,w)dx, \tag{2.9}$$

which holds for any pair of divergence-free vector fields $u,w$. The identity (2.7) thus underlines that the non-linearity of Navier-Stokes has the structure of a cross-product, and that its weak (distributional) form (2.9) is a determinant that involves the vorticity, the velocity and a test function.

Of course, this formulation is related to the vorticity equation. One has

$$\mathbf{C}\mathbb{P}\big((\mathbf{C}u)\times u\big)=\mathbb{P}\mathbf{C}\big((\mathbf{C}u)\times u\big)=\mathbf{C}\big((\mathbf{C}u)\times u\big)=(u\cdot\nabla)\mathbf{C}u-\big((\mathbf{C}u)\cdot\nabla\big)u.$$

Therefore, applying $\mathbf{C}$ to (2.8) directly implies the vorticity equation

$$\frac{\partial\omega}{\partial t}+(u\cdot\nabla)\omega+\nu\mathbf{C}^2\omega=(\omega\cdot\nabla)u \tag{2.10}$$

with $\omega=\mathbf{C}u$. In the line of (2.8), note that the nonlinear term of the vorticity equation inherits the structure of a cross-product: $(u\cdot\nabla)\omega-(\omega\cdot\nabla)u=\mathbf{C}(\omega\times u)$. The vortex-stretching term $(\omega\cdot\nabla)u=(\omega\cdot\nabla)|D|^{-2}\mathbf{C}\omega$ is of order zero but highly non-local in $\omega$. On average, it is orthogonal to $u$ (see (3.12) below). The vortex-stretching term plays a central role in the cascade of energy towards smaller scales in 3D turbulent flows by thinning the girth of vortex tubes.

**Remark 2.2.** The Navier-Stokes equation can also be rewritten as

$$\frac{\partial u}{\partial t}+\mathbb{P}(u\cdot\nabla)\mathbb{P}u+\nu\,\mathrm{curl}^2 u=0 \tag{2.11}$$

to put the emphasis on the transport-diffusion aspect of the Navier-Stokes system. However, due to the embedded pressure, the transport part is not the divergence-free vector field $u\cdot\nabla$, but the non-local skew-adjoint operator

$$\mathbb{P}(u\cdot\nabla)\mathbb{P}.$$

For a time-independent and divergence-free vector field $U$, the flow of that operator, i.e. the solution of $\partial_t\phi=\mathbb{P}(U\cdot\nabla)\mathbb{P}\phi$, is given by the Fourier Integral Operator

$$\phi(t)=\exp^{it\mathbb{P}(U\cdot D)\mathbb{P}}\phi_0,$$

where $U\cdot D=-iU\cdot\nabla$; under rather mild assumptions of regularity, this operator is self-adjoint (unbounded) on $L^2(\mathbb{R}^3;\mathbb{R}^3)$.

This flow is strikingly different from that of the vector field $V$, i.e. $\psi(t)=\exp^{it(U\cdot D)}\psi_0$. The difference induced by a projector "sandwich" is already striking among matrices. For example, in $\mathbb{R}^2$, let us consider a self-adjoint matrix $A$ and a self-adjoint projection $P$ onto a non-eigenvector of $A$

$$A=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix},\quad P=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=P^T,$$

then

$$e^{itA} = \begin{pmatrix} e^{it\lambda} & 0 \\ 0 & e^{it\mu} \end{pmatrix} \quad \text{while} \quad e^{itPAP} = \mathrm{Id} + \left( e^{it\frac{\lambda+\mu}{2}} - 1 \right) P.$$

The presence of the projector $P$ changes the evolution radically: the linear parts differ as $t \to 0$ (the later being the $P$-projection of the former) and the long-term behaviors are obviously completely different.

## 2.3 Decomposition of curl as a superposition of signed operators

Let us denote by $\mathbb{P}L^2$ the subspace of $L^2(\mathbb{R}^3)$ composed of vector fields that are divergence-free. As recalled in Subsection 2.1, the curl operator is self-adjoint and elliptic on $\mathbb{P}L^2$. We now want to decompose $\mathbb{P}L^2$ into an orthogonal direct sum of subspaces on which $\mathbf{C} = \mathrm{curl}$ is signed. The definition of these subspaces involves the following non-local operators associated with the "square root" of $\mathbb{P}$.

**Lemma 2.1.** *One can decompose $\mathbb{P} = \mathbb{Q}_+ + \mathbb{Q}_-$ where*

$$\mathbb{Q}_\pm = \frac{1}{2} \left( \mathbb{P} \pm \mathbf{C} |D|^{-1} \right). \tag{2.12}$$

*The operators $\mathbb{Q}_\pm$ satisfy $\mathbb{Q}_\pm^* = \mathbb{Q}_\pm = \mathbb{Q}_\pm^2$ and $\mathbb{Q}_+\mathbb{Q}_- = \mathbb{Q}_-\mathbb{Q}_+ = 0$.*

*Proof.* The main computation is

$$\mathbb{Q}_\pm^2 = \frac{1}{4} \Big( \mathbb{P}^2 + \mathbf{C}^2 |D|^{-2} \pm \left( \mathbb{P}\mathbf{C}|D|^{-1} + \mathbf{C}|D|^{-1}\mathbb{P} \right) \Big).$$

Applying (2.4) ensures the simplifications $\mathbb{P}^2 = \mathbb{P}$ and $[\mathbb{P}, \mathbf{C}|D|^{-1}] = 0$. As $\mathbb{P}\mathbf{C} = \mathbf{C}$, we obtain $\mathbb{Q}_\pm^2 = \mathbb{Q}_\pm$. The other properties follow immediately. □

Let us define the following signed curl operators:

$$\mathbf{C}_+ = \mathbf{C}\mathbb{Q}_+, \quad \mathbf{C}_- = -\mathbf{C}\mathbb{Q}_-. \tag{2.13}$$

These operators play a central role in this article.

**Lemma 2.2.** *One can decompose $\mathbf{C} = \mathbf{C}_+ - \mathbf{C}_-$. The operators $\mathbf{C}_\pm$ satisfy*

$$\mathbf{C}_\pm^* = \mathbf{C}_\pm \geq 0, \tag{2.14}$$

$$\mathbf{C}_+\mathbf{C}_- = \mathbf{C}_-\mathbf{C}_+ = 0, \tag{2.15}$$

$$\mathbf{C}_+ = |D|\mathbb{Q}_+ = \mathbb{Q}_+|D|\mathbb{Q}_+ = \mathbb{Q}_+\mathbf{C}\mathbb{Q}_+, \tag{2.16}$$

$$\mathbf{C}_- = |D|\mathbb{Q}_- = \mathbb{Q}_-|D|\mathbb{Q}_- = -\mathbb{Q}_-\mathbf{C}\mathbb{Q}_-. \tag{2.17}$$

*Proof.* Since $[\mathbb{P},\mathbf{C}]=0$ we have $[\mathbf{C},\mathbb{Q}_\pm]=0$ so that $\mathbf{C}\mathbb{Q}_\pm=\mathbf{C}\mathbb{Q}_\pm^2=\mathbb{Q}_\pm\mathbf{C}\mathbb{Q}_\pm$. The properties

$$\mathbf{C}=\mathbb{P}\mathbf{C}=\mathbb{Q}_+\mathbf{C}+\mathbb{Q}_-\mathbf{C}=\mathbf{C}_+-\mathbf{C}_-,\quad \mathbf{C}_+\mathbf{C}_-=\mathbf{C}_-\mathbf{C}_+=0,\quad \mathbf{C}_\pm^*=\mathbf{C}_\pm$$

follow from the corresponding ones for $\mathbb{Q}_\pm$. We also have

$$\begin{aligned}\mathbf{C}\mathbb{Q}_+&=\frac{1}{2}\big(\mathbf{C}\mathbb{P}+\mathbf{C}^2|D|^{-1}\big)=\frac{1}{2}\big(\mathbf{C}+\mathbb{P}|D|\big)=|D|\frac{1}{2}\big(\mathbb{P}+\mathbf{C}|D|^{-1}\big)=|D|\mathbb{Q}_+,\\ \mathbf{C}\mathbb{Q}_-&=\frac{1}{2}\big(\mathbf{C}\mathbb{P}-\mathbf{C}^2|D|^{-1}\big)=\frac{1}{2}\big(\mathbf{C}-\mathbb{P}|D|\big)=-|D|\frac{1}{2}\big(\mathbb{P}-\mathbf{C}|D|^{-1}\big)=-|D|\mathbb{Q}_-.\end{aligned}$$

Observing that $|D|\mathbb{Q}_\pm=\mathbb{Q}_\pm|D|\mathbb{Q}_\pm$ ensures the positivity of these operators. □

**Remark 2.3.** The previous lemma ensures that the respective restrictions of $\mathbf{C}_\pm$ to $\mathbb{Q}_\pm L^2$ both coincide with $|D|$. The kernel of $\mathbf{C}_\pm$ in $\mathbb{P}L^2$ is $\mathbb{Q}_\mp L^2$. In the orthogonal decomposition $\mathbb{P}L^2=\mathbb{Q}_+L^2\oplus\mathbb{Q}_-L^2$, the matrix of the curl operator is thus

$$\begin{pmatrix}|D| & 0\\ 0 & -|D|\end{pmatrix}$$

i.e. $\mathbf{C}=|D|\circ(\mathbb{Q}_+-\mathbb{Q}_-)$ is the diagonalization of the curl operator. This formula highlights a profound geometric connection between the curl and the pseudo-derivative $|D|$: both operators are images of one another by a symmetry of $\mathbb{P}L^2$. Note also that, by functional calculus, one may define fractional operators

$$\mathbf{C}_\pm^s=|D|^s\mathbb{Q}_\pm \tag{2.18}$$

for any $s\in\mathbb{R}$; the corresponding $\mathbf{C}^s=|D|^s\circ(\mathbb{Q}_++e^{si\pi}\mathbb{Q}_-)$ is however not self-adjoint if $s\in\mathbb{R}\backslash\mathbb{Z}$.

In view of these properties, we are led to introduce the following definition.

**Definition 2.1.** *A divergence-free vector field in $L^2(\mathbb{R}^3)$ is said to have positive (resp. negative) spin if it belongs to the subspace $\mathbb{Q}_+L^2$ (resp. $\mathbb{Q}_-L^2$). We say that $u$ is spin-definite if it has either positive or negative spin.*

According to Remark 2.3, a square integrable field $u$ has positive spin (up to a gradient field) if and only if $\mathbf{C}u=|D|\mathbb{P}u$ and negative spin if $\mathbf{C}u=-|D|\mathbb{P}u$. In general, a divergence-free vector field is not spin-definite; however, Lemma 2.1 ensures that any $\mathbb{P}L^2$ field is always the (direct) sum of two spin-definite vector fields with opposite spins.

**Remark 2.4.** The notion of spin-definite field has been known in physics literature under the denomination helical decomposition and dates back to Lesieur [61]. It has occasionally been used in theoretical and numerical investigations, e.g. Constantin & Majda [31], Cambon & Jacquin [18], Waleffe [76], Alexakis [1]. See also the discussion in Subsection 3.6 below.

**Example 2.1.** The spin-definite fields that are spectrally supported on a sphere are examples of Beltrami flows. If $\widehat{W}(\xi)$ is a distribution supported on $\{|\xi|=\lambda\}$, then $|D|W=\lambda W$; in this case, $W$ is spin-definite if and only if $\mathbf{C}W=\pm\lambda W$. In the periodic setting (or if one drops the square integrability on $\mathbb{R}^3$), the simplest non-trivial example is of the form

$$W^{\pm}_{\lambda,\phi}(x)=\cos(\lambda x\cdot\vec{e_1}+\phi)\vec{e_2}\mp\sin(\lambda x\cdot\vec{e_1}+\phi)\vec{e_3} \tag{2.19}$$

for some orthonormal basis $(\vec{e_1},\vec{e_2},\vec{e_3})$, a frequency $\lambda>0$ and a phase shift $\phi\in[0,2\pi)$; $W^{+}_{\lambda,\phi}$ is spin-positive and $W^{-}_{\lambda,\phi}$ is spin-negative. The fields $e^{-\nu t\lambda^2}W^{\pm}_{\lambda,\phi}(x)$ are exact solutions of the Navier-Stokes equation. They are a transient planar wave and a shear flow where the main direction of the shear rotates (resp. right- of left-handedly) as one travels along the axis $\mathbb{R}\vec{e_1}$. It is the hydrodynamical equivalent of a circularly polarized electromagnetic wave (for further results on Beltrami flows, see e.g. [27,44]).

Let us comment on the "microlocal" meaning of this definition. It is common knowledge that all complex vector spaces (even of higher dimension) are canonically oriented by the initial choice of one square root of $-1$ among the two choices $\pm i$. For $\xi\neq 0$, the subspace $\xi^{\perp}$ of $\mathbb{C}^3$ is of complex dimension 2; according to (2.4), the matrix $|\xi|^{-1}\mathbf{C}(\xi)\in\mathcal{M}_{3,3}(\mathbb{C})$ defined by (2.2) is a square root of the orthogonal projector $\mathbb{P}(\xi)=I-|\xi|^{-2}(\xi\otimes\xi)$ of $\mathbb{C}^3$ onto $\xi^{\perp}$. The pair $(\mathbb{P}(\xi),-i|\xi|^{-1}\mathbf{C}(\xi))$ defines a complex structure with conjugate coordinates $\mathbb{Q}_{\pm}(\xi)$. A field has positive spin if, at each frequency $\xi\in\mathbb{R}^3\backslash\{0\}$, the complex vector $\hat{u}(\xi)$ belongs to $\operatorname{ran}\mathbb{Q}_{+}(\xi)$.

