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WEAK SOLUTIONS CONSTRUCTED BY SEMI-DISCRETIZATION ARE SUITABLE: THE CASE OF SLIP BOUNDARY CONDITIONS

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Dedicated to William (Bill) Layton on the occasion of his 60th birthday

Abstract. We consider the initial boundary value problem for the three dimensional Navier-Stokes equations with Navier-type slip boundary conditions. After having properly formulated the problem, we prove that weak solutions constructed by approximating the time-derivative by backward finite differences (with Euler schemes) are suitable. The main novelty is the proof of the local energy inequality in the case of a weak solution constructed by time discretization. Moreover, the problem is analyzed with boundary conditions which are of particular interest in view of applications to turbulent flows.

Key words. Navier-Stokes equations, Euler scheme, local energy inequality, slip boundary conditions.

1. Introduction

In this paper we consider the three dimensional Navier-Stokes equations, with unit viscosity and zero external force (assumptions which are nevertheless unessential) in a bounded domain $\Omega \subset \mathbb{R}^3$, with a smooth boundary $\Gamma = \partial\Omega$ and under "curl based" Navier-type slip boundary conditions. Namely we consider the following initial boundary value problem

$$
\begin{cases}
\partial_t v - \Delta v + (v \cdot \nabla)\, v + \nabla q = 0 & (t,x) \in]0,T[\times\Omega, \\
\nabla \cdot v = 0 & (t,x) \in]0,T[\times\Omega, \\
v \cdot n = 0 & (t,x) \in]0,T[\times\Gamma, \\
\omega \times n = 0 & (t,x) \in]0,T[\times\Gamma, \\
v(0,x) = v_0(x) & x \in \Omega,
\end{cases} \tag{1}
$$

where $v : [0,T]\times\Omega \to \mathbb{R}^3$ is the unknown velocity, $\omega := \operatorname{curl} v$ the vorticity field, and $q : [0,T] \to \mathbb{R}$ the kinematic pressure. The role of the above boundary conditions in the mathematical theory of Euler and Navier-Stokes equations is emphasized in Xiao and Xin [35] and Beirão da Veiga and Crispo [6]. Interesting applications of the above conditions to turbulence modeling, especially for the description of unsteady phenomena, can be found in Layton [26], and the review paper [7] (see also Ref. [10] for a two-dimensional related problem linked with the detection of time-transient phenomena).

In this paper we continue and extend previous work from [13] (done in the space-periodic setting, as well as the forthcoming [9] concerning the space-time discrete problem) and we observe that the analysis of the time-discretization, is a topic which did not attract a lot of attention, in the context of construction of solutions

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satisfying the local energy inequality. Starting from the celebrated papers by Scheffer [30] and Caffarelli, Kohn, and Nirenberg [16] concerning the partial regularity for the Navier-Stokes equations, the notion of *suitable weak solution* became a concept of paramount importance in the mathematical theory of the Navier-Stokes equations. We recall that Leray-Hopf weak solutions satisfy a "global energy inequality," while the results of partial regularity require (beside technical conditions on the pressure) the so-called "local energy inequality", see (4) and the next section for precise definitions. In [16] authors introduced an approximation scheme with time-retarded mollifiers, in order to prove the local energy inequality and to estimate the pressure in appropriate Lebesgue spaces. The role of the regularity of the pressure has been later considered in Lin [27] and Vasseur [34]. Combined with the lack of uniqueness of weak solutions, the notion of local energy inequality raised the question to determine which solutions are suitable, see Beirão da Veiga [2, 3, 4] (on the other hand local-in-time strong solutions clearly satisfy the local energy inequality). Especially the question whether or not solutions obtained by the Faedo-Galerkin method satisfy the local energy inequality turned out to be a particular difficult problem. This has been left open for twenty years and a first partial solution to this problem came with the two companion papers by Guermond [22, 23]. In the above references it has been proved that if projectors over the finite element spaces used to discretize (with respect to the space variables x) velocity and pressure satisfy certain *commutation* properties, then weak solutions constructed in the limit of vanishing mesh-size are suitable. In particular, these results cover the MINI element and the Taylor-Hood one. I wrote that this result is partial since –at present– the case of the Fourier-Galerkin method in the space periodic setting is still open, see also Biryuk, Craig, and Ibrahim [15]. The question is also of relevance for applications, because the notion of suitable should be satisfied by any reasonable solution (called "physically relevant") obtained with approximation by Large Eddy Simulation methods, see Guermond *et al.* [24, 25]. Other recent related results can be found in [13, 14, 19].

In this paper, we continue in the spirit of connecting results from mathematical analysis with those from numerical analysis, and we focus on understanding when discrete-time approximations produce suitable solutions, as the time-step-size $\kappa > 0$ goes to zero. We treat the boundary value problem with certain slip conditions, while the Dirichlet problem seems to require a completely different and much more technical treatment, which is object of a still ongoing research. **Note added in proof:** After the paper being accepted we have been aware that in Sec. 5 of Ref [21] the time-discrete problem in the implicit case is studied in the Dirichlet case, by using techniques of semigroup theory.

