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ON THE VALIDITY OF THE LOCAL FOURIER ANALYSIS*

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Abstract

Local Fourier analysis (LFA) is a useful tool in predicting the convergence factors of geometric multigrid methods (GMG). As is well known, on rectangular domains with periodic boundary conditions this analysis gives the *exact* convergence factors of such methods. When other boundary conditions are considered, however, this analysis was judged as been heuristic, with limited capabilities in predicting multigrid convergence rates. In this work, using the Fourier method, we extend these results by proving that such analysis yields the exact convergence factors for a wider class of problems, some of which cannot be handled by the traditional rigorous Fourier analysis.

Mathematics subject classification: 65N55, 65T50
Key words: Local Fourier analysis, Multigrid, Fourier method.

1. Introduction

There exist two main approaches to quantitatively analyze the convergence of geometric multigrid algorithms: the rigorous Fourier analysis (RFA) [1, 2] (also called model problem analysis) and the local Fourier analysis [3] (or local mode analysis). Both techniques use error expansion in terms of the eigenvectors of a discrete differential operator, followed by study of the behavior of the multigrid error transfer operator when acting on these components. The main difference is that the RFA takes into account the boundary conditions, while the LFA neglects the effect of boundary conditions by assuming that the discrete differential operator is defined on an infinite grid. Clearly, in order to perform a RFA it is necessary to find a basis of eigenvectors for the discretized boundary value problem, that is, the basis elements must satisfy the boundary conditions. The convergence rates predicted by RFA then are exact, but also such procedure limits its applicability since to find such a basis may be impossible. Therefore, RFA gives the exact convergence rates of the GMG, but only for a small class of model problems. The LFA, on the other hand, works on an infinite grid and uses a basis of complex valued, exponential functions, which makes it applicable to a much wider class of discretized differential operators. It is well known [4] that the LFA provides accurate approximations of the

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asymptotic convergence factors of the GMG algorithms for many problems, and, moreover, it is exact for problems with periodic boundary conditions. In this work we focus on the question whether the LFA can be made exact (rigorous) for a wider class of discretized boundary value problems, not necessarily with periodic boundary conditions. As it turns out, we can answer this question positively. Our approach relies on the embedding of the model problem into a periodic problem. Similar ideas have also been explored in works on circulant preconditioners for elliptic problems [5, 6] and also for preconditioning the indefinite Helmholtz equation [7]. We introduce a class of operators called LFA-compatible operators here and prove that for such operators the LFA gives the exact multigrid convergence factors. Our studies include the Dirichlet, the Neumann and the mixed boundary condition problem for a constant coefficient, reaction-diffusion equation on a d-dimensional tensor product grid.

2. Preliminaries

2.1. The Dirichlet problem and its discretization

We consider a reaction-diffusion problem in d spatial dimensions on the domain $\Omega^D = (0,1)^d$,

$$-\Delta u(\boldsymbol{x}) + cu(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^D, \quad \text{and} \quad u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega^D, \tag{2.1}$$

where $c > 0$ is a constant. First, let us consider the simplest case when $d = 1$ (one dimensional problem). The computational domain then is the interval $\Omega^D = (0,1)$ and the corresponding two-point boundary value problem (2.1) is:

$$-u''(x) + cu(x) = f(x), \quad x \in \Omega^D, \quad u(0) = u(1) = 0. \tag{2.2}$$

For $d = 1$, we introduce a uniform grid $\Omega_h^D = \{x_k = kh\}_{k=0}^n$, with step size $h = 1/n$, $n \in \mathbb{N}$ and we discretize this problem by the standard central difference scheme. As a result, we obtain the linear system of algebraic equations with tri-diagonal matrix:

$$A_h^D \boldsymbol{u} = \boldsymbol{f} \text{ where } A_h^D = T_h^D + cI_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}, \tag{2.3}$$

where $\boldsymbol{u} = (u_1, \ldots, u_{n-1})^T$, $\boldsymbol{f} = (f_1, \ldots, f_{n-1})^T$, $I_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$ is the identity matrix, and

$$T_h^D = \frac{1}{h^2} \ \mathrm{diag}(-1, 2, -1) \in \mathbb{R}^{(n-1)\times(n-1)}. \tag{2.4}$$