**Lemma 2.3.** *For $\xi\in\mathbb{R}^3\backslash\{0\}$ and the matrix $\mathbf{C}(\xi)\in\mathcal{M}_{3,3}(\mathbb{C})$ defined by* (2.2)*, we have*

$$\ker\mathbf{C}(\xi)=\mathbb{C}\xi,\quad \operatorname{ran}\mathbf{C}(\xi)=\xi^{\perp}=\left\{\eta\in\mathbb{C}^3;\eta\cdot\xi=0\right\}, \tag{2.20}$$

$$\operatorname{Spec}\mathbf{C}(\xi)=\{0,\pm|\xi|\},\quad \ker\big(\mathbf{C}(\xi)\mp|\xi|\big)=\operatorname{ran}\mathbb{Q}_{\pm}(\xi). \tag{2.21}$$

*In particular,* $\operatorname{ran}\mathbb{Q}_{\pm}(\xi)$ *is one-dimensional if $\xi\neq 0$. One has $\mathbb{Q}_{-}(\xi)=\mathbb{Q}_{+}(-\xi)=\overline{\mathbb{Q}_{+}(\xi)}$. In local coordinates, the non-trivial eigenvectors are given, e.g. away from the* *axis* $\xi_2=\xi_3=0$, *by*

$$\delta_\pm(\xi)=\frac{1}{2|\xi|^2}\begin{pmatrix}\xi_2^2+\xi_3^2\\ -\xi_1\xi_2\pm i\xi_3|\xi|\\ -\xi_1\xi_3\mp i\xi_2|\xi|\end{pmatrix} \tag{2.22}$$

*and one has*

$$\ker\mathbb{Q}_\pm(\xi)=\mathrm{Span}_{\mathbb{C}}\{\xi,\delta_\mp(\xi)\},\quad \mathrm{ran}\,\mathbb{Q}_\pm(\xi)=\mathbb{C}\delta_\pm(\xi). \tag{2.23}$$

*Proof.* Let $\xi$ be in $\mathbb{R}^3\backslash\{0\}$. If for $a,b\in\mathbb{R}^3$ we have $i\xi\times(a+ib)=0$, we obtain that $\xi\times a=\xi\times b=0$, which is equivalent to $a\wedge\xi=b\wedge\xi=0$, i.e. $(a+ib)\in\mathbb{C}\xi$. On the other hand, the two-dimensional $\xi^\perp$ contains the two-dimensional range of $\mathbf{C}(\xi)$. Properties (2.21) follow from Lemma 2.2, which implies that

$$\mathbf{C}(\xi)=|\xi|\mathbb{Q}_+(\xi)-|\xi|\mathbb{Q}_-(\xi),$$

where $\mathbb{Q}_\pm(\xi)$ are the rank-one projections defined by

$$\mathbb{Q}_\pm(\xi)=\frac{1}{2}\big(\underbrace{I-|\xi|^{-2}(\xi\otimes\xi)}_{\text{real symmetric}}\pm\underbrace{|\xi|^{-1}\mathbf{C}(\xi)}_{\substack{\text{purely imaginary}\\ \text{anti-symmetric}}}\big).$$

The operators $\mathbb{Q}_\pm$ are the Fourier multipliers $\mathbb{Q}_\pm(D)$. The formula for $\delta_\pm(\xi)$ is obtained by choosing the first column of $\mathbb{Q}_\pm(\xi)$. □

**Remark 2.5.** The previous choice for $\delta_\pm(\xi)$ becomes singular along the axis $\xi_2=\xi_3=0$. To perform computations near this axis, one should instead choose another column of $\mathbb{Q}_\pm(\xi)$ as basis vectors.

With these local coordinates, the general expression of the Fourier reconstruction of a divergence-free vector field is

$$u(x)=\int_{\mathbb{R}^3}\big[\vartheta_+(\xi)\delta_+(\xi)+\vartheta_-(\xi)\delta_-(\xi)\big]e^{ix\cdot\xi}d\xi \tag{2.24}$$

for some spectral weights $\vartheta_\pm(\xi)\in\mathbb{C}$ defined almost everywhere and obtained in a unique way by the decomposition of $\hat{u}(\xi)$ on the basis $(\delta_+(\xi),\delta_-(\xi))$ of $\xi^\perp$. As $u$ is real-valued, the weights have to satisfy

$$\vartheta_\pm(-\xi)=\overline{\vartheta_\pm(\xi)}.$$

One can easily compute

$$\mathbf{C}u(x)=\int_{\mathbb{R}^3}|\xi|\big[\vartheta_+(\xi)\delta_+(\xi)-\vartheta_-(\xi)\delta_-(\xi)\big]e^{ix\cdot\xi}d\xi$$

and

$$|D|u(x)=\int_{\mathbb{R}^3}|\xi|\left[\vartheta_+(\xi)\delta_+(\xi)+\vartheta_-(\xi)\delta_-(\xi)\right]e^{ix\cdot\xi}d\xi.$$

A field $u$ has positive (resp. negative) spin if and only if $\vartheta_-\equiv 0$ (resp. $\vartheta_+\equiv 0$).

**Corollary 2.1.** *The spin is a chiral notion: the mirror image of a field with positive spin by a planar symmetry of $\mathbb{R}^3$ is a field of negative spin.*

*Proof.* Without impeding on the generality, one may assume that $v(x_1,x_2,x_3)=u(x_1,x_2,-x_3)$. It is then clear from (2.24) and (2.22) that the two fields $u$ and $v$ have opposite spins. □

The family of spin-definite vector fields is quite rich and appears to have a tubular jet-structure, where the sign of the spin reflects whether the forward motion is right- or left-handed. For example,

$$\begin{aligned}
u_1(x)=&-\frac{1}{2}\big(\cos(x_1-x_2)+2\sin(x_2+x_3)\big)\vec{e_1}\\
&-\frac{1}{2}\big(\cos(x_1-x_2)+\sqrt{2}\cos(x_2+x_3)\big)\vec{e_2}\\
&+\frac{\sqrt{2}}{2}\big(\sin(x_1-x_2)+\cos(x_2+x_3)\big)\vec{e_3},\\
u_2(x)=&-\frac{1}{5}\big(4\cos(x_1-2x_2)+5\sin(x_2+x_3)\big)\vec{e_1}\\
&-\frac{1}{10}\big(4\cos(x_1-2x_2)+5\sqrt{2}\cos(x_2+x_3)\big)\vec{e_2}\\
&+\frac{1}{10}\big(4\sqrt{5}\sin(x_1-2x_2)+5\sqrt{2}\cos(x_2+x_3)\big)\vec{e_3}
\end{aligned}$$

are divergence-free and have positive spin i.e. $\mathbf{C}u_j=|D|u_j$. They are illustrated in Fig. 2.

Note that $\mathbf{C}u_1=\sqrt{2}u_1$ so this example is a Beltrami flow; $\hat{u}_1$ is supported on the spectral sphere of radius $\sqrt{2}$. On the contrary, $\mathbf{C}(\mathbf{C}u_2\times u_2)\neq 0$ so this second example is not even a generalized Beltrami flow; $\hat{u}_2$ involves frequencies of magnitudes $\sqrt{2}$ and $\sqrt{5}$. However, both are clearly the superposition of two planar Beltrami waves of positive spin (i.e. flows from Example 2.1) that progress in different directions. Both flows have a similar structure: they swirl in a right-hand fashion, the center of each vortex is a zone of low pressure and high dissipation, the four hyperbolic corners of each cell (where the convection diverges) are axes of high pressure with minimal dissipation. Accounting for box-periodicity, these two examples display one single continuous vortex tube.

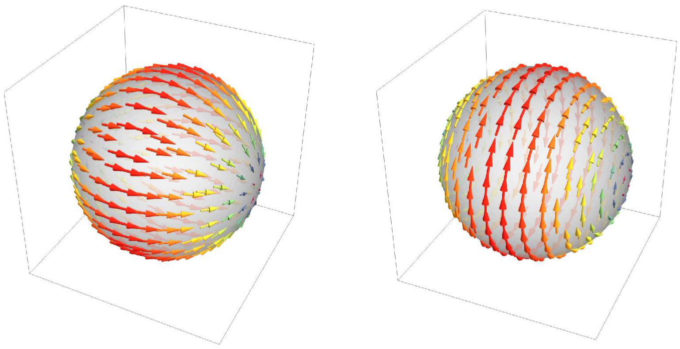

Figure 1: Real (left) and imaginary (right) parts of $\delta_+(\xi)$ on the unit sphere $|\xi|=1$. Multiplication by a suitable prefactor in $\mathbb{C}$ can rotate the axis (and the apparent singularity) of $\delta_+(\xi)$ to any point on the sphere (the axis for the real and imaginary parts are the same). One obtains $\delta_-(\xi)$ by complex conjugation of $\delta_+(\xi)$; therefore, the imaginary part of the Fourier field "flows" the other way around in $\mathbb{C}^3$.

If we superpose three or more planar Beltrami waves of positive spin, one can build more refined flows with positive spin that contain an intricate network of vortex tubes. The positive spin imposes that the movement remains exclusively right-handed at all scales. In the example shown in Fig. 3, four distinct regions (accounting for periodicity) of high vorticity appear to be disconnected, i.e. one generates vortex tubes of finite length.

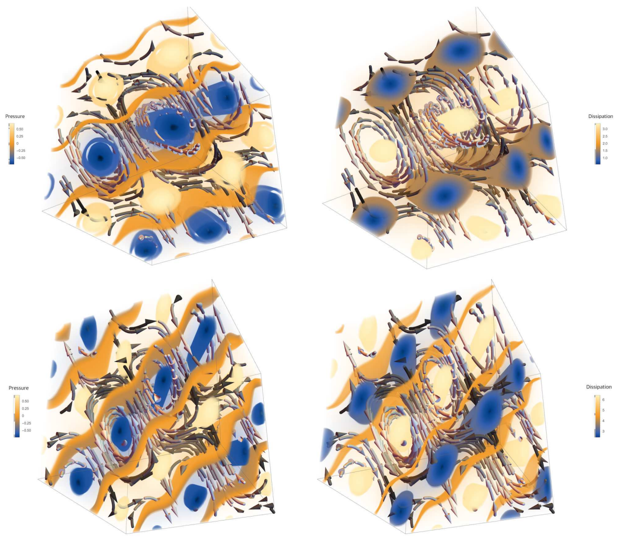

Figure 2: Two examples of non-trivial divergence-free fields, with positive spin in the periodic setting $x\in\mathbb{T}^3$. Above: field $u_1$ (Beltrami); below: field $u_2$ (not generalized Beltrami). Left: streamlines of $u_j(x)$ over the pressure field. Right: streamlines of $u_j(x)$ over the intensity of the dissipation field. Units are arbitrary. Observe the right-hand side motion.

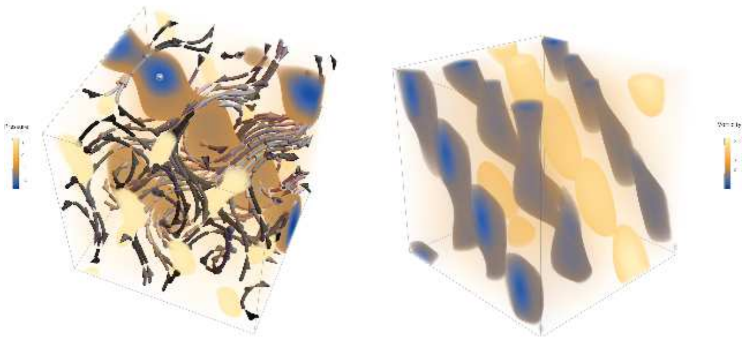

Figure 3: A third example of a non-trivial divergence-free field, with positive spin in the periodic setting $x\in\mathbb{T}^3$. The field is constructed as the superposition of three planar Beltrami waves with linearly independent directions. Left: streamlines of $u_j(x)$ over the pressure field. Right: intensity of the vorticity field. Units are arbitrary. The viewpoint is slightly different for better legibility. Four vortex filaments occur in the high-pressure region.

These examples suggest that the family of spin-definite flows is structurally simple (superposition of planar Beltrami waves) and yet quite rich. It is the building blocks of intricate vortex structures and deserves to be studied specifically, as we will now do.

**Remark 2.6.** The question of defining a microlocal notion of spin is legitimate[§], albeit non-trivial because the operators $\mathbb{C}\pm|D|$ are non-local. If $u$ is a divergence-free field, there exists a stream vector (i.e. vector potential) $\Psi$ such that $u=\mathbf{C}\Psi$. It is given by $\Psi=|D|^{-2}\mathbf{C}u+\nabla q$ where $\nabla q$ is an arbitrary irrotational component, e.g. $q=0$. If one is interested only in the local behavior of the flow near a point $x_0\in\mathbb{R}^3$, one could consider a smooth cut-off function $\chi\in\mathcal{D}(\mathbb{R}^3)$ supported in a ball of radius $r>0$ and such that $\chi(x)=1$ if $|x|\leq\frac{r}{2}$. The field

$$\tilde{u}=\mathbf{C}\left(\chi(x-x_0)\Psi(x)\right)=\chi(x-x_0)u(x)+\underbrace{\nabla\chi(x-x_0)\times\Psi}_{\substack{\text{recirculation around}\\ \text{the cutout zone}}}$$

remains divergence-free, coincides with $u$ on the ball $B(x_0,\frac{r}{2})$ and is compactly supported on $B(x_0,r)$. The two spin-definite components of $\tilde{u}$ can be seen as a local expression of the spin of the original field $u$ near $x_0$. However, the recirculation of $\tilde{u}$ near the edge of the cutoff zone may shadow the meaning of the spin at low frequencies, so a secondary microlocal cutout to isolate frequencies $|\xi|\gg r^{-1}$ may be necessary. We will not investigate this question further in this article.

[§]The notion of spin introduced in this article could then reasonably be called Fourier spin to insist on its global nature.

# 3 Two integral quantities preserved by the Navier-Stokes evolution

In this section, we revisit the classical energy balance for Navier-Stokes in the light of the aforementioned properties of the curl operator over $\mathbb{P}L^2 = \mathbb{Q}_+L^2 \oplus \mathbb{Q}_-L^2$.