In particular, we analyze the following single step scheme:

Algorithm. (Euler implicit) Let be given a time-step-size $\kappa > 0$ and the corresponding net $I^M = \{t_m\}_{m=0}^M$, with $M = [T/\kappa] \in \mathbb{N}$ and $t_m := m\kappa$. Then, for $m \geq 1$ and for v^{m-1} given from the previous step with $v^0 = v_0$, compute the iterate v^m as follows: Solve

$$
\begin{cases}
d_t v^m - \Delta v^m + (v^m \cdot \nabla)\, v^m + \nabla q^m = 0 & \text{in } \Omega, \\
\nabla \cdot v^m = 0 & \text{in } \Omega, \\
v^m \cdot n = 0 & \text{on } \Gamma, \\
\operatorname{curl} v^m = 0 & \text{on } \Gamma,
\end{cases}
\tag{2}
$$

where $d_t v^m := \frac{v^m - v^{m-1}}{\kappa}$ denotes the backward finite difference.

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Remark 1.1. *For each $m = 1, \dots, M$ we have to solve a problem very close to the stationary Navier-Stokes equations with slip conditions, for which we have more or less standard results of existence of weak solutions v^m. The most relevant point is to prove estimates independent of κ.*

For each m, we also associate to v^m a corresponding pressure q^m. The use of slip conditions is crucial at this point since, instead of using some abstract argument, we can directly construct the pressure by solving a Poisson problem with Neumann conditions, see Lemma 4.3 below. This fact allows us to get precise estimates on the pressure. Then, as usual, to the sequence $\{v^m, q^m\}_{m=1,\dots,M}$ we can associate the piecewise functions (v_M, u_M, q_M) defined in $[0, T]$ as follows:

$$\begin{cases} v_M(t) = v^{m-1} + \dfrac{t - t_{m-1}}{\kappa}(v^m - v^{m-1}) & \text{for } t \in [t_{m-1}, t_m[, \\ v_M(t_M) = v^M & \\ u_M(t_0) = v_0 & \\ u_M(t) = v^m & \text{for } t \in]t_{m-1}, t_m], \\ q_M(t) = q^m & \text{for } t \in]t_{m-1}, t_m], \end{cases} \tag{3}$$

in such a way that $v_M(t_m) = u_M(t_m)$, for all $m = 0, \dots, M$.

The main result of this paper is the following:

Theorem 1.1. *Let be given $v_0 \in H^1(\Omega)$, which is divergence-free and tangential to the boundary. Then, there exist $(v, q) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \times L^{5/3}(0, T; L^{5/3}(\Omega))$ and a sequence $\kappa \to 0$ ($M \to +\infty$) such that the functions v_M and u_M both converge to v, and q_M converges to q. The vector field v is a Leray-Hopf weak solutions to the Navier-Stokes equations* (1).

Moreover, for all $\phi \in C_0^\infty((0, T) \times \Omega)$ such that $\phi \geq 0$, the couple (v, q) satisfies the local energy inequality

$$\begin{aligned} &\int_\Omega |v(x,t)|^2 \, \phi(x,t) \, dx + 2 \int_0^t \int_\Omega |\nabla v(x,\tau)|^2 \, \phi(x,\tau) \, dx d\tau \\ &\qquad \leq \int_0^t \int_\Omega \Big(|v(x,\tau)|^2 (\partial_t \phi(x,\tau) - \Delta \phi(x,\tau)) \\ &\qquad\qquad + (|v(x,\tau)|^2 + 2q(x,\tau)) \, v(x,\tau) \cdot \nabla \phi(x,\tau) \Big) \, dx d\tau, \end{aligned} \tag{4}$$

for all $t \in [0, T]$.

The existence part is a simple variation of a rather standard result known for the Dirichlet case, see Temam [33, Ch. III, § 4], while the most original contribution is the analysis of the pressure and the proof of the local energy inequality (4).

Remark 1.2. *By an appropriate additional smoothing argument the hypothesis on the initial datum can be relaxed to the more natural condition of square integrability. We are assuming more regularity to keep the proof as simple as possible and without technicalities. We also observe that this point (see also* [29, Ch. 13]*) is generally not treated or overlooked in the literature, and requires to handle a further technical step.*

2. Notation

We briefly fix the notation, and in the sequel we shall use the customary Lebesgue spaces $(L^p(\Omega), \|\,.\,\|_p)$ and Sobolev spaces $(W^{k,p}(\Omega), \|\,.\,\|_{k,p})$ and $H^s(\Omega) := W^{s,2}(\Omega)$ (for simplicity we shall do not distinguish between scalar and vector valued functions and we denote the L^2-norms simply by $\| \, . \, \|$). The subscript "σ" will denote the subspace of solenoidal vector fields, obtained by using the Leray projection operator over tangential and divergence-free vector fields, see Galdi [20] and Sohr [31]. In particular, it is standard to introduce the following spaces

$$L^2_\sigma(\Omega) := \{u \in L^2(\Omega) : \nabla \cdot u = 0, \ u \cdot n = 0 \text{ on } \Gamma\},$$
$$H^1_\sigma(\Omega) := \{u \in H^1(\Omega) : \nabla \cdot u = 0, \ u \cdot n = 0 \text{ on } \Gamma\}.$$