This is the simple, but very important, one dimensional case. In the case of higher spatial dimensions and on a uniform grid with the same step size $h = 1/n$ in all the directions the linear systems are written in compact form by using the standard tensor product $\otimes$ for matrices. We recall the following properties of the tensor product

$$(X + Y) \otimes Z = (X \otimes Z) + (Y \otimes Z), \quad (X_1 \otimes X_2)(Y_1 \otimes Y_2) = (X_1Y_1 \otimes X_2Y_2). \tag{2.5}$$

We further denote the k-th tensor power of a matrix X by $X^{\otimes k} = \underbrace{X \otimes \ldots \otimes X}_{k}$. Finally, let us note that the generalization to different step sizes in different directions is straightforward.

With this notation, the standard second order central difference scheme for discretization of the Dirichlet problem (2.1) results in the linear system

$$A_h^D \boldsymbol{u} = \boldsymbol{f}, \quad A_h^D = \sum_{j=1}^{d} \left(I_{n-1}^{\otimes(j-1)} \otimes T_h^D \otimes I_{n-1}^{\otimes(d-j)} \right) + cI_{n-1}^{\otimes d} \in \mathbb{R}^{(n-1)^d \times (n-1)^d}. \tag{2.6}$$

2.2. A periodic problem

We now consider a finite difference discretization on a grid with step size $h = 1/n$ of a periodic problem on $\Omega^P = (0,2)$:

$$A_h^P \widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{f}}, \quad \text{where} \quad A_h^P = T_h^P + cI_N \in \mathbb{R}^{N\times N}, \tag{2.7}$$

with $N = 2n$ and $T_h^P = \frac{1}{h^2}\left(\text{diag}(-1,2,-1) - \boldsymbol{e}_1^N(\boldsymbol{e}_N^N)^T - \boldsymbol{e}_N^N(\boldsymbol{e}_1^N)^T\right) \in \mathbb{R}^{N\times N}$. Here, we have denoted $\widetilde{\boldsymbol{u}} = (\widetilde{u}_1,\dots,\widetilde{u}_N)^T$, $\widetilde{\boldsymbol{f}} = \left(\widetilde{f}_1,\dots,\widetilde{f}_N\right)^T$, and $\boldsymbol{e}_k^m$ is the k-th canonical Euclidean basis vector in $\mathbb{R}^m$. Finally, let us point out that by a periodic problem here we mean the problem (2.2) defined on Ω^P with boundary conditions $u(0) - u(2) = u'(0) - u'(2) = 0$.

The extension to higher dimension $d > 1$ is obvious and we have the linear system $A_h^P\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{f}}$, with

$$A_h^P = \sum_{j=1}^{d}\left(I_N^{\otimes(j-1)} \otimes T_h^P \otimes I_N^{\otimes(d-j)}\right) + cI_N^{\otimes d} \in \mathbb{R}^{N^d\times N^d}. \tag{2.8}$$

2.3. Relation between the Dirichlet and the periodic problem

Our goal now is to describe how the discretized Dirichlet problem relates to the periodic problem defined in section 2.2. To begin, we consider the 1-dimensional case given in (2.3) and we define the *odd extension operator* as the linear operator $E_{o,h} : \mathbb{R}^{n-1} \to \mathbb{R}^N$, $N = 2n$ such that

$$E_{o,h}\boldsymbol{e}_i^{n-1} = \boldsymbol{e}_i^N - \boldsymbol{e}_{N-i}^N, \quad i = 1,\dots,n-1. \tag{2.9}$$

The *restriction operator* $R_{o,h}$ is defined as $R_{o,h} = \frac{1}{2}E_{o,h}^T$. It is easy to see that the following relations hold in the one dimensional case: $R_{o,h}E_{o,h} = I_{n-1}$, and $E_{o,h}R_{o,h}\boldsymbol{u} = \boldsymbol{u}$, for all $\boldsymbol{u} \in \text{range}(E_{o,h})$. Notice also that $\text{range}(E_{o,h}) = \{\boldsymbol{u} \in \mathbb{R}^N \mid u_n = u_N = 0, u_j = -u_{N-j}, j = 1,\dots,n-1\}$ and $\widetilde{\boldsymbol{f}} = E_{o,h}\boldsymbol{f}$. For $d > 1$ the restriction and extensions are $R_{o,h}^{\otimes d}$ and $E_{o,h}^{\otimes d}$ and we have:

$$R_{o,h}^{\otimes d}E_{o,h}^{\otimes d} = I_{n-1}^{\otimes d}, \quad \text{and} \quad E_{o,h}^{\otimes d}R_{o,h}^{\otimes d}\boldsymbol{u} = \boldsymbol{u}, \quad \text{for all} \quad \boldsymbol{u} \in \text{range}(E_{o,h}^{\otimes d}). \tag{2.10}$$