## 3.1 Classical energy method

Leray's method was introduced in 1934 in the seminal article [58]. It consists in multiplying (2.8) by $u$ to get

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2\big\langle \mathbb{P}(\mathbf{C}u\times u), u\big\rangle_{L^2} + 2\nu\|\mathbf{C}u\|_{L^2}^2 = 0.$$

Since $\mathbb{P}^* u = \mathbb{P}u = u$, the non-linear term formally cancels out

$$\big\langle \mathbb{P}(\mathbf{C}u\times u), u\big\rangle = \langle \mathbf{C}u\times u, u\rangle = \det(\mathbf{C}u, u, u) = 0. \tag{3.1}$$

This leads to the classical energy balance

$$\|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\mathbf{C}u\|_{L^2}^2 dt' = \|u(0)\|_{L^2}^2,$$

which, truthfully, only holds for smooth solutions in the three-dimensional case. As Leray solutions are obtained as limits of compact sequences $(u_n)_{n\in\mathbb{N}}$ that satisfy the energy equality but converge to $u$ only weakly in $H^1$, Fatou's lemma implies

$$\|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\mathbf{C}u\|_{L^2}^2 dt' \le \|u(0)\|_{L^2}^2. \tag{3.2}$$

The possibility of anomalous dissipation, i.e. a strict inequality in (3.2), was envisioned by Onsager [67] and formalized e.g. in [37]. Onsager's conjecture on the minimal regularity assumption on $u$ that is necessary to ensure (3.2) was solved recently by the conjunction of the works of Isett [46] and Constantin *et al.* [32]. Soon afterwards, its importance was renewed by the construction of Buckmaster, Vicol [14] of wild (i.e. non-Leray) solutions of Navier-Stokes that defy any physically reasonable energy balance (their energy profile can even be prescribed arbitrarily) even though they belong to a reasonable function space $C_t^0(H_x^\sigma)$ for some $\sigma > 0$, typically $\sigma \simeq 2^{-18}$. For further details, see Subsection 3.4 below.

## 3.2 Conservation law associated with the signed curl

Let us define the following quantities:

$$N_\pm(u,t)=\left\|\mathbf{C}_\pm^{1/2}u(t)\right\|_{L^2}^2+2\nu\int_0^t\left\|\mathbf{C}_\pm^{3/2}u\right\|_{L^2}^2dt'. \tag{3.3}$$

Thanks to the results of Subsection 2.3, the sum $N_+(u,t)+N_-(u,t)$ is equivalent, for divergence-free vector fields, to the square of the norm of $u$ in $L_t^\infty\dot{H}_x^{1/2}\cap L_t^2\dot{H}_x^{3/2}$. Inspired by the negative sign of the curl on $\mathbb{Q}_-L^2$, let us now turn our attention to the Krein [54] "norm" $N_+(u,t)-N_-(u,t)$.

**Proposition 3.1.** *Let $u$ be a smooth solution of* (2.8)*. The following conservation law then holds:*

$$N_+(u,t)-N_-(u,t)=N_+(u,0)-N_-(u,0). \tag{3.4}$$

*Proof.* Thanks to the self-adjointness of the curl, one has $\langle\mathbf{C}u,\partial_t u\rangle_{L^2}=\langle\partial_t\mathbf{C}u,u\rangle_{L^2}$ pointwise in time. Let us multiply Eq. (2.8) by $2\mathbf{C}u$. We get

$$\frac{d}{dt}\big\langle\mathbf{C}u(t),u(t)\big\rangle_{L^2}+2\big\langle\mathbb{P}(\mathbf{C}u\times u),\mathbf{C}u\big\rangle_{L^2}+2\nu\big\langle\mathbf{C}^3u,u\big\rangle_{L^2}=0.$$

For smooth vector fields, the cubic term vanishes since

$$\big\langle\mathbb{P}(\mathbf{C}u\times u),\mathbf{C}u\big\rangle=\langle\mathbf{C}u\times u,\mathbf{C}u\rangle=\det(\mathbf{C}u,u,\mathbf{C}u)=0. \tag{3.5}$$

The lemma then follows, with $k=1$ or 3, from the identities

$$\mathbf{C}^k=(\mathbf{C}_+-\mathbf{C}_-)^k=\mathbf{C}_+^k+(-1)^k\mathbf{C}_-^k$$

and

$$\big\langle\mathbf{C}^k u,u\big\rangle_{L^2}=\left\|\mathbf{C}_+^{k/2}u\right\|_{L^2}^2+(-1)^k\left\|\mathbf{C}_-^{k/2}u\right\|_{L^2}^2,$$

which are a consequence of the diagonalization of the curl obtained in Subsection 2.3. □

For Leray solutions, the pendant of the conservation law (3.4) is not obvious. For example, it is not clear how $N_+(u,t)-N_-(u,t)$ compares to $N_+(u,0)-N_-(u,0)$ for all Leray solutions (see Subsection 3.5 below). However, if one considers the first singularity event, the following result expresses that singularities for the 3D Navier-Stokes equation can only occur as the result of a direct conflict of spin.

**Theorem 3.1.** *If $u$ is a smooth solution of Navier-Stokes on $[0,T^*)$ with a maximal lifetime $T^*<\infty$, then*

$$\limsup_{t\to T^*} N_\pm(u,t) = +\infty, \quad \lim_k \frac{N_+(u,t_k)}{N_-(u,t_k)} = 1 \tag{3.6}$$

*for some increasing sequence of times $t_k \to T^*$.*

An attempt at a physical interpretation of this result is proposed in Subsection 3.6 below.

*Proof.* As $u$ is smooth on $[0,T^*)$, the conservation law (3.4) holds for any $t<T^*$ and

$$|N_+(u,t) - N_-(u,t)| \le C_0$$

with e.g. $C_0 = |\int_{\mathbb{R}^3} \omega_0 \cdot u_0|$ according to (3.7) below. Thanks to [41], the sum $N_+(u,t) + N_-(u,t)$ and therefore at least one of the two norms $N_\pm(u,t)$ must diverge in lim-sup as $t\to T^*$. As the difference remains bounded, both norms $N_\pm(u,t)$ must diverge simultaneously. One obtains an increasing sequence $t_k \to T^*$ such that

$$N_+(u,t_k) \ge C_0 + k$$

and therefore $N_-(u,t_k) \ge k$. Then $|N_+(u,t_k)/N_-(u,t_k) - 1| \le C_0/k \to 0$. □

## 3.3 Helicity

Using the properties of $\mathbf{C}_\pm$ exposed in Subsection 2.3, one recovers the helicity

$$\mathcal{H}(t) = \int_{\mathbb{R}^3} \omega \cdot u = \langle (\mathbf{C}_+ - \mathbf{C}_-)u, u\rangle_{L^2} = \left\| \mathbf{C}_+^{1/2} u(t) \right\|_{L^2}^2 - \left\| \mathbf{C}_-^{1/2} u(t) \right\|_{L^2}^2. \tag{3.7}$$

More generally, the quantity $N_+ - N_-$ can be written as a conservation law for helicity

$$\begin{aligned} &N_+(u,t) - N_-(u,t) \\ &= \int_{\mathbb{R}^3} \omega \cdot u - 2\nu \int_0^t \int_{\mathbb{R}^3} \omega \cdot \Delta u = \int_{\mathbb{R}^3} \omega \cdot u + 2\nu \int_0^t \int_{\mathbb{R}^3} \nabla\omega \cdot \nabla u. \end{aligned} \tag{3.8}$$

The previous results imply that, for smooth solutions of the Euler equation, $\mathcal{H}(t)$ is conserved and that for smooth solutions of Navier-Stokes, the quantity (3.8) is invariant. The benefit of using the non-local diagonalization of the curl operator (i.e. the $\mathbf{C}_\pm$ operators) is that this new point of view isolates two distinct signed quantities $N_\pm$ in the balance of helicity, which is really not obvious in the right-hand side of (3.8). Helicity thus appears as a measure of the balance between the spin-definite components of $u$.

Let us also point out that (3.7) and Lemma 2.2 imply immediately

$$|\mathcal{H}(t)| \le \left\| \mathbf{C}_+^{1/2} u(t) \right\|_{L^2}^2 + \left\| \mathbf{C}_-^{1/2} u(t) \right\|_{L^2}^2 = \|u\|_{\dot{H}^{\frac{1}{2}}}^2. \tag{3.9}$$

One recovers the classical estimate

$$|\mathcal{H}(t)| = \left| \left\langle |D|^{-1/2} \mathbf{C} u, |D|^{1/2} u \right\rangle \right| \le C \|u\|_{H^{\frac{1}{2}}}^2,$$

which relies on the fact that the operator $|D|^{-1}\mathbf{C}$ is obviously bounded on $L^2$.

**Remark 3.1.** Note that, contrary to the phrasing of most proofs, the conservation of helicity does not result from a global cancellation of terms; instead, each term (and sub-term) given by the respective evolution equations for $u$ and $\omega$ vanishes on its own

$$\int_{\mathbb{R}^3} (\partial_t \omega) \cdot u + \nu \langle \nabla u, \nabla \omega \rangle_{L^2} = \left\langle (u \cdot \nabla) \omega, u \right\rangle_{L^2} + \left\langle (\omega \cdot \nabla) u, u \right\rangle_{L^2} = 0 + 0 = 0,$$
$$\int_{\mathbb{R}^3} (\partial_t u) \cdot \omega + \nu \langle \nabla u, \nabla \omega \rangle_{L^2} = \left\langle (u \cdot \nabla) u, \omega \right\rangle + \left\langle \nabla p, \omega \right\rangle = 0 + 0 = 0.$$

Indeed, using the self-adjointness of the curl twice, the identity (2.6) implies (either formally or for smooth $u$) that, on average, the convection term $(u \cdot \nabla) u$ is orthogonal to the vorticity

$$\begin{aligned} \left\langle w, (u \cdot \nabla) u \right\rangle_{L^2} &= \left\langle \operatorname{curl} u, (u \cdot \nabla) u \right\rangle_{L^2} = \left\langle u, \operatorname{curl}[(u \cdot \nabla) u] \right\rangle_{L^2} \\ &= \left\langle u, \operatorname{curl}[\omega \times u] \right\rangle_{L^2} = \langle \omega, \omega \times u \rangle_{L^2} = 0. \end{aligned} \tag{3.10}$$

If $u$ is divergence-free, one has the well known identity

$$\left\langle \omega, (u \cdot \nabla) u \right\rangle_{L^2} + \left\langle u, (u \cdot \nabla) \omega \right\rangle_{L^2} = -\langle \operatorname{div} u, u \cdot \omega \rangle = 0.$$

Combining this last identity with (3.10), one gets that the transport term $(u \cdot \nabla)\omega$ is, on average, orthogonal to the velocity field

$$\left\langle u, (u \cdot \nabla) \omega \right\rangle_{L^2} = 0. \tag{3.11}$$

Finally, as $\operatorname{div} \omega = 0$, and assuming enough decay at infinity

$$\left\langle (\omega \cdot \nabla) u, u \right\rangle_{L^2} = \frac{1}{2} \int_{\mathbb{R}^3} (\omega \cdot \nabla) |u|^2 = \frac{1}{2} \int_{\mathbb{R}^3} \operatorname{div} \left( |u|^2 \omega \right) = 0. \tag{3.12}$$

The identities (3.10)-(3.12) provide a simple derivation of the conservation of helicity for the Euler equation, which holds as long as it is legitimate to test the equation for vorticity (2.10) against $u$ itself.

The connection with helicity provides the following uniform integral bounds that imply that the two spin-definite components of $u$ must have, on average, a comparable size in $\dot{H}^{1/2}$. Note however that the last result of Subsection 3.2 provides a stronger insight at the time of first singularity.

**Proposition 3.2.** *For any Leray solution of Navier-Stokes, one has*

$$\int_0^\infty |\mathcal{H}(t)|^2 dt = \int_0^\infty \left( \left\| \mathbf{C}_+^{1/2} u(t) \right\|_{L^2}^2 - \left\| \mathbf{C}_-^{1/2} u(t) \right\|_{L^2}^2 \right)^2 dt \leq \frac{\|u_0\|_{L^2}^4}{8\nu} \tag{3.13}$$

*and*

$$\|u\|^2_{L_t^4 \dot{H}_x^{\frac{1}{2}}} = \int_0^\infty \left( \left\| \mathbf{C}_+^{1/2} u(t) \right\|_{L^2}^2 + \left\| \mathbf{C}_-^{1/2} u(t) \right\|_{L^2}^2 \right)^2 dt \leq \frac{\|u_0\|_{L^2}^2}{4\sqrt{2\nu}}. \tag{3.14}$$

*Proof.* The helicity is globally square-integrable in time because

$$\int_0^\infty |\mathcal{H}(t)|^2 dt \leq \int_0^\infty \|\omega(t)\|_{L^2}^2 \|u(t)\|_{L^2}^2 dt \leq \|\omega\|^2_{L_t^2 L_x^2} \|u\|^2_{L_t^\infty L_x^2} \leq \frac{\|u_0\|_{L^2}^4}{8\nu}.$$

For the last step, we used the energy inequality (3.2) and $ab \leq \frac{c^2}{4\beta}$ if $a,b,c,\beta > 0$ with $a + \beta b \leq c$. The full $L_t^4 \dot{H}_x^{1/2}$ norm of $u$ is controlled by interpolation between $L_t^\infty L_x^2$ and $L_t^2 \dot{H}_x^1$. □

## 3.4 Onsager's conjecture anew

In this section, we investigate briefly the minimal regularity that is required to ensure respectively the conservation of energy or the balance of helicity.

Onsager's famous conjecture [67] states that unless $u \in C_x^\alpha$ with $\alpha > \frac{1}{3}$, there may be an energy miscount at spectral infinity and that $u$ itself is not an admissible test function. The heuristic leading to that exponent is that the minimal regularity required to make sense of (3.1) consists in spreading one derivative across the three factors, hence the $C_x^{1/3}$ critical space. For the Euler equation, Constantin *et al.* [32] indeed proved the conservation of energy for $\alpha > \frac{1}{3}$ while Isett [46], using convex integration, recently showed its failure for $\alpha < \frac{1}{3}$ and solved the problem that had been open for 69 years.