Let be given a Banach space $(X, \| \, . \, \|_X)$. We will also use the Bochner spaces $L^p(0, T; X)$, endowed with the norm

$$\|f\|_{L^p(0,T;X)} := \begin{cases} \left(\displaystyle\int_0^T \|f(s)\|_X^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \displaystyle\sup_{0 \leq s \leq T} \|f(s)\|_X & \text{if } p = +\infty, \end{cases}$$

see for instance Constantin and Foias [17] and Temam [33]. To deal with discrete problems we shall make use of the spaces $l^p(I^M; X)$, which are the natural discrete counterpart of $L^p(0, T; X)$. The Banach space $l^p(I^M; X)$ consists of X-valued sequences $\{a_m\}_{m=0}^M$, defined on the net I^M and endowed with the weighted norm (recall that $\kappa = t_{m+1} - t_m$)

$$\|a_m\|_{\ell^p(I^M;X)} := \begin{cases} \left(\kappa \displaystyle\sum_{m=0}^M \|a_m\|_X^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \displaystyle\max_{0 \leq m \leq M} \|a_m\|_X & \text{if } p = +\infty. \end{cases}$$

We now give the precise definitions and some technical facts needed when dealing with the Navier-Stokes equations with the Navier-type slip boundary conditions, see also the overview (end extensions to the Boussinesq equations) in Ref. [12].

Definition 2.1. *We say that* $v \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1(\Omega))$ *is a (Leray-Hopf) weak solution of the NSE* (1) *(with curl-based Navier-type slip boundary-conditions) if the following holds true*

$$\int_0^T \int_\Omega \big(- v \cdot \phi_t + \nabla v : \nabla \phi - (v \cdot \nabla)\, \phi \cdot v \big)\, dxdt + \int_0^T \int_\Gamma v \cdot (\nabla n)^T \cdot \phi \, dSdt = \int_\Omega v_0 \cdot \phi(0)\, dx,$$

for all vector-fields $\phi \in C^\infty_0([0, T[\times \overline{\Omega})$ *such that* $\nabla \cdot \phi = 0$ *in* $[0, T[\times \Omega$*, and* $\phi \cdot n = 0$ *on* $[0, T[\times \Gamma$*. Moreover, the following energy balance*

$$\frac{1}{2}\|v(t)\|^2 + \int_0^t \|\nabla v(s)\|^2 \, ds + \int_0^t \int_\Gamma v(s, x) \cdot (\nabla n(x))^T \cdot v(s, x)\, dSds \leq \frac{1}{2}\|v_0\|^2, \tag{5}$$

is satisfied for all $t \in [0, T]$.

With this definition we have the following result.

Theorem 2.1. *Let be given any positive* $T > 0$ *and* $v_0 \in L^2_\sigma(\Omega)$*. Then, there exists at least a weak solution* v *of the Navier-Stokes equations* (1) *on* $[0, T]$.

The proof of the above result of global existence for weak solution in the sense of the Definition 2.1 can be found for instance in [35, § 6]. We observe that an equivalent formulation can be given, which is relevant to obtain a-priori estimates. To this end we recall the following formulas for integration by parts (see [5, Lemma 2.1] for the proofs).

Lemma 2.1. *Let u, $\phi : \Omega \to \mathbb{R}^3$ be two smooth enough vector fields, tangential to the boundary Γ. Then, it follows that*

$$-\int_\Omega \Delta u \cdot \phi \, dx = \int_\Omega \nabla u : \nabla \phi \, dx - \int_\Gamma (curl\, u \times n)\, \phi \, dS + \int_\Gamma u \cdot (\nabla n)^T \cdot \phi \, dS.$$

Moreover, if $\nabla \cdot u = 0$, then $-\Delta u = curl(curl\, u)$, and also

$$-\int_\Omega \Delta u \cdot \phi \, dx = \int_\Omega curl\, u \cdot curl\, \phi \, dx + \int_\Gamma \big[(curl\, u) \times n\big] \cdot \phi \, dS.$$

With the above formulas, the weak formulation of Leray-Hopf solutions can be also written as follows:

$$\int_0^T \int_\Omega \big(- v \cdot \phi_t + \omega \cdot \mathrm{curl}\, \phi - (v \cdot \nabla)\, \phi \cdot v \big)\, dx d\tau = \int_\Omega v_0 \cdot \phi(0)\, dx,$$

for all vector-fields $\phi \in C_0^\infty([0, T[\times\overline{\Omega})$ such that $\nabla \cdot \phi = 0$ in $[0, T[\times\Omega$, and $\phi \cdot n = 0$ on $[0, T[\times\Gamma$. Moreover, the following energy estimate holds true

$$\frac{1}{2}\|v(t)\|^2 + \int_0^t \|\omega(s)\|^2 \, ds \leq \frac{1}{2}\|v_0\|^2 \qquad \forall\, t \in [0, T]. \tag{6}$$