2.3.1. LFA-compatibility

We now clarify the relation between the Dirichlet and the periodic problem. We begin with a very general definition of LFA-compatibility.

Definition 2.1. *Let $R_{o,h}$ and $E_{o,h}$ be operators satisfying* (2.10). *We say that the pair of operators (M_h^D, M_h^P) is an* LFA-compatible *pair if and only if $M_h^D = R_{o,h}M_h^PE_{o,h}$ and $M_h^P\boldsymbol{v} \in$* range$(E_{o,h})$ *for all $\boldsymbol{v} \in$* range$(E_{o,h})$.

The LFA-compatibility is, in some sense, the minimal requirement which allows to build relations between solutions to a periodic and the corresponding Dirichlet problems, or the iterates constructed in an iterative method for these problems. In a more abstract setting, the operators M_h^P and M_h^D do not have to be a periodic or a Dirichlet problem, they only need to be connected via a compatibility relation based on operators $E_{o,h}$ and $R_{o,h}$ satisfying the relations in (2.10). In the following, however, we only use $E_{o,h}$ and $R_{o,h}$ as defined above.

Now we prove several results, which follow directly from the definition of LFA-compatibility.

Lemma 2.1. *Let A_h^D and A_h^P be the coefficient matrices related to the Dirichlet and periodic problems. Then, (A_h^D, A_h^P) is an LFA-compatible pair.*

Proof. The standard properties of the tensor product imply that

$$\begin{aligned} R_{o,h}^{\otimes d} A_h^P E_{o,h}^{\otimes d} &= R_{o,h}^{\otimes d} \left(\sum_{j=1}^{d} \left(I_N^{\otimes(j-1)} \otimes T_h^P \otimes I_N^{\otimes(d-j)} \right) + c I_N^{\otimes d} \right) E_{o,h}^{\otimes d} \\ &= R_{o,h}^{\otimes d} \left(\sum_{j=1}^{d} \left(E_{o,h}^{\otimes(j-1)} \otimes T_h^P E_{o,h} \otimes E_{o,h}^{\otimes(d-j)} \right) + c E_{o,h}^{\otimes d} \right) \\ &= \sum_{j=1}^{d} \left(I_{n-1}^{\otimes(j-1)} \otimes R_{o,h} T_h^P E_{o,h} \otimes I_{n-1}^{\otimes(d-j)} \right) + c I_{n-1}^{\otimes d}. \end{aligned}$$

Further, taking into account that $R_{o,h} T_h^P E_{o,h} = T_h^D$, we also have $A_h^D = R_{o,h}^{\otimes d} A_h^P E_{o,h}^{\otimes d}$. If $\boldsymbol{u} \in \text{range}(E_{o,h}^{\otimes d})$, then there exists $\boldsymbol{v} \in \mathbb{R}^{(n-1)^d}$ such that $\boldsymbol{u} = E_{o,h}^{\otimes d} \boldsymbol{v}$ and we have

$$\begin{aligned} A_h^P \boldsymbol{u} &= \left(\sum_{j=1}^{d} I_N^{\otimes(j-1)} \otimes T_h^P \otimes I_N^{\otimes(d-j)} \right) E_{o,h}^{\otimes d} \boldsymbol{v} + c I_N^{\otimes d} E_{o,h}^{\otimes d} \boldsymbol{v} \\ &= \left(\sum_{j=1}^{d} E_{o,h}^{\otimes(j-1)} \otimes T_h^P E_{o,h} \otimes E_{o,h}^{\otimes(d-j)} \right) \boldsymbol{v} + c E_{o,h}^{\otimes d} \boldsymbol{v}. \end{aligned}$$

A straightforward computation shows that $T_h^P \boldsymbol{u} \in \text{range}(E_{o,h})$ for any $\boldsymbol{u} \in \text{range}(E_{o,h})$, and this completes the proof. □