For the Navier-Stokes equation, the conservation of energy for Leray solutions was proved by Serrin under an $L_t^q L_x^p$ assumption with $\frac{2}{q} + \frac{3}{p} = 1$, $p \geq 3$, which also implies smoothness (see criterion (1.12) above). Lions [62], Ladyzhenskaya [56] and Shinbrot [70] also proved the conservation of energy when

$$\frac{2}{q}+\frac{2}{p}\leq 1, \quad p\geq 4$$

so in particular for $L_t^4L_x^4$. This intermediary scaling ($\frac{2}{4}+\frac{3}{4}=\frac{5}{4}$) is of particular interest because it is both too low to be a guaranteed bound for all Leray solutions, but also too high to automatically imply the smoothness of the solution. Kukavica [55] weakened this assumption to a local $L^2_{t,x}$ bound on the pressure (recall that $p$ is obtained by a Calderón-Zygmund operator applied to $u\times u$). The last gap in scaling was closed by Cheskidov *et al.* [24], who proved that any Leray solution in $L^3([0,T];H^{5/6})$ conserves energy (see also Leslie and Shvydkoy [60]).

To understand why the space $L_t^3\dot{H}_x^{5/6}$ is exactly consistent with Onsager's heuristic, let us point out that, even with a loose Leibniz rule, one cannot expect to make sense of

$$\int_0^T\int_{\mathbb{R}^3}\det(\mathbf{C}u,u,u)=0 \quad \text{unless} \quad \int_0^T\int_K\left||D|^{\frac{1}{3}}u(t,x)\right|^3dxdt<\infty$$

for any compact subset $K\subset\mathbb{R}^3$. The Navier-Stokes (i.e. parabolic) scaling of $L_t^3\dot{W}_x^{1/3,3}$ is $\frac{2}{3}+\frac{3}{3}-\frac{1}{3}=1+\frac{1}{3}$, which matches that of $L_t^3\dot{H}_x^{5/6}\subset L_t^3\dot{W}_x^{1/3,3}$. The local integrability at this scale is ensured in the following way. For a triple $s_1+s_2+s_3\geq\frac{3}{2}$ with $s_j\geq 0$ and at least two non-zero regularity indices and $K\subset\mathbb{R}^3$ bounded, Hölder law and the Sobolev embeddings imply (see Constantin-Foias [30])

$$\begin{aligned}\int_K|(u\cdot\nabla)v\cdot w|&\leq c_K\|u\|_{L^{6/(3-2s_1)_+}}\|\nabla v\|_{L^{6/(3-2s_2)_+}}\|w\|_{L^{6/(3-2s_3)_+}}\\&\leq C_K\|u\|_{H^{s_1}}\|\nabla v\|_{H^{s_2}}\|w\|_{H^{s_3}}.\end{aligned}$$

At Onsager's scaling, the difficulty is that, when $u\in H^{5/6}$, then $\nabla u\in H^{-1/6}$ may fail to be locally integrable. Very elegantly, Cheskidov *et al.* [24] used a frequency decomposition $u=u_l+u_h$ with an arbitrary spectral threshold $\kappa$ and controlled the non-trivial terms with Bernstein's inequalities to transfer the singularity across the trilinear interaction, effectively loosening Leibniz's rule

$$\begin{aligned}\int_K|(u\cdot\nabla)u_l\cdot u|&\leq\int_K|(u_h\cdot\nabla)u_l\cdot u_h|+\int_K|(u_l\cdot\nabla)u_l\cdot u_h|+0\\&\leq\|u_h\|^2_{H^{\frac{1}{2}}}\|u_l\|_{H^{\frac{3}{2}}}+\|u_l\|_{H^{\frac{5}{6}}}\|u_l\|_{H^1}\|u_h\|_{H^{\frac{2}{3}}}\\&\lesssim\left(\kappa^{-\frac{1}{3}}\|u_h\|_{H^{\frac{5}{6}}}\right)^2\left(\kappa^{\frac{2}{3}}\|u_l\|_{H^{\frac{5}{6}}}\right)\end{aligned}$$

$$+\|u_l\|_{H^{\frac{5}{6}}}\left(\kappa^{\frac{1}{6}}\|u_l\|_{H^{\frac{5}{6}}}\right)\left(\kappa^{-\frac{1}{6}}\|u_h\|_{H^{\frac{5}{6}}}\right)$$
$$\lesssim \|u\|^3_{H^{\frac{5}{6}}}.$$

This computation ensures that the cancellation

$$\lim_{\kappa\to\infty}\int_{\mathbb{R}^3}(u\cdot\nabla)u_l\cdot u=0$$

is legitimate.

The other side of Onsager's conjecture for Navier-Stokes is still open. A historical breakthrough[¶] was achieved very recently by Buckmaster & Vicol [14,15] and with Colombo [11]. They showed that a small positive regularity $C^0_tH^\sigma_x$ with $\sigma\simeq 2^{-18}$ is not enough to prevent the existence of non-conservative viscous flows. They constructed flows in that class whose energy profile can be prescribed arbitrarily. Such strange flows are weak solutions of the Navier-Stokes equation but are not Leray solutions. While this pathology may seem to be of a purely mathematical nature, it does have a deep connection with turbulence [16,35]. These flows display a persistent low-frequency shadow of a vanishing high-frequency forcing, which was first observed for Euler [12]. This reverse cascade ends up to be stronger than what the viscosity can diffuse. In the absence of viscosity [13], one can even push the regularity of the pathologies to $\sigma=\frac{1}{2^-}$.

In the same spirit as Onsager's original conjecture, one can ask which minimal regularity will ensure the balance of the helicity, i.e. the conservation of $N_+-N_-$ defined above. Roughly speaking, in order to use $\mathbf{C}u$ as a test function and ensure (3.5), one would need to spread two derivatives across three factors, which would place the bar at $C^{2/3}_x$. This threshold is sometimes known as Onsager's conjecture for helicity. In the case of Euler's equation, Onsager's conjecture for helicity was essentially resolved by Cheskidov *et al.* [23]. Recently, Luigi de Rosa [36] investigated the possibility of splitting the assumption between $u\in L^{q_1}_tC^{\alpha_1}_x$ and $\operatorname{curl}u\in L^{q_2}_tW^{\alpha_2,1}_x$ with $\frac{2}{q_1}+\frac{1}{q_2}=1$ and $2\alpha_1+\alpha_2\geq 1$, which suggests that, for helicity, subtle plays with scaling are possible.

Because of the higher regularity threshold, the estimate in the case of Navier-Stokes is simpler than the one presented above. For example, having $u\in L^3_t(\dot{H}^{7/6}_x)$

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[¶] This article is the result of three years of reflection inspired by Vlad Vicol's remarkable talk at the CIRM of Marseille, in December 2018, which brought the two authors together. We are grateful to Prof. Vicol for his kind advice at that time and when we met again at the IHES in Gif-sur-Yvette in early 2020 [75]. Our meditation on the Beltrami waves that were used in the original proof [14] ultimately led us to Definition 2.1 of spin-definite fields and convinced us of the importance of this notion for hydrodynamics.

provides enough integrability

$$\int_0^t \int_{\mathbb{R}^3} |(u\cdot\nabla)u\cdot \mathbf{C}u| \le \|u\|_{L^3(L^9)} \|\nabla u\|_{L^3(L^{9/4})} \|\mathbf{C}u\|_{L^3(L^{9/4})} \\ \le \|u\|_{L^3(\dot{H}^{7/6})} \|\nabla u\|^2_{L^3(\dot{H}^{1/6})} \le \|u\|^3_{L^3(\dot{H}^{7/6})}$$

and thus legitimizes (3.5). Note that the scaling of $L^3_t(\dot{H}^{7/6}_x)$ is consistent with $\frac{1}{3}$ more derivative than that of $L^3_t(\dot{H}^{5/6}_x)$, which was critical for the conservation of energy. This scaling is thus coherent, in spirit, with Onsager's conjecture for helicity. The scaling of $L^3(\dot{H}^{7/6})$ differs from that of $L^\infty(L^2)\cap L^2(\dot{H}^1)$ by $\frac{7}{6}-\frac{2}{3}=\frac{1}{2}$ derivative; such a control is similar in scaling to $L^\infty(\dot{H}^{1/2})\cap L^2(\dot{H}^{3/2})$ and is therefore not known (and possibly not expected) for the most general Leray solutions.

**Remark 3.2.** Formally, there are two other known conserved integrals for Euler and Navier-Stokes: the momentum

$$P(t) = \int_{\mathbb{R}^3 \text{ or } \mathbb{T}^3} u(t,x)dx, \tag{3.15}$$

and the angular momentum

$$L(t) = \int_{\mathbb{R}^3 \text{ or } \mathbb{T}^3} x\times u(t,x)dx. \tag{3.16}$$

However, on $\mathbb{R}^3$, the decay of the velocity field that is necessary to define the momentum is not benign; for example, $P(t)$ is identically zero for any integrable divergence-free field. Similarly, the weighted integrability

$$u\in L^1\big((1+|x|)dx\big)$$

happens to be the critical one that cannot be propagated by the flow because the generic profile of a well localized flow decays exactly as $|x|^{-d-1}$ at infinity along most directions, which is due to the non-local effect of the pressure field (see Brandolese-Vigneron [10]). Therefore $P$ and $L$ are not the most useful conservation laws for flows on the full space $\mathbb{R}^3$.

## 3.5 Non-explosion criteria

The Navier-Stokes system can be written for the decomposition $u=u_+ + u_-$ where $u_\pm = \mathbb{Q}_\pm u$ are the two spin-definite components of $u$ (see Definition 2.1)

$$\begin{cases} \dfrac{\partial u_+}{\partial t} + \mathbb{Q}_+\big(\mathbf{C}u\times u\big) + \nu \mathbf{C}_+^2 u_+ = 0, \quad u_+(0) = \mathbb{Q}_+ u_0, \\ \dfrac{\partial u_-}{\partial t} + \mathbb{Q}_-\big(\mathbf{C}u\times u\big) + \nu \mathbf{C}_-^2 u_- = 0, \quad u_-(0) = \mathbb{Q}_- u_0. \end{cases} \tag{3.17}$$

However, the coupling of the two equations through $\mathbf{C}u\times u$ is highly intricate. The point of this section is to investigate how this coupling relates to issues of regularity.

Let us briefly explain the technical difficulty that one encounters when one attempts to generalize the conservation law (3.4) to the framework of Leray solutions. Let $u$ be a Leray solution of Navier-Stokes with $u_0\in H^{\frac{1}{2}}$ and $u_k$ a sequence of Galerkine approximations of $u$ that are spin-definite. It is common knowledge (see e.g. [57]) that the convergence of $u_k$ to $u$ holds in the strong topology of $L^\infty([0,T];H^{-1})\cap L^2([0,T],H^s)$ for any $T>0$ and any arbitrary but fixed value $s<1$. In particular, with $s=\frac{1}{2}$, one gets that

$$\lim\left\langle \mathbf{C}_\pm u_k(t),u_k(t)\right\rangle_{L^2}=\left\langle \mathbf{C}_\pm u(t),u(t)\right\rangle_{L^2}$$

for almost every $t\geq 0$. The proof of (3.4) can be reproduced for the smooth functions $u_k$ leading to

$$\begin{aligned}&\left\langle \mathbf{C}_+ u_k(t),u_k(t)\right\rangle_{L^2}+2\nu\int_0^t\left\langle \mathbf{C}_+^3 u_k,u_k\right\rangle_{L^2}+\left\langle \mathbf{C}_- u_k(0),u_k(0)\right\rangle_{L^2}\\&=\left\langle \mathbf{C}_+ u_k(0),u_k(0)\right\rangle_{L^2}+\left\langle \mathbf{C}_- u_k(t),u_k(t)\right\rangle_{L^2}+2\nu\int_0^t\left\langle \mathbf{C}_-^3 u_k,u_k\right\rangle_{L^2}\end{aligned}$$

i.e. $N_+(u_k,t)+N_-^0=N_+^0+N_-(u_k,t)$. However, in general, Fatou's lemma can only guarantee that

$$N_\pm(u,t)\leq\liminf_{k\to\infty}N_\pm(u_k,t),$$

which is not in our favor if we want to pass to the limit in the previous identity.

It is possible to circumvent this difficulty in the case of spin-definite solutions.

**Theorem 3.2.** *If $u$ is a Leray solution of Navier-Stokes stemming from $u_0\in H^{1/2}$, then $u$ is smooth as long as it remains spin-definite.*

*Proof.* Let $u$ be a Leray solution of Navier-Stokes with $u_0\in H^{1/2}$ and $T_1>0$ such that $u$ is spin-definite on $[0,T_1]$. Without impeding the generality, one can apply a planar symmetry if necessary and assume positive spin. It is common knowledge that $u$ is smooth on some non-trivial interval $[0,T_2]$. One considers

$$T=\sup\left\{t\in[0,T_1];u\text{ is smooth on }[0,t]\right\}\geq T_2.$$

Reasoning by contradiction, let us assume that $T\leq T_1$. Then (3.4) and the fact that $u$ has positive spin imply $N_+(u,t)=N_+(u,0)$ for all $t\in[0,T)$ i.e.

$$\|u(t)\|^2_{\dot H^{\frac{1}{2}}}+2\nu\int_0^t\|u\|^2_{\dot H^{\frac{3}{2}}}dt'=\left\|\mathbf{C}_+^{1/2}u(t)\right\|^2_{L^2}+2\nu\int_0^t\left\|\mathbf{C}_+^{3/2}u\right\|^2_{L^2}dt'=\|u_0\|^2_{\dot H^{\frac{1}{2}}}.$$

In particular, $u\in L^\infty([0,T);\dot{H}^{1/2})$ and [41] implies that $T$ cannot be a singular time; consequently, one has $T>T_1$. $\square$

**Remark 3.3.** According to (3.17), a solution $u$ remains spin-definite if and only if $\mathbb{P}(\mathbf{C}u\times u)$ has the same spin as $u$. In general, it is not clear that this property is propagated by the flow. At least, this is the case for generalized Beltrami flows, i.e. when $\mathbf{C}(\mathbf{C}u\times u)=0$ because then $\mathbb{P}(\mathbf{C}u\times u)=0$.