We also observe that, by using integration by parts (see [5, Lemma 2.7] for the proof) it can be also proved that there exists $C = C(\Omega)$ depending only on Ω such that

$$\|\nabla u\|^2 \leq 2\|\mathrm{curl}\, u\|^2 + C(\Omega)\|u\|^2,$$

for all u which are divergence-free, smooth enough, and satisfying the boundary conditions as in (1). In particular, the latter inequality applied to the vector field v –for which we have a bound in $L^\infty(0, T; L^2(\Omega))$, by the global energy inequality (6)– implies that

$$\|\nabla v\|^2 \leq 2\|\omega\|^2 + C(\Omega)\|v_0\|^2, \tag{7}$$

hence, the control of the vorticity field ω in $L^2(\Omega)$ becomes equivalent to that of ∇v in $L^2(\Omega)$.

3. On the numerical scheme

We first recall a lemma about existence of discrete solutions, which is a variation of the classical one (valid in the case of Dirichlet conditions) from Temam [33, Ch. III, Lemma 4.3].

Lemma 3.1. *There exists at least one sequence $\{v^m\}_{m=0}^M$ defined by the algorithm (2) with $v^0 = v_0$.*

Proof. We have $M - 1$ coupled systems and in particular, given $v^{m-1} \in H^1_\sigma(\Omega)$ and $\kappa > 0$, the function $v^m \in H^1_\sigma(\Omega)$ can be obtained as solution of the modified steady Navier-Stokes system

$$\begin{cases} \dfrac{v^m}{\kappa} - \Delta v^m + (v^m \cdot \nabla)\, v^m + \nabla q^m = \dfrac{v^{m-1}}{\kappa} & x \in \Omega, \\ \nabla \cdot v^m = 0 & x \in \Omega, \\ v^m \cdot n = 0 & x \in \Gamma, \\ \mathrm{curl}\, v^m \times n = 0 & x \in \Gamma. \end{cases}$$

By testing with v^m itself (a reasoning that can be made rigorous by a Faedo-Galerkin approximation) one obtains

$$\frac{\|v^m\|^2}{2\kappa} + \|\mathrm{curl}\, v^m\|^2 \leq \frac{\|v^{m-1}\|^2}{2\kappa}.$$

Then, it is enough to apply in a standard way the Brouwer fixed point theorem (see Lions [28]) to infer existence of at least one solution $v^m \in H^1_\sigma(\Omega)$. The existence of the associated q^m is then obtained by De Rham theorem. Generally the pressure q^m is normalized by the constraint of zero mean value. Observe that –at this step– for the solution $\{v^m\}$ of the discrete problem we did not prove neither uniqueness, neither estimates independent of $\kappa > 0$. $\square$

Since the possible non-uniqueness for the discrete approximate problem may be considered as a limitation in certain situations, we may also consider the following scheme:

Algorithm. (Euler semi-implicit) Let be given a time-step-size $\kappa > 0$ and the corresponding net $I^M = \{t_m\}_{m=0}^M$, with $M = [T/\kappa] \in \mathbb{N}$ and $t_m := m\kappa$. Then, for $m \geq 1$ and for v^{m-1} given from the previous step with $u^0 = v_0$, compute the iterate v^m as follows: Solve

$$
\begin{cases}
\dfrac{v^m}{\kappa} - \Delta v^m + (v^{m-1}\cdot\nabla)\, v^m + \nabla q^m = \dfrac{v^{m-1}}{\kappa} & x \in \Omega, \\
\nabla\cdot v^m = 0 & x \in \Omega, \\
v^m \cdot n = 0 & x \in \Gamma, \\
\operatorname{curl} v^m \times n = 0 & x \in \Gamma.
\end{cases}
\tag{8}
$$

For the above system we have the following result

Lemma 3.2. *There exists a unique sequence $\{v^m\}_{m=0}^M$ defined by the algorithm* (8) *with $u^0 = v_0$.*

Proof. For system (8) we have to solve for each $m = 1, \dots, M$ a linear system for which the *a-priori* estimate is the same as for the implicit scheme (2), since for $v^{m-1} \in H^1_\sigma(\Omega)$ it follows that $\int_\Omega (v^{m-1}\cdot\nabla)\, v^m \cdot v^m\, dx = 0$. Finally uniqueness follows from linearity of the problem due to a semi-implicit algorithm. $\square$

The rest of the paper can be easily adapted also to consider the semi-implicit algorithm.