Lemma 2.2. *If $\boldsymbol{u}$ satisfies $A_h^D \boldsymbol{u} = \boldsymbol{f}$, then $A_h^P (E_{o,h}^{\otimes d} \boldsymbol{u}) = E_{o,h}^{\otimes d} \boldsymbol{f}$.*

Proof. Using that $A_h^D = R_{o,h}^{\otimes d} A_h^P E_{o,h}^{\otimes d}$, we have that $R_{o,h}^{\otimes d} A_h^P E_{o,h}^{\otimes d} \boldsymbol{u} = \boldsymbol{f}$. Applying $E_{o,h}^{\otimes d}$ on the left and taking into account that $A_h^P E_{o,h}^{\otimes d} \boldsymbol{u} \in \text{range}(E_{o,h}^{\otimes d})$, completes the proof. □

Theorem 2.3. *The pair $((A_h^D)^{-1}, (A_h^P)^{-1})$ is LFA-compatible.*

Proof. We consider $\boldsymbol{f} \in \text{range}(E_{o,h}^{\otimes d})$. Then, there exists $\boldsymbol{g} \in \mathbb{R}^{(n-1)^d}$ such that $E_{o,h}^{\otimes d} \boldsymbol{g} = \boldsymbol{f}$. If $\boldsymbol{u} = (A_h^D)^{-1} \boldsymbol{g}$, by using Lemma 2.2 we have that $E_{o,h}^{\otimes d} \boldsymbol{u} = (A_h^P)^{-1} \boldsymbol{f}$, which implies that $(A_h^P)^{-1} \boldsymbol{f} \in \text{range}(E_{o,h}^{\otimes d})$. Next, again from Lemma 2.2, it follows that if $\boldsymbol{u} = (A_h^D)^{-1} \boldsymbol{f}$, then

$$(A_h^P)^{-1} E_{o,h}^{\otimes d} \boldsymbol{f} = E_{o,h}^{\otimes d} \boldsymbol{u}.$$

Hence,

$$R_{o,h}^{\otimes d} (A_h^P)^{-1} E_{o,h}^{\otimes d} \boldsymbol{f} = R_{o,h}^{\otimes d} E_{o,h}^{\otimes d} \boldsymbol{u} = \boldsymbol{u},$$

and the proof is complete. □

3. Linear Iterative Methods and Multigrid

Let us consider a general stationary iterative method for the Dirichlet and the periodic problems:

$$\boldsymbol{u}^{k+1} = \boldsymbol{u}^k + B_h^D(\boldsymbol{f} - A_h^D \boldsymbol{u}^k), \quad \widetilde{\boldsymbol{u}}^{k+1} = \widetilde{\boldsymbol{u}}^k + B_h^P(\widetilde{\boldsymbol{f}} - A_h^P \widetilde{\boldsymbol{u}}^k), \tag{3.1}$$

where $B_h^{D,P}$ are linear operators (called iterators). We have the following theorem which shows that the LFA-compatibility of the iterators provides a relation between the iterates.

Theorem 3.1. *Let* (B_h^D, B_h^P) *be an LFA-compatible pair and* $\widetilde{\boldsymbol{f}} = E_{o,h}^{\otimes d}\boldsymbol{f}$*. If* $\widetilde{\boldsymbol{u}}^0 = E_{o,h}^{\otimes d}\boldsymbol{u}^0$*, then* $\widetilde{\boldsymbol{u}}^k = E_{o,h}^{\otimes d}\boldsymbol{u}^k,\ k = 1, 2, \dots$

Proof. We prove the result by showing that if $\widetilde{\boldsymbol{u}}^k = E_{o,h}^{\otimes d}\boldsymbol{u}^k$ then $\widetilde{\boldsymbol{u}}^{k+1} = E_{o,h}^{\otimes d}\boldsymbol{u}^{k+1}$. Clearly, from (3.1), and the fact that $\widetilde{\boldsymbol{u}}^k = E_{o,h}^{\otimes d}\boldsymbol{u}^k$, we have

$$\widetilde{\boldsymbol{u}}^{k+1} = E_{o,h}^{\otimes d}\boldsymbol{u}^k + B_h^P(E_{o,h}^{\otimes d}\boldsymbol{f} - A_h^P E_{o,h}^{\otimes d}\boldsymbol{u}^k).$$