The assumptions of the previous statement are somewhat exorbitant. In the rest of this section, we investigate instead how the respective sizes of the spin-definite components $\mathbb{Q}_\pm u$ of a smooth solution $u$ are related to the emergence of singularities. However, as we only need smoothness to ensure the conservation of $N_+(u,t)-N_-(u,t)$, we will preserve some generality by assuming instead that $u$ is a Leray solution such that $|N_+(u,t)-N_-(u,t)|\le C_0$.

The following lemma will be useful to bound a pair of close numbers from a common lower bound.

**Lemma 3.1.** *For $\alpha,\beta\in\mathbb{R}_+$ and positive $C_j,\varepsilon_j$, we have*

$$C_0\ge|\alpha-\beta|\ge -C_1+\varepsilon_1\min(\alpha^{\varepsilon_2},\beta^{\varepsilon_2}) \implies \max(\alpha,\beta)\le C_0+\left(\varepsilon_1^{-1}(C_0+C_1)\right)^{\frac{1}{\varepsilon_2}},$$
$$C_0\ge|\alpha-\beta|\ge -C_1+\varepsilon_1\min(\log\alpha,\log\beta) \implies \max(\alpha,\beta)\le C_0+\exp\left(\varepsilon_1^{-1}(C_0+C_1)\right).$$

*Proof.* Since the assumption and the conclusion are symmetrical in $\alpha,\beta$, we may assume that $0\le\beta\le\alpha$. We then have

$$C_0+\beta+C_1\ge\alpha+C_1\ge\beta+\varepsilon_1\beta^{\varepsilon_2}$$

so

$$\varepsilon_1\beta^{\varepsilon_2}\le C_0+C_1 \quad \text{i.e.} \quad \beta\le\left(\varepsilon_1^{-1}(C_0+C_1)\right)^{\frac{1}{\varepsilon_2}}.$$

Consequently,

$$\max(\alpha,\beta)=\alpha\le\beta+C_0\le\left(\varepsilon_1^{-1}(C_0+C_1)\right)^{\frac{1}{\varepsilon_2}}+C_0.$$

The second claim can be obtained in a similar way. $\square$

At a time of first singularity, we have already mentioned (see the last result of Subsection 3.2) that $N_+(u,t)$ and $N_-(u,t)$ will simultaneously diverge to $+\infty$ and at the same rate. As Leray's flow goes on, the value of $N_+(u,t)-N_-(u,t)$ may be altered through each singular event. If the conflicts of spins were resolved (possibly in a non-unique way) by favoring one over the other, this could lead to a substantial drift. The following result quantifies, in this general setting, that even a logarithmic in-balance between the two spins is enough to deter singularities.

**Theorem 3.3.** *If $u$ is a Leray solution of Navier-Stokes such that*

$$
\begin{aligned}
C_0 &\geq |N_+(u,t) - N_-(u,t)| \\
&\geq -C_1 + \varepsilon \min\big(\log N_+(u,t), \log N_-(u,t)\big), \quad \forall t \in [0,T)
\end{aligned}
\tag{3.18}
$$

*for some constants $C_0, C_1, \varepsilon > 0$. Then $u$ remains smooth on $[0,T^*)$ with $T^* > T$.*

*Proof.* The previous lemma implies $N_\pm(u,t) \leq \exp[\varepsilon_1^{-1}(C_0+C_1)]$ on $[0,T)$ and in particular

$$
\|u\|^2_{L^\infty([0,T];\dot{H}^{1/2})} \leq \sup_{t\in[0,T]} N_+(u,t) + N_-(u,t) \leq 2\exp\left[\varepsilon_1^{-1}(C_0+C_1)\right]
$$

thus $u(T)$ is smooth thanks to [41] and the solution can be extended slightly beyond that point by the standard local well-posedness argument. □

## 3.6 Comparison with the dimension $n=2$

Let us conclude this section by a brief investigation of the case of dimension 2. The general expression of a divergence-free real-valued 2D vector field is

$$
\vec{u}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \vartheta(\xi) \overrightarrow{\delta(\xi)} e^{ix\cdot\xi} d\xi, \tag{3.19}
$$

where $\overrightarrow{\delta(\xi)} = \xi^\perp / |\xi| \in \mathbb{R}^2$ and $\vartheta(\xi) \in \mathbb{C}$ satisfies $\vartheta(\xi) = -\overline{\vartheta(\xi)}$; note the anti-Hermitian symmetry because of the anti-symmetric nature of $\overrightarrow{\delta(\xi)}$ in 2D. Exceptionally, we write the arrows as a visual cue to distinguish between vector and scalar quantities. It is obvious that

$$
|D|\vec{u} = \frac{1}{2\pi}\int_{\mathbb{R}^2} |\xi| \vartheta(\xi) \overrightarrow{\delta(\xi)} e^{ix\cdot\xi} d\xi, \quad \omega = \operatorname{curl} \vec{u} = \frac{1}{2\pi}\int_{\mathbb{R}^2} i|\xi|\vartheta(\xi) e^{ix\cdot\xi} d\xi. \tag{3.20}
$$

Even though $|D|\vec{u} \in \mathbb{R}^2$ is non-local while $\omega \in \mathbb{R}$ and is local, on the spectral side, the two operators are conjugate of one another

$$
\omega = i\overrightarrow{\delta(D)} \cdot |D|\vec{u}, \quad |D|\vec{u} = -i\overrightarrow{\delta(D)}(\omega). \tag{3.21}
$$

This property means that, in 2D, the structure of the curl is not as rich as its 3D analog (compare with Remark 2.3) and that, consequently, the conflict of two 2D contra-rotating vortices is not as profound as a conflict of spins in 3D.

In 2D, the resolution of such a conflict can only lead to a plain redistribution of the amplitude $\vartheta(\xi)$ in (3.19) and as the geometry of the equation does not leave any room for microlocal compensations, the flow either "has to" make a choice in favor of one direction of rotation or, in the case of perfect balance, let the viscosity eat up the singularity attempt. As we know, the Navier-Stokes equation is well-posed in 2D and the qualitative behavior of the vorticity [42] matches this heuristic.

In 3D, a redistribution among the pair of amplitudes $(\vartheta_+(\xi),\vartheta_-(\xi))$ in (2.24) also means favoring one spin over the other. However, the richer geometry provides the flow with a new way of "not choosing": it can amplify both spins simultaneously instead of letting the viscosity take over, which results in an escalating conflict of spin. Singularities, if they occur, are thus the byproduct of this unresolved microlocal game of chicken.

Of course, in the physical realm, the presence of sticky boundaries (i.e. with Dirichlet conditions) can produce numerous cases of spin imbalance and couplings, which gives the boundary layer its driving role in turbulence, regardless of whether or not true singularities or only quasi-singularities occur. One also has to wonder whether the late resolution of physically admissible extreme events of this type (i.e. conflict of spins that have escalated for a long time) favors subsequent cancellations, which could be the mechanism that drives intermittency.

**Remark 3.4.** We encourage the reader to consider the recent numerical simulations of Alexakis [1]. Our colleagues in physics study the energy and helicity fluxes of turbulent flows, by decomposing the influence of all possible interactions among spin-definite components. The numerical evidence hints at multiple non-trivial facts: the total energy flux can be split into three spin-related fluxes that remain independently constant in the inertial range; one of them amounts to 10% of the total energy flux and is a (hidden) backwards energy cascade, which subsists even in fully developed 3D turbulence. The helicity flux can be decomposed in a similar fashion into two fluxes that remain constant in the inertial range.

# 4 Critical determinants and non-local aspects of the regularity theory

In this section, we investigate the idea of computing energy estimates for $\mathbf{C}^\theta u$ with various values of $\theta>0$. Each computation leads to a determinant whose average sign plays a key role both in the growth of the regularity norms in the case of a potential blow-up and in their control as long as the flow remains smooth. It is worth insisting on the fact that geometric and non-local estimates seem to play a central role in the question of the regularity of the solutions of 3D Navier-Stokes. This study also leads to a geometric criterion for the uniqueness of Leray solutions and a slight variant of the Beale-Kato-Majda criterion.

## 4.1 Example: a geometric drive for enstrophy

Let us first investigate the well-known case of the enstrophy

$$\mathcal{E}(t)=\int_{\mathbb{R}^3}|\nabla u|^2=\int_{\mathbb{R}^3}|\omega|^2. \tag{4.1}$$

The equivalence between the two formulations follows e.g. from $\mathbf{C}^2=-\Delta$ for divergence-free fields.

Assuming regularity, one uses $\omega$ as a test function in the vorticity equation (2.10) and takes advantage of the cross-product structure of the nonlinearity, i.e.

$$(u\cdot\nabla)\omega-(\omega\cdot\nabla)u=\mathbf{C}(\omega\times u).$$

One is led to the following balance:

$$\|\omega(t)\|_{L^2}^2+2\nu\int_0^t\|\mathbf{C}\omega(t')\|_{L^2}^2dt'+2\int_0^t\int_{\mathbb{R}^3}\det\big(u,\mathbf{C}u,\Delta u\big)\,dxdt'=\|\omega_0\|_{L^2}^2. \tag{4.2}$$

Note that

$$\langle\omega\times u,\mathbf{C}\omega\rangle_{L^2}=\int_{\mathbb{R}^3}\det\big(u,\mathbf{C}u,\Delta u\big)\,dx.$$

This computation is a typical example involving a critical determinant: the average sign of the determinant is responsible for the variations of the norms measuring the regularity of the flow, here in terms of enstrophy. When $\nu=0$, i.e. for (smooth) 3D Euler flows, the space-time average of $\det(u,\mathbf{C}u,\Delta u)$ is the sole geometrical drive of the variations of enstrophy.

The best known a priori upper bounds for enstrophy is a Riccati-type control by Lu & Doering [63]

$$\mathcal{E}'(t)\le C\mathcal{E}^3(t). \tag{4.3}$$

It is obtained by estimating the critical determinant mentioned above and diverges in finite time. For advanced numerical experiments on the growth of enstrophy for 3D viscous flows, see e.g. [5, 49, 50] and the numerous references to the numerical literature therein.

An immediate corollary of (4.2) is a geometric criterion for regularity

$$\int_0^T\int_{\mathbb{R}^3}\det(u,\mathbf{C}u,\Delta u)dxdt\ge 0 \quad\Longrightarrow\quad u\in L^\infty\big([0,T];\dot{H}^1\big)\cap L^2\big([0,T];\dot{H}^2\big). \tag{4.4}$$

For example, one recovers in this manner that all irrotational flows are smooth because the critical determinant vanishes identically (they are indeed the gradients of solutions of the heat equation).

As a slightly more involved application, let us investigate the case of 3D fields with 2D symmetry, i.e.

$$v=\begin{pmatrix} v_1(x_1,x_2)\\ v_2(x_1,x_2)\\ 0\end{pmatrix},\quad \omega=\begin{pmatrix}0\\0\\ \partial_1 v_2-\partial_2 v_1\end{pmatrix}.$$

For such a field, one has

$$\det(v,\mathbf{C}v,\Delta v)=-\begin{vmatrix}0 & v_1 & \Delta v_1\\ 0 & v_2 & \Delta v_2\\ \partial_1 v_2-\partial_2 v_1 & 0 & 0\end{vmatrix}=(\partial_2 v_1-\partial_1 v_2)(v_1\Delta v_2-v_2\Delta v_1).$$

If one introduces the stream function $\psi(x_1,x_2)$ such that $v_1=\partial_2\psi$ and $v_2=-\partial_1\psi$

$$\det(v,\mathbf{C}v,\Delta v)=\big(-(\partial_2\psi)(\partial_1\Delta\psi)+(\partial_1\psi)(\partial_2\Delta\psi)\big)\Delta\psi,$$

which has no particular reason to vanish but leads to a global cancellation for any $t>0$

$$\begin{aligned}\int_{\mathbb{R}^3}\det(v,\mathbf{C}v,\Delta v)&=\frac{1}{2}\Big(\big\langle\partial_2\psi,-\partial_1(\Delta\psi)^2\big\rangle_{L^2}+\big\langle\partial_1\psi,\partial_2(\Delta\psi)^2\big\rangle_{L^2}\Big)\\ &=\frac{1}{2}\big\langle\partial_1\partial_2\psi-\partial_2\partial_1\psi,(\Delta\psi)^2\big\rangle_{L^2}=0.\end{aligned}$$

In particular, (4.4) implies the global regularity of such solutions, which has been known since Leray [58]. As the balance law (4.2) also holds for smooth solutions of the Euler equation, the previous computation implies the conservation of enstrophy for smooth 2D Euler flows.

**Remark 4.1.** For a general 3D divergence-free flow $u$, invoking the vector potential $u=\mathbf{C}\Psi$ and computing the critical determinant in (4.2) brings out 288 terms involving the product of a first, second and third order derivative of the components of $\Psi$, with no obvious compensations through space averages. This remark illustrates the huge gap in complexity between 2D and 3D flows.

## 4.2 General case

Let us go back to the Navier-Stokes equation written in the form (2.8). The weak form of the nonlinear term is, as mentioned in (2.9), a determinant

$$\langle\partial_t u,w\rangle_{L^2}+\nu\big\langle\mathbf{C}^2u,w\big\rangle_{L^2}+\int_{\mathbb{R}^3}\det(\mathbf{C}u,u,w)dx=0 \tag{4.5}$$

for all divergence-free test fields $w$. Assuming $u$ is smooth, one can collect various balance laws for Navier-Stokes by choosing $w$ appropriately as a function of $u$. The two standard choices are either $w=u$, which gives Leray's energy equality for smooth solutions, and $w=\mathbf{C}u$, which we explored in Subsection 3.2 and which relates to the balance of helicity. Taking $w=\mathbf{C}^2u$ leads to (4.2) and the balance of enstrophy with, this time, a non-trivial critical determinant.