Remark 3.1. *The role of the semi-implicit approximation is emphasized for instance in* [8]*, where it represents a critical tool to obtain optimal convergence rates for certain shear-dependent fluids with nonlinear viscosities (and consequently with the lack of the standard regularity results known for strong solutions to the Navier-Stokes equations). See also Diening, Ebmeyer, and Růžička* [18] *for general parabolic system.*

4. Proof of the main result

In order to show that solutions constructed by the algorithm (2) converge to suitable weak solutions, we need to pass to the limit as $\kappa \to 0^+$ (and consequently $M \to +\infty$). This can be obtained by means of appropriate a-priori estimates independent of κ. To this end, we multiply the equations (2) by v^m itself and we use a slightly different argument to prove the following lemma. Observe that the same argument will also work for the algorithm (8). We mainly consider (2), since it is the most basic time-discretization, which can be found in many textbooks when proposing alternate proofs of existence of weak solutions by semi discretization. (Clearly the analysis of more accurate or more stable time-discretizations will be needed in a further study)

Lemma 4.1. *Let be given $v_0 \in H^1_\sigma(\Omega)$. Then, there exists a constant $C > 0$, (independent of κ) such that*

$$\|v^m\|_{\ell^\infty(I^M;L^2_\sigma(\Omega))\cap\ell^2(I^M;H^1_\sigma(\Omega))} \leq C.$$

Proof. We test the equations (2) by v^m. By integration by parts and with the aid of the elementary algebra equality

$$(a-b)a = \frac{a^2-b^2}{2} + \frac{(a-b)^2}{2} \qquad \forall\, a,\, b \in \mathbb{R},$$

we easily get

$$\frac{1}{2}d_t\|v^m\|^2 + \frac{\kappa}{2}\|d_t v^m\|^2 + \|\operatorname{curl} v^m\|^2 = 0.$$

Next, by multiplying by $\kappa > 0$ and, summing up over $m = 1, \dots, M$, we obtain

$$\frac{1}{2}\|v^M\|^2 + \frac{\kappa^2}{2}\sum_{m=1}^{M}\|d_t v^m\|^2 + \kappa\sum_{m=1}^{M}\|\operatorname{curl} v^m\|^2 \leq \frac{1}{2}\|v_0\|^2. \tag{9}$$

From the latter inequality we can easily deduce by using (7) that

$$v^m \in \ell^\infty(I^M; L^2_\sigma(\Omega)) \cap \ell^2(I^M; H^1_\sigma(\Omega)).$$

The definition of the weighted spaces $\ell^p(I^M; X)$ allows us to use the standard Hölder inequality and the convex interpolation, obtaining then

$$v^m \in \ell^{10/3}(I^M; L^{10/3}(\Omega)).$$

□

As we previously observed in Lemma 3.1, to each v^m we can also associate a discrete pressure q^m, by using De Rham theorem. Here, we want to have a more precise information about the regularity of q^m and this is the crucial point where we need to use the slip boundary conditions. The main difference with respect to the Dirichlet case is the treatment of the pressure, which is now much simpler. In particular, we recall to the reader that the use of the Navier-type conditions allow us to infer the following lemma, see Refs. [6, 11].

Lemma 4.2. *Let $v : \Omega \to \mathbb{R}^3$ be a smooth enough vector field satisfying $(curl\, v) \times n = 0$ on Γ. Then, $\zeta = curl(curl\, v)$ is a vector field tangential to the boundary, i.e., $(\zeta \cdot n)_{|\Gamma} = 0$.*

In particular, in the case of algorithm (2), since $\nabla \cdot v^m = 0$, then we have $curl(curl\, v^m) = -\Delta v^m$ in Ω. Moreover since $v^m \cdot n = 0$ on Γ, then $d_t v^m \cdot n = 0$ on Γ. We finally get that

$$\Delta v^m \cdot n = 0 \quad \text{on } \Gamma.$$

For a detailed proof see [7, Lemma 7.4]. In that reference many other results on the Navier conditions are reviewed.

Let $\mathcal{U} \subset \mathbb{R}^3$ be a neighbourhood of Γ and $n : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth function with compact support in $\mathcal{U}$ and such that $n\big|_\Gamma$ is the normal vector to Γ.

Lemma 4.3. *Let (v^m, q^m) be a smooth solution to (2). Then, the pressure q^m satisfies the following Neumann problem*

$$\begin{cases} -\Delta q^m = \displaystyle\sum_{i,j=1}^{n} \partial_i\partial_j(v_i^m v_j^m) & x \in \Omega, \\ \dfrac{\partial q^m}{\partial n} = v^m \cdot (\nabla n) \cdot v^m & x \in \Gamma. \end{cases} \tag{10}$$

Consequently, there exists $C > 0$, depending only on Ω (hence independent of v^m), such that the following estimate holds true:

$$\|q^m\|_{\ell^{5/3}(I^M;L^{5/3}(\Omega))} \le C.$$

Proof. By taking the divergence of the momentum equation, since $v^m \in H^1_\sigma(\Omega)$ we get

$$-\Delta q^m = \nabla\cdot\big((v^m\cdot\nabla)\,v^m\big) = \sum_{i,j=1}^{n} \partial_i\partial_j(v_i^m v_j^m),$$

and by using classical interpolation inequalities we have that $(v^m\cdot\nabla)\,v^m$ is uniformly (with respect to m) bounded in $\ell^{\frac{5}{3}}(I^M;L^{\frac{15}{14}}(\Omega))$. This holds true because when applying standard interpolation we use just that $v^m \in \ell^\infty(I^M;L^2(\Omega))\cap\ell^2(I^M;H^1(\Omega))$ to prove the estimate. (This it the regularity inherited by Leray-Hopf weak solutions)