Next, we use Lemmas 2.1–2.3 to obtain that,

$$\widetilde{\boldsymbol{u}}^{k+1} = E_{o,h}^{\otimes d}\boldsymbol{u}^k + B_h^P(E_{o,h}^{\otimes d}\boldsymbol{f} - E_{o,h}^{\otimes d}R_{o,h}^{\otimes d}A_h^P E_{o,h}^{\otimes d}\boldsymbol{u}^k) = E_{o,h}^{\otimes d}\boldsymbol{u}^k + B_h^P E_{o,h}^{\otimes d}(\boldsymbol{f} - A_h^D\boldsymbol{u}^k).$$

Since, $E_{o,h}^{\otimes d}(\boldsymbol{f} - A_h^D\boldsymbol{u}^k) \in \operatorname{range}(E_{o,h}^{\otimes d})$, and $B_h^P E_{o,h}^{\otimes d}(\boldsymbol{f} - A_h^D\boldsymbol{u}^k) \in \operatorname{range}(E_{o,h}^{\otimes d})$ we have that

$$\widetilde{\boldsymbol{u}}^{k+1} = E_{o,h}^{\otimes d}\boldsymbol{u}^k + E_{o,h}^{\otimes d}R_{o,h}^{\otimes d}B_h^P E_{o,h}^{\otimes d}(\boldsymbol{f} - A_h^D\boldsymbol{u}^k).$$

Finally, we use that (B_h^D, B_h^P) is an LFA-compatible pair to obtain that

$$\widetilde{\boldsymbol{u}}^{k+1} = E_{o,h}^{\otimes d}(\boldsymbol{u}^k + B_h^D(\boldsymbol{f} - A_h^D\boldsymbol{u}^k)) = E_{o,h}^{\otimes d}\boldsymbol{u}^{k+1}$$

which is what we wanted to show. □

Remark 3.1. Clearly, this theorem implies that LFA compatible iterators (B_h^D, B_h^P) result in error transfer operators which are invariant on $\operatorname{range}(E_{o,h}^{\otimes d})$. Therefore, when the initial guess for the periodic problem is in $\operatorname{range}(E_{o,h}^{\otimes d})$ we have the following immediate corollaries from Theorem 3.1:

- Every iterate for the periodic problem is an extension of the same iterate for the Dirichet problem.
- The (asymptotic) convergence factors for the iterative methods applied to the periodic and the Dirichlet problems are identical.

3.1. Two grid methods

We now consider the two-grid and multigrid methods. We begin by defining the coarse grids for the Dirichlet and periodic problems in one spatial dimension ($d = 1$). In a standard fashion, we define

$$\Omega_{2h}^D = \Big\{x_i = 2ih \mid i = 0, \dots, n/2\Big\}, \quad \text{and} \quad \Omega_{2h}^P = \Big\{x_i = 2ih \mid i = 0, \dots, n\Big\}.$$

We denote by $\mathcal{G}(\Omega_h^{D,P})$, $\mathcal{G}(\Omega_{2h}^{D,P})$ the subspaces of grid-functions defined on $\Omega_h^{D,P}$ and $\Omega_{2h}^{D,P}$, respectively. On such coarse grid, we also define $A_{2h}^{D,P}$ by (2.6) but with $2h$ instead of h. The extension to higher spatial dimensions is done using standard tensor products of grids and operators.

We now consider the two-grid algorithms, which are linear iterative methods already defined in (3.1) with special iterators $B_{TG} = B_{TG}^{D,P}$ as follows:

$$B_{TG} = \big(I - (I - I_{2h,h}(A_{2h})^{-1} I_{h,2h} A_h)(I - S_h A_h)\big)\,(A_h)^{-1}, \tag{3.2}$$