Leray's energy identity can be extended given a first integral of the flow, i.e. $\alpha$ such that $(u\cdot\nabla)\alpha=0$. Then $w=\alpha u$ is a divergence-free field and we have

$$\langle \partial_t u,\alpha u\rangle_{L^2}+\nu\left\langle \mathbf{C}^2 u,\alpha u\right\rangle_{L^2}=0,$$

and thus

$$\langle \alpha u,u\rangle_{L^2}-\int_0^t\langle \dot{\alpha}u,u\rangle_{L^2}+2\nu\int_0^t\left\langle \mathbf{C}^2 v,\alpha u\right\rangle_{L^2}=\left\langle \alpha(0)u_0,u_0\right\rangle_{L^2}. \tag{4.6}$$

For example, with $\alpha(t)=e^{-2\lambda t}$, we get a family of conservation laws indexed by $\lambda>0$

$$\begin{aligned} &e^{-2\lambda t}\|u(t)\|_{L^2}^2+2\lambda\int_0^t e^{-2\lambda t'}\|u(t')\|_{L^2}^2dt' \\ &\quad +2\nu\int_0^t e^{-2\lambda t'}\|\nabla u(t')\|_{L^2}^2dt'=\|u_0\|_{L^2}^2, \end{aligned} \tag{4.7}$$

which is a weighted time-integral (gauge transform) of the classical energy balance that puts $t'\sim 1/(2\lambda)$ into focus. Similarly, for $w=e^{-2\lambda t}\mathbf{C}u$, one gets a variant of (3.4)

$$\begin{aligned} &e^{-2\lambda t}\langle u,\mathbf{C}u\rangle_{L^2}+2\lambda\int_0^t e^{-2\lambda t'}\langle u,\mathbf{C}u\rangle_{L^2} \\ &\quad +2\nu\int_0^t e^{-2\lambda t'}\left\langle \mathbf{C}^2u,\mathbf{C}u\right\rangle_{L^2}=\langle u_0,\mathbf{C}u_0\rangle_{L^2}. \end{aligned} \tag{4.8}$$

Note that

$$\begin{aligned} \langle u,\mathbf{C}u\rangle_{L^2}&=\left\|\mathbf{C}_+^{1/2}u\right\|_{L^2}^2-\left\|\mathbf{C}_-^{1/2}u\right\|_{L^2}^2, \\ \left\langle \mathbf{C}^2u,\mathbf{C}u\right\rangle_{L^2}&=\left\|\mathbf{C}_+^{3/2}u\right\|_{L^2}^2-\left\|\mathbf{C}_-^{3/2}u\right\|_{L^2}^2. \end{aligned}$$

Let us now investigate the more interesting case where $w=\mathbf{C}_\pm u$.

**Proposition 4.1.** *If $u$ is a smooth solution of Navier-Stokes, one has the following balance laws:*

$$\left\|\mathbf{C}_\pm^{1/2}u(t)\right\|^2+2\nu\int_0^t\left\|\mathbf{C}_\pm^{3/2}u\right\|^2dt'$$

$$+\int_0^t\int_{\mathbb{R}^3}\det(\mathbf{C}u,u,|D|u)dxdt' = \big\|\mathbf{C}_\pm^{1/2}u_0\big\|^2. \tag{4.9}$$

*Note that the critical determinant is identical in both cases, which is a new proof of* (3.4)*. One has also*

$$\|u(t)\|^2_{\dot H^{\frac12}} + 2\nu\int_0^t \|u(t')\|^2_{\dot H^{\frac12}}dt' + \int_0^t\int_{\mathbb{R}^3}\det(\mathbf{C}u,u,|D|u)dxdt' = \|u_0\|^2_{\dot H^{\frac12}}. \tag{4.10}$$

*Proof.* The only non-trivial point is the critical determinant. One has

$$\det(\mathbf{C}u,u,\mathbf{C}_+u)dx = \det(\mathbf{C}u,u,\mathbf{C}u+\mathbf{C}_-u) = \det(\mathbf{C}u,u,\mathbf{C}_-u)$$

and thus

$$\det(\mathbf{C}u,u,|\mathbf{C}|u)dx = \det(\mathbf{C}u,u,\mathbf{C}_+u) + \det(\mathbf{C}u,u,\mathbf{C}_-u) = 2\det(\mathbf{C}u,u,\mathbf{C}_\pm u).$$

Finally, since $|\mathbf{C}| = |D|\mathbb{P}$, we can replace $|\mathbf{C}|u$ by $|D|u$. Subtracting the two identities gives (3.4), while adding them up provides the last claim. ☐

**Remark 4.2.** Thanks to Lemma 2.2, one can rewrite this critical determinant as

$$\begin{aligned}\det(\mathbf{C}u,u,|D|u) &= \det\big((\mathbf{C}_+-\mathbf{C}_-)u,u,(\mathbf{C}_++\mathbf{C}_-)u\big)\\ &= -2\det(u,\mathbf{C}_+u,\mathbf{C}_-u).\end{aligned} \tag{4.11}$$

This determinant is a geometrical drive for the growth of the $\dot H^{1/2}$ norm. Among possible cancellations, it vanishes for Beltrami waves ($\mathbf{C}u$ proportional to $u$), for flows spectrally supported on a sphere ($|D|u$ proportional to $u$) and, most importantly, for spin-definite flows ($\mathbf{C}u$ proportional to $|D|u$).

To handle fractional powers, it is simplest to split the spin-definite components to avoid problems with the lack of self-adjointness. Using $w = \mathbf{C}_\pm^{2\theta}u$ for some $\theta > 0$ and the properties established in Subsection 2.3, one gets

$$\begin{aligned}&\big\|\mathbf{C}_\pm^\theta u(t)\big\|^2_{L^2} + 2\nu\int_0^t\big\|\mathbf{C}_\pm^{\theta+1}u(t')\big\|^2_{L^2}dt'\\ &\quad+2\int_0^t\int_{\mathbb{R}^3}\det\big(\mathbf{C}u,u,\mathbf{C}_\pm^{2\theta}u\big)dxdt' = \big\|\mathbf{C}_\pm^\theta u_0\big\|^2_{L^2}.\end{aligned} \tag{4.12}$$

This time, the cancellation takes the form

$$\det\big(\mathbf{C}u,u,\mathbf{C}_+^{2\theta}u\big) + \det\big(\mathbf{C}u,u,\mathbf{C}_-^{2\theta}u\big) = \det\big(\mathbf{C}u,u,|D|^{2\theta}u\big).$$

For integer values of $2\theta$, one has

$$\det\left(\mathbf{C}u,u,\mathbf{C}_+^{2\theta}u\right)-\det\left(\mathbf{C}u,u,\mathbf{C}_-^{2\theta}u\right)=\det\left(\mathbf{C}u,u,\mathbf{C}^{2\theta}u\right).$$

The determinants $\det(\mathbf{C}u,u,\mathbf{C}_\pm^{2\theta}u)$ are the geometric drive for the growth of the $\dot{H}^\theta$ norm of the spin-definite components of $u$. In particular, we have proven the following statement.

**Proposition 4.2.** *If u is a smooth solution of Navier-Stokes, one has the following balance laws:*

$$\begin{aligned}&\|u(t)\|_{\dot{H}^\theta}^2+2\nu\int_0^t\|u(t')\|_{\dot{H}^{\theta+1}}^2dt'\\&\quad+2\int_0^t\int_{\mathbb{R}^3}\det\left(\mathbf{C}u,u,|D|^{2\theta}u\right)dxdt'=\|u_0\|_{\dot{H}^\theta}^2\end{aligned}\tag{4.13}$$

*for any $\theta>0$, and the spin-definite variants* (4.12)*; when $\theta\in\mathbb{N}$, one can replace $|D|^{2\theta}$ by $(-\Delta)^\theta$. For any $n\in\mathbb{N}^*$, one has also*

$$\begin{aligned}&N_+^n(u,t)-N_-^n(u,t)+2\int_0^t\int_{\mathbb{R}^3}\det(\mathbf{C}u,u,\mathbf{C}^nu)dxdt'\\&=\|u_0^+\|_{\dot{H}^{\frac{n}{2}}}^2-\|u_0^-\|_{\dot{H}^{\frac{n}{2}}}^2,\end{aligned}\tag{4.14}$$

*where the definition* (3.3) *is extended by*

$$N_\pm^n(u,t)=\|u^\pm(t)\|_{\dot{H}^{\frac{n}{2}}}^2+2\nu\int_0^t\|u^\pm(t')\|_{\dot{H}^{\frac{n}{2}+1}}^2dt'\tag{4.15}$$

*and $u_0^\pm=\mathbb{Q}_\pm u_0$.*

The case $\theta=0$ is of special interest because, as for $\theta=\frac{1}{2}$, both critical determinants coincide.

**Proposition 4.3.** *If u is a smooth solution of Navier-Stokes, one has the following balance laws:*

$$\begin{aligned}&\|u^\pm(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla u^\pm(t')\|_{L^2}^2dt'\\&\quad\pm2\int_0^t\int_{\mathbb{R}^3}\det(\mathbf{C}u,u^-,u^+)dxdt'=\|u_0^\pm\|_{L^2}^2.\end{aligned}\tag{4.16}$$

*In particular, the balance between the spin-definite components is ruled by*

$$\begin{aligned}&N_+^0(u,t)-N_-^0(u,t)+4\int_0^t\int_{\mathbb{R}^3}\det(\mathbf{C}u,u^-,u^+)dxdt'\\&=\|u_0^+\|_{L^2}^2-\|u_0^-\|_{L^2}^2.\end{aligned}\tag{4.17}$$

*Proof.* Using $u^{\pm}$ as a test function, one has

$$\det(\mathbf{C}u,u,u^{\pm}) = \det(\mathbf{C}u,u^{+}+u^{-},u^{\pm}) = \pm\det(\mathbf{C}u,u^{-},u^{+}).$$

This completes the proof. $\square$

## 4.3 Applications

Regularity on $[0,T]\times\mathbb{R}^3$ is assured when the following inequality holds:

$$\exists\theta \geq \frac{1}{2}, \quad \int_0^T \int_{\mathbb{R}^3} \det\big(\mathbf{C}u,u,|D|^{2\theta}u\big)\,dxdt \geq 0. \tag{4.18}$$

Of course, giving sense to the previous integral requires some a priori knowledge that the solution is smooth. However, if the inequality is satisfied for some $\theta \geq \frac{1}{2}$ along a sequence of, e.g., Galerkine approximations that converge to a given Leray solution $u$, then $u$ enjoys a uniform bound in $L^\infty([0,T];\dot{H}^\theta)$ and therefore, according to [41], is smooth on $[0,T]$. To avoid making an assumption on approximating sequences, one can require instead the slightly stronger property on a general Leray solution

$$\exists\theta \geq \frac{1}{2}, \quad \text{a.e.} \quad t \in [0,T] \quad \int_{\mathbb{R}^3} \det\big(\mathbf{C}u,u,|D|^{2\theta}u\big)\,dx \geq 0 \tag{4.19}$$

with $u_0 \in H^\theta$. Then one can proceed as in the proof of Theorem 3.2 and show that the first time of singularity cannot occur before $T$.

### 4.3.1 Uniqueness criterion based on critical determinants

In this section, we revisit the weak-strong uniqueness result and investigate how the associated stability estimate can be expressed in a more geometric way. We refer the reader to [43] and the references therein for an in-depth discussion of weak-strong uniqueness for Navier-Stokes.

Let us consider two Leray solutions $u_j$, $j=1,2$ of the incompressible Navier-Stokes equation (2.8) and their difference $\delta = u_1 - u_2$. Using the energy inequality for each, one gets

$$\begin{aligned} &\|\delta(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla\delta\|_{L^2}^2 \\ \leq\ &\|u_1(0)\|_{L^2}^2 + \|u_2(0)\|_{L^2}^2 - 2\left(\langle u_1(t),u_2(t)\rangle_{L^2} + 2\nu\int_0^t \langle\nabla u_1,\nabla u_2\rangle_{L^2}\right). \end{aligned}$$

The standard argument in favor of weak-strong uniqueness consists in observing that each equation tested against (a regularized version of) the other field ultimately gives

$$\begin{aligned}&\big\langle u_1(t),u_2(t)\big\rangle_{L^2}+2\nu\int_0^t\langle\nabla u_1,\nabla u_2\rangle_{L^2}\\=&\big\langle u_1(0),u_2(0)\big\rangle_{L^2}-\int_0^t\big\langle(\delta\cdot\nabla)u_1,\delta\big\rangle_{L^2},\end{aligned}$$

which implies

$$\|\delta(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla\delta\|_{L^2}^2\le\|\delta(0)\|_{L^2}^2+2\int_0^t\big\langle(\delta\cdot\nabla)u_1,\delta\big\rangle_{L^2} \tag{4.20}$$

and, with Gronwall's inequality

$$\|\delta(t)\|_{L^2}^2\le\|\delta(0)\|_{L^2}^2\exp\left(\int_0^t\|\nabla u_1(t')\|_{L^\infty}dt'\right). \tag{4.21}$$

This control is enough to ensure the uniqueness of all Leray solutions stemming from $u_1(0)$ as long as $u_1$ remains smooth. It remains nonetheless quite crude.