By multiplying the momentum equation (restricted to Γ) by n and by using the fact that $v^m\cdot n = 0$ on Γ we get

$$\frac{\partial q^m}{\partial n} = \Big(\Delta v^m - d_t v^m - (v^m\cdot\nabla)\,v^m\Big)\cdot n = v_i^m v_j^m \partial_j n_i,$$

where we have used Lemma 4.2 and the equality $(v^m\cdot\nabla)\,v^m\cdot n = -v^m\cdot\nabla n\cdot v^m$ which immediately holds true on Γ. This follows since $v^m\cdot\nabla(v^m\cdot n)$ vanishes on Γ being a tangential derivative of a constant function, see Ref. [11]. Then, (v^m, q^m) satisfies (10). By using a trace theorem and the fact that $v^m\otimes v^m$ is in $\ell^{\frac{5}{3}}(I^M;W^{1,\frac{15}{14}}(\Omega))$ we have that $v^m\cdot\nabla n\cdot v^m \in \ell^{\frac{5}{3}}(I^M;W^{1-\frac{14}{15}}(\Gamma))$, uniformly in M. Then, by classical L^p estimates for the scalar Neumann problem, see [1, 32] we get

$$\|\nabla q^m\|_{\ell^{5/3}(I^M;L^{15/14}(\Omega))} \le C,$$

with $C > 0$ independent of M. The whole argument can be made completely rigorous through a Galerkin approximation of (v^m, q^m). By using a Sobolev embedding inequality, and since q^m is with zero mean value, we finally end the proof. $\square$

From the estimate (9) we have the following inequality holds true. We state it as a Lemma, because it will be crucial in the proof of the Theorem 1.1.

Lemma 4.4. *The following estimate holds true for the sequence $\{v^m\}_{m=1,\dots,M}$*

$$\kappa^2\sum_{m=1}^{M}\|d_t v^m\|^2 = \sum_{m=1}^{M}\|v^m - v^{m-1}\|^2 \le \|v_0\|^2.$$

We observe that this estimate is obtained because the "*natural multiplier*" (the one which cancels out the convective term) is v^m, and the estimate comes from algebraic manipulation of the integral $\int_\Omega d_t v^m\cdot v^m\,dx$. We observe that the same estimates can be proved for the semi-implicit scheme (8), since again the correct multiplier is v^m.

We show now some properties of the step function u_M and of the piecewise linear function v_M.

Lemma 4.5. *Let $v_0 \in H^1_\sigma(\Omega)$ be given. Then, there exists a constant $C > 0$ (independent of κ, hence independent of M) such that*

$$\begin{aligned}
&\|v_M\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le C,\\
&\|u_M\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le C,\\
&\|q_M\|_{L^{5/3}(0,T;L^{5/3}(\Omega))} \le C,\\
&\|\partial_t v_M\|_{L^{4/3}(0,T;(H^1_\sigma(\Omega))')} \le C.
\end{aligned}$$

Moreover, we also have the following identity

$$\|u_M - w_M\|^2_{L^2(0,T;L^2(\Omega))} = \frac{\kappa}{3}\sum_{m=1}^{M} \|v^m - v^{m-1}\|^2. \tag{11}$$

Proof. The results of Lemma 4.1 imply that the functions v_M defined in (3) are, for each positive M, Lipschitz functions $[0,T] \mapsto H^1_\sigma(\Omega)$. Moreover, for each M, the function v_M satisfies, in the sense of distributions over $]0,T[$, the following equality

$$\frac{d}{dt}\int_\Omega v_M\cdot\psi\,dx + \int_\Omega \operatorname{curl} u_M\cdot\operatorname{curl}\psi + (u_M\cdot\nabla)\,u_M\cdot\psi\,dx = 0 \qquad \forall\,\psi\in H^1_\sigma(\Omega). \tag{12}$$

Note that, by their definition (v_M, u_M, q_M) have the regularity stated in the lemma, which derives directly from the analogous one valid for the sequence $\{v^m, q^m\}_{m=1,\dots,M}$.

Remark 4.1. *Observe that the proof of existence of weak solutions in Temam* [33] *is based on estimates for the discrete time-derivative in the Hilbert space* $L^2(0,T;H^{-s}(\Omega))$, *for* $s \geq 3/2$. *The idea of obtaining such an estimates in negative spaces (but also considering negative Sobolev spaces in the time variable) represents also the core of the results in Guermond* [22, 23]. *Here, due to the particular setting, we can follow a more standard path.*

The estimate $\partial_t v_M \in L^{4/3}(0,T;(H^1_\sigma(\Omega))')$ is obtained by a comparison argument. It remains to prove estimate (11). We have

$$v_M(t) - u_M(t) = \frac{(t-t_{m-1})}{k}(v^m - v^{m-1}) + v^{m-1} - v^m \qquad \forall\,t\in]t_{m-1},t_m].$$

Then

$$\begin{aligned}\int_0^T \|v_M(t)-u_M(t)\|^2\,dt &= \sum_{m=1}^{M}\int_{t_{m-1}}^{t_m} \|v_M(t)-u_M(t)\|^2\,dt\\ &= \sum_{m=1}^{M}\|v^m - v^{m-1}\|^2 \int_{t_{m-1}}^{t_m}\left(\frac{t-t_{m-1}}{k}-1\right)^2 dt\\ &= \frac{\kappa}{3}\sum_{m=1}^{M}\|v^m-v^{m-1}\|^2.\end{aligned}$$

□

We can finally prove the main result

Proof of Theorem 1.1. The sequence $\{v_M\}_M$ is bounded uniformly in the Hilbert space $L^2(0,T;H^1_\sigma(\Omega))$ and it is such that $\partial_t v_M \in L^{4/3}(0,T;(H^1_\sigma(\Omega))')$, again with bound independent of M. From a standard application of the Aubin-Lions compactness argument, we can extract a (relabelled) sub-sequence $v_M \to v$ in $L^2(0,T;L^2(\Omega))$. Moreover, by using (11) and Lemma 4.4 we have also that

$$u_M - v_M \to 0 \text{ in } L^2(0,T;L^2(\Omega)),$$

and hence also the sequence u_M converges strongly to v in $L^2(0,T;L^2(\Omega))$. By using standard interpolation inequalities and the previous strong convergence, it is now standard to show also that

$$\begin{aligned} v_M &\to v \quad \text{strongly in } L^3(0,T;L^3(\Omega)),\\ u_M &\to v \quad \text{strongly in } L^3(0,T;L^3(\Omega)).\end{aligned}$$

Moreover, since q_M is uniformly bounded in $L^{5/3}((0,T)\times\Omega)$, up to extraction of a further sub-sequence, we have that

$$q_M \rightharpoonup q \text{ weakly in } L^{5/3}((0,T)\times\Omega).$$

Finally, as in [33], we have that v is a weak solutions of the Navier-Stokes equations with associated pressure q.

We show now that (v,q) satisfies the local energy inequality. To this end we test the equations (12) by $u_M\phi$, where $\phi\geq 0$ is smooth and with space-time compact support. The first term regarding the time-derivative is the most relevant for our purposes. We have

$$\begin{aligned}\int_0^T\int_\Omega \partial_t v_M\cdot u_M\phi\,dxd\tau &= \int_0^T\int_\Omega \partial_t v_M\cdot(v_M - v_M + u_M)\,\phi\,dxd\tau\\ &= \int_0^T\int_\Omega \partial_t v_M\cdot v_M\,\phi\,dxd\tau + \int_0^T\int_\Omega \partial_t v_M\cdot(u_M - v_M)\,\phi\,dxd\tau\\ &= I_1 + I_2.\end{aligned}$$

We start with the first term I_1. By splitting the integral over $[0,T]$ with the sum of integrals over $[t_{m-1},t_m]$ and, by performing integration by parts, we immediately obtain

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t v_M\cdot v_M\phi\,dxd\tau = \sum_{m=1}^M\int_{t_{m-1}}^{t_m}\int_\Omega \partial_t v_M\cdot v_M\phi\,dxd\tau\\ &= \sum_{m=1}^M\int_{t_{m-1}}^{t_m}\int_\Omega \frac{1}{2}\partial_t|v_M|^2\,\phi\,dx\\ &= \frac{1}{2}\sum_{m=1}^M\int_\Omega |v^m|^2\,\phi(t_m) - |v^{m-1}|^2\,\phi(t_{m-1})\,dx - \sum_{m=1}^M\int_{t_{m-1}}^{t_m}\int_\Omega \frac{1}{2}|v_M|^2\,\partial_t\phi\,dxd\tau,\end{aligned}$$

where we used that $\partial_t v_M(t) = \frac{v^m - v^{m-1}}{\kappa}$, for all $t\in[t_{m-1},t_m[$. Next, we observe that the sum telescopes and consequently we have

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t v_M\cdot v_M\,\phi\,dxd\tau\\ &\qquad= \int_\Omega \frac{1}{2}|u^M|^2\,\phi(T) - \frac{1}{2}|u_0|^2\,\phi(0)\,dx - \sum_{m=1}^M\int_{t_{m-1}}^{t_m}\int_\Omega \frac{1}{2}|v_M|^2\,\partial_t\phi\,dxd\tau\\ &\qquad= -\int_0^T\int_\Omega \frac{1}{2}|v_M|^2\,\partial_t\phi\,dt.\end{aligned}$$

By the strong convergence of $v_M\to v$ in $L^2(0,T;L^2(\Omega))$, as $M\to+\infty$, we can conclude that

$$\lim_{M\to+\infty}\int_0^T\int_\Omega \partial_t v_M\cdot v_M\phi\,dxd\tau = -\frac{1}{2}\int_0^T\int_\Omega |v|^2\,\partial_t\phi\,dxd\tau.$$