Instead, using (2.9), let us rewrite the crucial step in a more geometric way

$$\begin{aligned}&\big\langle u_1(t),u_2(t)\big\rangle_{L^2}+2\nu\int_0^t\langle\nabla u_1,\nabla u_2\rangle_{L^2}\\&\quad+\int_0^t\det(\mathbf{C}u_1,u_1,u_2)+\det(\mathbf{C}u_2,u_2,u_1)\\=&\big\langle u_1(0),u_2(0)\big\rangle_{L^2}.\end{aligned}$$

Observe that

$$\det(\mathbf{C}u_1,u_1,u_2)+\det(\mathbf{C}u_2,u_2,u_1)=\det(\mathbf{C}\delta,u_1,u_2)=(u_1\times u_2)\cdot\mathbf{C}\delta.$$

As $\operatorname{div}\delta=0$, one has $\|\mathbf{C}\delta\|_{L^2}=\|\nabla\delta\|_{L^2}$ and one can completely absorb the offending derivative

$$\|\delta(t)\|_{L^2}^2\le\|\delta(0)\|_{L^2}^2+\frac{1}{2\nu}\int_0^t\|u_1\times u_2\|_{L^2}^2. \tag{4.22}$$

In particular, we have the following statement.

**Theorem 4.1.** *If $u_1$ and $u_2$ are two Leray solutions such that*

$$\|u_1\times u_2\|_{L^2}^2\le\gamma(t)\|u_1-u_2\|_{L^2}^2\quad\textit{with}\quad\gamma\in L^1([0,T]), \tag{4.23}$$

*then for any $t\in[0,T]$, one has*

$$\|\delta(t)\|_{L^2}^2\le\|\delta(0)\|_{L^2}^2\exp\left(\int_0^T\gamma(t')dt'\right). \tag{4.24}$$

For example, if $u_1 \in L^2_t L^\infty_x$ we can apply this result because $u_1 \times u_2 = -u_1 \times \delta$ and we recover a well-known case of weak-strong uniqueness. However, the geometric assumption (4.23) is a priori weaker if, for example, the two fields tend to line up when one of them grows unbounded.

### 4.3.2 A variant of BKM based on critical determinants

Formally, the standard argument for the Beale-Kato-Majda criterion [8, 68] consists in writing the equation for vorticity (2.10) in weak form against $\omega$ itself, which gives

$$\|\omega(t)\|^2_{L^2} + 2\nu \int_0^t \|\mathbf{C}\omega\|^2_{L^2} = \|\omega_0\|^2 + 2\int_0^t \big\langle (\omega\cdot\nabla)u, \omega\big\rangle_{L^2}$$

and thus, in particular

$$\|\omega(t)\|^2_{L^2} \leq \|\omega_0\|^2 + 2\int_0^t \|\omega\|^2_{L^2}\|\omega\|_{L^\infty}.$$

Combined with Gronwall lemma, this ensures that the solution (of either Euler or Navier-Stokes) remains smooth as long as

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < +\infty. \tag{4.25}$$

Let us present a variant of this computation, inspired by the previous critical determinants.

Our starting point is similar, but we write the non-linear term slightly differently

$$\|\omega(t)\|^2_{L^2} + 2\nu \int_0^t \|\mathbf{C}\omega\|^2_{L^2} + 2\int_0^t \langle \omega\times u, \mathbf{C}\omega\rangle_{L^2} = \|\omega_0\|^2_{L^2}.$$

Now, if one splits $\nu = \nu_1 + \nu_2$ with arbitrary values $\nu_j > 0$, one gets

$$\begin{aligned} &\|\omega(t)\|^2_{L^2} + 2\nu_1 \int_0^t \left\| \mathbf{C}\omega + \frac{1}{2\nu_1}(\omega\times u)\right\|^2_{L^2} + 2\nu_2 \int_0^t \|\mathbf{C}\omega\|^2_{L^2} \\ = &\frac{1}{2\nu_1}\int_0^t \|\omega\times u\|^2_{L^2} + \|\omega_0\|^2_{L^2}. \end{aligned}$$

In particular, one obtains an estimate that is now specific to Navier-Stokes

$$\|\omega(t)\|^2_{L^2} + 2\nu_2 \int_0^t \|\mathbf{C}\omega\|^2_{L^2} \leq \|\omega_0\|^2_{L^2} + \frac{1}{2\nu_1}\int_0^t \|\omega\times u\|^2_{L^2}. \tag{4.26}$$

Consequently, as

$$\|\omega\times u\|_{L^2}^2\le\|\omega\|_{L^2}^2\|u\|_{L^\infty}^2,$$

Gronwall's lemma ensures the regularity of the flow on $[0,T]$ provided that

$$\int_0^T\|u(t)\|_{L^\infty}^2dt<+\infty. \tag{4.27}$$

This condition is the endpoint of the Prodi-Serrin $L_t^qL_x^p$ family with $\frac{2}{q}+\frac{3}{p}=1$.

Let us finally point out that an interesting connection between the Beale-Kato-Majda criterion and the theory of turbulence was established by Cheskidov & Shvydkoy [26], who showed that a condition

$$\int_0^T\left\|\omega_{\le Q(t)}(t)\right\|_{B^0_{\infty,\infty}}dt<\infty \tag{4.28}$$

ensures the regularity of the flow on $[0,T]$. The dynamic wave-number $2^{Q(t)}$ separates high-frequency modes where viscosity prevails over the non-linear term from the low-frequency modes where the Euler dynamics is dominant. It is defined by

$$Q(t)=\min\left\{q\in\mathbb{N};\,\forall p>q,\,2^{-p}\|\Delta_pu\|_{L^\infty}<c_0\nu\right\}. \tag{4.29}$$

The constant $c_0>0$ is absolute. The operators $\Delta_p$ are the Littlewood-Paley projection on the $p$-th dyadic shell and $\omega_{\le Q}$ denotes the corresponding projection on the spectral ball of radius $2^Q$. Using this criterion and a relation between the time-average of $2^{Q(t)}$ and Kolmogorov's dissipation wave-number, the authors of [26] provide a strong analytical support to the fact that most turbulent flows (i.e. even mildly intermittent ones) are actually regular solutions of Navier-Stokes.

In retrospect, this last observation makes the denomination of turbulent solution given by Leray [58] to his weak solutions a now unnecessarily confusing linguistic choice and it may be unwise to propagate it in the modern literature: mathematical singularities, if they exist, will be violent events that are likely to be of turbulent nature; however, most turbulent flows of practical interest for engineering purposes are smooth, albeit less smooth (e.g. in terms of analyticity radius) than the laminar flows, and only display quasi-singularities. Of course, this remark does not intend to denigrate in any way the admirable work of Jean Leray, who was greatly ahead of his era and whose entire life [19,64] was a tribute to what a great mind can achieve in adversity, when it is moved by an unquenchable curiosity and a strong sense of humanism.

# Appendix A

In this appendix, we recall some well known facts that bridge the standard vector calculus with its geometric foundations. We denote by $\langle\cdot,\cdot\rangle$ the canonical Euclidian scalar product of $\mathbb{R}^3$ and by $(\vec{\mathbf{e}_1},\vec{\mathbf{e}_2},\vec{\mathbf{e}_3})$ the canonical orthonormal basis. For a comprehensive introduction to geometrical hydrodynamics, we refer the reader to Arnold's works [3,4].

## A.1 Some vector calculus formulas

Let us start with the defining identity for the vector product in $\mathbb{R}^3$.

**Claim A.1.** Let $A,B,C$ be vectors in $\mathbb{R}^3$. Then we have

$$\langle A\times B,C\rangle_{\mathbb{R}^3}=\det(A,B,C). \tag{A.1}$$

In particular if $\mathcal{R}$ is a $3\times 3$ matrix, we have

$${}^t\mathcal{R}\big(\mathcal{R}A\times\mathcal{R}B\big)=(\det\mathcal{R})(A\times B). \tag{A.2}$$

*Proof.* Both sides are bilinear antisymmetric in $A,B$ thus one can reduce the identity to the sole case $A=\vec{\mathbf{e}_1}$ and $B=\vec{\mathbf{e}_2}$, i.e.

$$c_3=\begin{vmatrix}1&0&c_1\\0&1&c_2\\0&0&c_3\end{vmatrix},$$

which is obviously true. □

**Claim A.2.** Let $A,B,C,X,Y$ be vectors in $\mathbb{R}^3$. Then we have

$$\det(A\times B,X,Y)=\langle A,X\rangle\langle B,Y\rangle-\langle B,X\rangle\langle A,Y\rangle \tag{A.3}$$

and the triple cross-product formula

$$(A\times B)\times C=\langle C,A\rangle B-\langle C,B\rangle A. \tag{A.4}$$

*Proof.* For each identity, both sides are bilinear antisymmetric in $A,B$. The formulas reduce respectively to

$$\begin{vmatrix}0&x_1&y_1\\0&x_2&y_2\\1&x_3&y_3\end{vmatrix}=x_1y_2-x_2y_1\quad\text{and}\quad\begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}-c_2\\c_1\\0\end{pmatrix},$$

which are obviously true. □

**Remark A.1.** Eq. (A.4) implies the Jacobi identity

$$(A\times B)\times C+(B\times C)\times A+(C\times A)\times B=0, \tag{A.5}$$

since the left-hand side of (A.5) is also

$$\underbrace{\langle C,A\rangle}B\underbrace{-\langle C,B\rangle A}+\langle A,B\rangle C\underbrace{-\langle A,C\rangle B}+\underbrace{\langle B,C\rangle A}-\langle B,A\rangle C=0.$$

## A.2 Some differential calculus formulas

An orientation of $\mathbb{R}^3$ is a choice of a non-trivial $\omega_0$ in the 3rd exterior power $\Lambda^3\mathbb{R}^3$, i.e. a non-degenerate alternating trilinear form on $\mathbb{R}^3$.

**Definition A.1.** *Let $w$ be a one-form in $\mathbb{R}^3$. We define the vector field* curl$w$ *by the identity*

$$\iota_{(\operatorname{curl} w)}\omega_0=dw, \tag{A.6}$$

*where $\iota$ stands for the interior product.*

**Remark A.2.** For $\omega_0=dx_1\wedge dx_2\wedge dx_3$ the interior product reads

$$\iota_X(\omega_0)=X_1dx_2\wedge dx_3-X_2dx_1\wedge dx_3+X_3dx_1\wedge dx_2 \tag{A.7}$$

and with $w=\sum w_j dx_j$ we recover the usual formula for the curl.

In particular, for a function $\alpha$, identifying a vector field $u$ to a one-form we find

$$\operatorname{curl}(\alpha u)=\alpha\operatorname{curl}u+\nabla\alpha\times u. \tag{A.8}$$

Next we investigate the curl of a general advection term and how these operators (do not) commute.

**Lemma A.1.** *Let $u\in W^{1,p}_{loc}$ and $v\in W^{2,p'}_{loc}$ be two vector fields on $\mathbb{R}^3$ for some $p\in[1,+\infty]$. We have*

$$\begin{aligned}\operatorname{curl}\big((u\cdot\nabla)v\big)=&(u\cdot\nabla)\operatorname{curl}v-\big((\operatorname{curl}v)\cdot\nabla\big)u+(\operatorname{div}u)(\operatorname{curl}v)\\&+\sum_{1\le j\le 3}(\nabla u_j\times\nabla v_j).\end{aligned} \tag{A.9}$$

*Proof.* We use a geometric approach because any direct attempt leads to nightmarish computations. We consider $u$ as a vector and $v$ as a 1-form and use Einstein summation convention freely

$$u=u_j\frac{\partial}{\partial x_j},\quad v=v_jdx_j.$$

With $\omega_0 = dx_1 \wedge dx_2 \wedge dx_3$, recall that $\iota_{(\mathrm{curl}\, v)}\omega_0 = dv$ i.e. curl$v$ is a vector and $dv$ is a 2-form. The Lie derivative $\mathcal{L}_u$ is defined by Elie Cartan's formula

$$\mathcal{L}_u(\omega) = \iota_u d\omega + d(\iota_u \omega). \tag{A.10}$$

The convective term can be expressed as a 1-form in the following way:

$$\begin{aligned}(u\cdot\nabla)v &= \mathcal{L}_u(v_j)dx_j = \mathcal{L}_u(v) - v_j\mathcal{L}_u(dx_j)\\ &= \mathcal{L}_u(v) - v_j d(\iota_u dx_j) = \mathcal{L}_u(v) - v_j du_j.\end{aligned}$$

As the Lie derivative commutes with exterior differentiation, one gets

$$d\big((u\cdot\nabla)v\big) = \mathcal{L}_u(dv) + du_j \wedge dv_j.$$

Proceeding by identification, one gets

$$\begin{aligned}\iota_{\mathrm{curl}((u\cdot\nabla)v)}\omega_0 &= d\big((u\cdot\nabla)v\big) = \mathcal{L}_u(\iota_{(\mathrm{curl}\, v)}\omega_0) + du_j\wedge dv_j\\ &= \iota_{(\mathrm{curl}\, v)}\mathcal{L}_u(\omega_0) + \iota_{\mathcal{L}_u(\mathrm{curl}\, v)}\omega_0 + du_j\wedge dv_j\\ &= (\mathrm{div}\, u)\iota_{\mathrm{curl}\, v}\omega_0 + \iota_{[u,\mathrm{curl}\, v]}\omega_0 + du_j \wedge dv_j,\end{aligned}$$

providing (A.9) since $du_j \wedge dv_j = \iota_{(\nabla u_j \times \nabla v_j)}\omega_0$. □

The geometrical reason that gives the convection term its cross-product structure (see identity (2.6) when $u = v$) is the following.