Then, we consider the second term. Since u_M is constant on the interval $[t_{m-1}, t_m[$ we can write

$$\begin{aligned}\int_0^T \int_\Omega \partial_t v_M \cdot (u_M - v_M)\, \phi \, dxd\tau \\ &= -\sum_{m=1}^{M} \int_{t_{m-1}}^{t_m} \int_\Omega \partial_t (v_M - u_M) \cdot (v_M - u_M)\, \phi \, dxd\tau \\ &= -\frac{1}{2} \sum_{m=1}^{M} \int_{t_{m-1}}^{t_m} \int_\Omega \partial_t |v_M - u_M|^2 \, \phi \, dxdt \\ &= \frac{1}{2} \sum_{m=1}^{M} \int_{t_{m-1}}^{t_m} \int_\Omega |v_M - u_M|^2 \, \partial_t \phi \, dxdt,\end{aligned}$$

where the equality in the last line follows integrating by parts. Observe that these integrations by parts (with respect to time over $[t_{m-1}, t_m]$) do not produce boundary terms because $v_M(t_m) = u_M(t_m)$ for all $m = 0, \dots, M$.

Then, since $u_M - v_M$ goes to 0 strongly in $L^2(0, T; L^2_\sigma(\Omega))$, we get that $I_2 \to 0$ as $M \to +\infty$ (or equivalently as $\kappa \to 0$).

By the usual reasoning (in this term the integration by parts is in the space variables, so there is no need for a special treatment) we have that

$$\begin{aligned}-\int_0^T \int_\Omega \Delta u_M \cdot u_M \phi \, dxdt &= \int_0^T \int_\Omega |\nabla u_M|^2 \phi \, dxdt + \frac{1}{2} \int_0^T \int_\Omega \nabla |u_M|^2 \, \nabla \phi \, dxdt \\ &= \int_0^T \int_\Omega |\nabla u_M|^2 \phi \, dxdt - \frac{1}{2} \int_0^T \int_\Omega |u_M|^2 \Delta \phi \, dxd\tau,\end{aligned}$$

and integrations by parts do not produce any boundary term, due to the fact that ϕ is with compact support. By using the lower semi-continuity of the norm and also that $\phi \geq 0$, we obtain:

$$\liminf_{M \to +\infty} \int_0^T \int_\Omega |\nabla u_M|^2 \phi \, dxd\tau \geq \int_0^T \int_\Omega |\nabla v|^2 \phi \, dxd\tau,$$

while again by the strong convergence in $L^2(0, T; L^2(\Omega))$

$$\lim_{M \to +\infty} \frac{1}{2} \int_0^T \int_\Omega |u_M|^2 \Delta \phi \, dxd\tau = \frac{1}{2} \int_0^T \int_\Omega |v|^2 \Delta \phi \, dxd\tau.$$

The convective term is treated again by integrating by parts with respect to the space variables. In fact, we obtain

$$\begin{aligned}\int_0^T \int_\Omega (u_M \cdot \nabla)\, u_M \cdot u_M \phi \, dxd\tau &= \frac{1}{2} \int_0^T \int_\Omega \nabla |u_M|^2 \cdot u_M \phi \, dxd\tau \\ &= -\frac{1}{2} \int_0^T \int_\Omega |u_M|^2 \, u_M \cdot \nabla \phi \, dxd\tau,\end{aligned}$$

and, by the strong convergence $u_M \to v$ in $L^3(0, T; L^3(\Omega))$, we get

$$\lim_{M \to +\infty} \int_0^T \int_\Omega (u_M \cdot \nabla)\, u_M \cdot u_M \phi \, dxd\tau = \frac{1}{2} \int_0^T \int_\Omega |v|^2 \, v \cdot \nabla \phi \, dxd\tau.$$

Finally, the term with the pressure is integrated by parts as follows

$$\int_0^T \int_\Omega \nabla q_M \cdot u_M \phi \, dxd\tau = -\int_0^T \int_\Omega q_M \, u_M \cdot \nabla \phi \, dxd\tau,$$

and, thanks to the weak convergence $q_M \rightharpoonup q$ in $L^{5/3}(0,T;L^{5/3}(\Omega))$ and again the strong convergence of u_M in $L^3(0,T;L^3(\Omega))$, in we get

$$\lim_{M\to+\infty}\int_0^T\int_\Omega \nabla q_M\cdot u_M\phi\,dxd\tau = -\int_0^T\int_\Omega q\,v\cdot\nabla\phi\,dxd\tau.$$

We finally proved that

$$\begin{aligned}2\int_0^T\int_\Omega &|\nabla v(x,\tau)|^2\phi(x,\tau)\,dxd\tau\\ &\leq \int_0^T\int_\Omega\Big(|v(x,\tau)|^2(\partial_t\phi(x,\tau)-\Delta\phi(x,\tau))+(|v(x,\tau)|^2\\ &\qquad\qquad + 2q(x,\tau))v(x,\tau)\cdot\nabla\phi(x,\tau)\Big)\,dxd\tau,\end{aligned}$$

for all smooth and non-negative ϕ, which are space-time with compact support.

By following the argument detailed in [16, p. 13], by with a further test-function only of the time variable and approximating a Dirac's delta at a given time $t\in]0,T]$, one can easily deduce from the latter the validity of the local energy inequality (4) for a.e. $t\in[0,T]$. □

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