**Lemma A.2.** *Let $u,v$ be vector fields in $\mathbb{R}^3$. Then we have*

$$(u\cdot\nabla)v + (v\cdot\nabla)u = \nabla(u\cdot v) - u\times \mathrm{curl}\, v - v\times\mathrm{curl}\, u. \tag{A.11}$$

*Proof.* We introduce $\tilde{u} = u_j dx_j$, $\tilde{v} = v_j dx_j$ the two one-forms associated to $u$ and $v$ and proceed as in the proof of the previous lemma

$$\begin{aligned}(u\cdot\nabla)\tilde{v} + (v\cdot\nabla)\tilde{u} &= \mathcal{L}_u(v_j)dx_j + \mathcal{L}_v(u_j)dx_j\\ &= \mathcal{L}_u(\tilde{v}) - v_j\mathcal{L}_u(dx_j) + \mathcal{L}_v(\tilde{u}) - u_j\mathcal{L}_v(dx_j)\\ &= \iota_u d\tilde{v} + \underbrace{d(\iota_u\tilde{v}) - v_j du_j} + \iota_v d\tilde{u} + d(\iota_v\tilde{u})\underbrace{-u_j dv_j}\\ &= d(\iota_v\tilde{u}) + \iota_u\iota_{\mathrm{curl}\, v}\omega_0 + \iota_v\iota_{\mathrm{curl}\, u}\omega_0.\end{aligned}$$

In the last expression, we used (A.6) to expand $d\tilde{u}$ and $d\tilde{v}$. The three underlined terms cancel each other out because $\iota_u\tilde{v} = u_j v_j$. Recall that the cross-product $u\times v$ is defined as a 1-form by the identity

$$(u\times v)\cdot w = \omega_0(u,v,w) \quad \text{i.e.} \quad u\times v = \iota_v\iota_u\omega_0. \tag{A.12}$$

We thus get

$$(u\cdot\nabla)\tilde{v}+(v\cdot\nabla)\tilde{u}=\nabla(u\cdot v)+\operatorname{curl} v\times u+\operatorname{curl} u\times v,$$

which is the sought result. □

## References


[1] A. Alexakis, *Helically decomposed turbulence*, J. Fluid Mech. 812 (2017), 752–770.

[2] M. Arnold and W. Craig, *On the size of the Navier-Stokes singular set*, Discrete Contin. Dyn. Syst. 28 (2010), 1165–1178.

[3] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics, Vol. 60, Springer-Verlag, 1989. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein.

[4] V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics*, Applied Mathematical Sciences, Vol. 125, Springer-Verlag, 1998.

[5] D. Ayala and B. Protas, *Extreme vortex states and the growth of enstrophy in 3D incompressible flows*, J. Fluid Mech. 818 (2017), 772–806.

[6] T. Barker, *Uniqueness Results for Viscous Incompressible Fluids*, PhD Thesis, University of Oxford, 2017.

[7] T. Barker and C. Prange, *Quantitative regularity for the Navier-Stokes equations via spatial concentration*, Preprint (2020).

[8] J. T. Beale, T. Kato, and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys. 94 (1984), 61–66.

[9] L. C. Berselli, *Three-Dimensional Navier-Stokes Equations for Turbulence*, Academic Press, 2021.

[10] L. Brandolese and F. Vigneron, *New Asymptotic Profiles of Nonstationary Solutions of the Navier-Stokes System*, Journal de Mathématiques Pures et Appliquées 88(1) (2007), 64–86.

[11] T. Buckmaster, M. Colombo and V. Vicol, *Wild solutions of the Navier-Stokes equations whose singular sets in time have Hausdorff dimension strictly less than 1*, J. Eur. Math. Soc. (2021).

[12] T. Buckmaster, C. De Lellis, P. Isett, and L. Székelyhidi Jr., *Anomalous dissipation for 1/5-Hölder Euler flows*, Ann. of Math. 181(1) (2015), 127–172.

[13] T. Buckmaster, N. Masmoudi, M. Novack, and V. Vicol, *Non-conservative $H^{1/2-}$ weak solutions of the incompressible 3D Euler equations*, Preprint (2021).

[14] T. Buckmaster and V. Vicol, *Nonuniqueness of weak solutions to the Navier-Stokes equation*, Ann. of Math. 189(2) (2019), 101–144.

[15] T. Buckmaster and V. Vicol, *Convex integration and phenomenologies in turbulence*, EMS Surveys in Mathematical Sciences 6(1/2) (2019), 173–263.

[16] T. Buckmaster and V. Vicol, *Convex integration constructions in hydrodynamics*, Bulletin of the AMS 58(1) (2021), 1–44.

[17] L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. 35 (1982), 771–831.

[18] C. Cambon and L. Jacquin, *Spectral approach to non-isotropic turbulence subjected to rotation*, J. Fluid Mech. 202 (1989), 295–317.

[19] J.-Y. Chemin, *Jean Leray et les fondements mathématiques de la turbulence*, Video recording `https://smf.emath.fr/node/27310`, in the series *Un texte, un mathématicien*, SMF, 2007.

[20] J.-Y. Chemin and I. Gallagher, *A nonlinear estimate of the life span of solutions of the three dimensional Navier-Stokes equations*, Tunis. J. Math. 1(2) (2019), 273–292.

[21] J.-Y. Chemin, I. Gallagher, and P. Zhang, *Some remarks about the possible blow-up for the Navier-Stokes equations*, Comm. Partial Differential Equations 44(12) (2019), 1387–1405.

[22] J.-Y. Chemin and N. Lerner, *Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes*, J. Differential Equations 121 (1995), 314–328.

[23] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy, *Energy conservation and Onsager's conjecture for the Euler equations*, Nonlinearity 21(6) (2008), 1233–1252.

[24] A. Cheskidov, S. Friedlander, and R. Shvydkoy, *On the energy equality for weak solutions of the 3D Navier-Stokes equations*, Adv. Math. Fluid Mech. (2010), 171–175.

[25] A. Cheskidov and R. Shvydkoy, *The regularity of weak solutions of the 3D Navier-Stokes equations in* $B^{-1}_{\infty,\infty}$, Arch. Rat. Mech. Anal. 195 (2010), 159–169.

[26] A. Cheskidov and R. Shvydkoy, *A unified approach to regularity problems for the 3D Navier-Stokes and Euler equations: the use of Kolmogorov's dissipation range*, J. Math. Fluid Mech. 16(2) (2014), 263–273.

[27] C. Chicone, *The topology of stationary curl parallel solutions of Euler's equations*, Israel J. Math. 39 (1981), 161–166.

[28] P. Constantin, *Geometric statistics in turbulence*, SIAM Review 36(1) (1994), 73–98.

[29] P. Constantin and C. Fefferman, *Direction of vorticity and the problem of global regularity for the Navier-Stokes equations*, Indiana Univ. Math. J. 42(3) (1993), 775–789.

[30] P. Constantin and C. Foias, *Navier-Stokes Equations*, Chicago Lectures in Mathematics, University of Chicago Press, 1988.

[31] P. Constantin and A. Majda, *The Beltrami spectrum for incompressible fluid flows*, Comm. Math. Phys. 115 (1988), 435–456.

[32] P. Constantin, E. Weinan, and E. S. Titi, *Onsager's conjecture on the energy conservation for solutions of Euler's equation*, Comm. Math. Phys. 165(1) (1994), 207–209.

[33] D. Coŕdoba, C. Fefferman, and R. de la Llave, *On squirt singularities in hydrodynamics*, SIAM J. Math. Anal. 36 (2004), 204–213.

[34] P.A. Davidson, *Turbulence: an Introduction for Scientists and Engineers*, Oxford University Press, 2015.

[35] C. De Lellis and L. Székelyhidi Jr., *On turbulence and geometry: from Nash to Onsager*, Notices Amer. Math. Soc. 66(5) (2019), 677–685.

[36] L. De Rosa, *On the Helicity conservation for the incompressible Euler equations*, Proc. Amer. Math. Soc. 148 (2020), 2969–2979.

[37] J. Duchon and R. Robert, *Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations*, Nonlinearity 13 (2000), 249–255.

[38] E. Ferretti, *Uniqueness in the Cauchy problem for parabolic equations*, Proc. Edinb. Math. Soc. 46 (2003), 329–340.

[39] C. Foias, C. Guillopé, and R. Temam, *New a priori estimates for Navier-Stokes equations in dimension 3*, Comm. Partial Differential Equations 6 (1981), 329–359.

[40] C. Foias, C. Guillopé, and R. Temam, *Lagrangian representation of a flow*, J. Differential Equations 57(3) (1985), 440–449.

[41] I. Gallagher, G. S. Koch, and F. Planchon, *Blow-up of critical Besov norms at a potential Navier-Stokes singularity*, Comm. Math. Phys. 343(1) (2016), 39–82.

[42] Th. Gallay and C. E. Wayne, *Long-time asymptotics of the Navier-Stokes equation in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$, Z. Angew. Math. Mech. 86 (2006), 256–267.

[43] P. Germain, *Multipliers, paramultipliers, and weak-strong uniqueness for the Navier-Stokes equations*, J. Differential Equations 226 (2006), 373–428.

[44] R. Ghrist and R. Komendarczyk, *Overtwisted energy-minimizing* curl *eigenfields*, Nonlinearity 19 (2006), 41–51.

[45] J. Guillod and V. Sverak, *Numerical investigations of non-uniqueness for the Navier-Stokes initial value problem in borderline spaces*, Preprint (2017).

[46] P. Isett, *A proof of Onsager's conjecture*, Annals of Mathematics 188(3) (2018), 871–963.

[47] L. Iskauriaza, G. A. Serëgin, and V. Sverak, $L^{3,\infty}$*-solutions of Navier-Stokes equations and backward uniqueness*, Uspekhi Mat. Nauk 58 (2003), 2–44.

[48] H. Jia and V. Sverak, *Are the incompressible 3D Navier-Stokes equations locally ill-posed in the natural energy space?*, J. Func. Anal. 268(12) (2015), 3734–3766.

[49] D. Kang and B. Protas, *Searching for Singularities in Navier-Stokes Flows Based on the Ladyzhenskaya-Prodi-Serrin Conditions*, Preprint (2021).

[50] D. Kang, D. Yun, and B. Protas, *Maximum amplification of enstrophy in three-dimensional Navier-Stokes flows*, J. Fluid Mech. 893 (2020), A22.

[51] T. Kato, *Nonlinear evolution equations in Banach spaces*, in: *Proceedings of the Symposium on Applied Mathematics*, 17 (1965), 50–67.

[52] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math. 157 (2001), 22–35.

[53] H. Kozono and Y. Taniuchi, *Bilinear estimates in BMO and the Navier-Stokes equations*, Math. Z. 235 (2000), 173–194.

[54] M. G. Krein, *Introduction to the geometry of indefinite J-spaces and the theory of operators in these spaces*, Second Math. Summer School, Kiev, 1 (1965), 15–92.

[55] I. Kukavica, *Role of the pressure for validity of the energy equality for solutions of the*

*Navier-Stokes equation*, J. Dyn. Differ. Equations 18(2) (2006), 461–482.

[56] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, 23, AMS, 1968.

[57] P. G. Lemarié-Rieusset, *The Navier-Stokes Problem in the 21st Century*, CRC Press, 2016.

[58] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. 63 (1934), 193–248.

[59] N. Lerner, Y. Morimoto, and C.-J. Xu, *Instability of the Cauchy-Kovalevskaya solution for a class of nonlinear systems*, Amer. J. Math. 132 (2010), 99–123.

[60] T. Leslie and R. Shvydkoy, *Conditions implying energy equality for weak solutions to the Navier-Stokes equation*, SIAM J. Math. Anal. 50(1) (2018), 870–890.

[61] M. Lesieur, *Décomposition d'un champ de vitesse non divergent en ondes d'hélicité*, Tech. Rep., Observatoire de Nice (1972).

[62] J. L. Lions, *Sur la régularité et l'unicité des solutions turbulentes des équations de Navier-Stokes*, Rend. Sem. Mat. Univ. Padova 30 (1960), 16–23.

[63] L. Lu and C. R. Doering, *Limits on enstrophy growth for solutions of the three-dimensional Navier-Stokes equations*, Indiana Univ. Math. J. 57 (2008), 2693–2727.

[64] J. Mawhin, *Éloge de Jean Leray (1906-1998)*, Bulletins de l'Académie Royale de Belgique 10(1-6), (1999), 89–98.

[65] Y. Meyer, *Wavelets, Paraproducts, and Navier-Stokes Equations*, in: *Current Developments in Mathematics*, Cambridge, MA, 1996, R. Bott et al. (Eds.), International Press, 1997.

[66] K. Ohkitani, *Characterization of blowup for the Navier-Stokes equations using vector potentials*, AIP Advances 7 (2017), 015211.

[67] L. Onsager, *Statistical hydrodynamics*, Nuovo Cimento (9), 6, Supplemento, 2, Convegno Internazionale di Meccanica Statistica (1949), 279–287.

[68] F. Planchon, *An extension of the Beale-Kato-Majda criterion for the Euler equations*, Comm. Math. Phys. 232 (2003), 319–326.

[69] G. Seregin, *A certain necessary condition of potential blow up for Navier-Stokes equations*, Commun. Math. Phys. 312 (2012), 833–845.

[70] M. Shinbrot, *The energy equation for the Navier-Stokes system*, SIAM J. Math. Anal. 5(6) (1974), 948–954.

[71] T. Tao, *Quantitative bounds for critically bounded solutions to the Navier-Stokes equations*, Preprint (2019).

[72] A. N. Tychonof, *A uniqueness theorem for the heat equation*, Mat. Sb. 42 (1935), 199–216.

[73] A. Vasseur, *Regularity criterion for 3D Navier-Stokes equations in terms of the direction of the velocity*, Applications of Mathematics 54(1) (2009), 47–52.

[74] A. Vasseur, *Higher derivatives estimate for the 3D Navier-Stokes equation*, Ann. I. H. Poincaré AN 27 (2010), 1189–1204.

[75] V. Vicol, *Wild Weak Solutions to Equations arising in Hydrodynamics*, Video recordings

`https://youtu.be/cgquRkiasv8` (1/3), `https://youtu.be/RXNrDLFAqYU` (2/3), `https://youtu.be/1m1YaWN4LgI` (3/3) from Hadamard lectures, IHES (2020).

[76] F. Waleffe, *The nature of triad interactions in homogeneous turbulence*, Phys. Fluids 4 (1992), 350–363.