In (3.2) all operators change depending on whether we consider Dirichlet or periodic problem, namely, we have A_h^D, A_h^P, $I_{2h,h}^D$, $I_{2h,h}^P$, etc. Here, $S_h^{D,P}$ are relaxation (smoothing) operators, $I_{h,2h}^{D,P} : \mathcal{G}(\Omega_h^{D,P}) \to \mathcal{G}(\Omega_{2h}^{D,P})$ are the restriction operators and $I_{2h,h}^{D,P} : \mathcal{G}(\Omega_{2h}^{D,P}) \to \mathcal{G}(\Omega_h^{D,P})$ are the prolongation operators. To prove the main result, we need to introduce LFA-compatible restriction and prolongation operators. We say that the pairs $(I_{2h,h}^D, I_{2h,h}^P)$ and $(I_{h,2h}^D, I_{h,2h}^P)$ are *LFA-compatible* if and only if

$$I_{h,2h}^D = R_{o,2h} I_{h,2h}^P E_{o,h}, \quad I_{h,2h}^P v \in \text{range}(E_{o,2h}), \quad \text{for all} \quad v \in \text{range}(E_{o,h}), \tag{3.3}$$
$$I_{2h,h}^D = R_{o,h} I_{2h,h}^P E_{o,2h}, \quad I_{2h,h}^P v \in \text{range}(E_{o,h}), \quad \text{for all} \quad v \in \text{range}(E_{o,2h}). \tag{3.4}$$

The multigrid iterator is obtained from the two grid by recursion, namely,

$$B_h = (I - (I - I_{2h,h} B_{2h} I_{h,2h} A_h)(I - S_h A_h))\,(A_h)^{-1}, \tag{3.5}$$

where $B_{nh} = A_{nh}^{-1}$ for both the Dirichlet and the periodic problem.

We have the following theorem, showing that the iterations via two grid are related.

Theorem 3.2. *If* (A_h^D, A_h^P), $((A_{2h}^D)^{-1}, (A_{2h}^P)^{-1})$, (S_h^D, S_h^P), $(I_{2h,h}^D, I_{2h,h}^P)$, $(I_{h,2h}^D, I_{h,2h}^P)$ *are LFA compatible, then* (B_h^D, B_h^P) *is LFA-compatible.*

Proof. We prove this theorem for the case $d = 1$ only and $B_h = B_{TG}$ as the general case follows from recursive application of this argument and the properties of tensor product listed earlier.

$$\begin{aligned} R_{o,h} B_{TG}^P E_{o,h} &= R_{o,h}\left(I - (I - I_{2h,h}^P (A_{2h}^P)^{-1} I_{h,2h}^P A_h^P) E_{o,h} R_{o,h} (I - S_h^P A_h^P)\right) E_{o,h} R_{o,h} (A_h^P)^{-1} E_{o,h} \\ &= \left(I - (I - R_{o,h} I_{2h,h}^P (A_{2h}^P)^{-1} I_{h,2h}^P A_h^P E_{o,h})(I - S_h^D A_h^D)\right)(A_h^D)^{-1}. \end{aligned}$$

Moreover, because of the invariant properties it follows that

$$R_{o,h} I_{2h,h}^P (A_{2h}^P)^{-1} I_{h,2h}^P A_h^P E_{o,h} = (R_{o,h} I_{2h,h}^P E_{2h})(R_{2h} (A_{2h}^P)^{-1} E_{2h})(R_{2h} I_{h,2h}^P E_{o,h})(R_{o,h} A_h^P E_{o,h}).$$

By using the properties in the assumptions in the theorem we have that $B_{TG}^D = R_{o,h} B_{TG}^P E_{o,h}$. The invariant property of B_{TG}^P follows from the invariant properties of all the operators involved in the two-grid method. □

4. Examples and Extensions

The compatibility result in Theorem 3.2 shows that the LFA, which is exact for periodic problems, provides rigorous results also for the Dirichlet problems. Of course, this is for particular choices of S_h, $I_{h,2h}$ and the rest of the operators involved in the analysis given above. LFA-compatible smoothers include the weighted Jacobi method, the Red-Black Gauss-Seidel, line relaxation methods and polynomial smoothers. The frequently used inter-grid transfer operators, such as full-weighting and bilinear interpolation, are LFA-compatible restriction and prolongation operators, respectively. As a conclusion, multigrid methods based on these components applied to problems with Dirichlet boundary conditions can be analyzed rigorously by LFA.

Moreover, problems with other boundary conditions can also be put into this framework. For example, all the results presented here are easily carried over to the pure Neumann problem by using an even extension operator instead the odd extension operator.

Another interesting case, which cannot be analyzed with a technique such as RFA is when mixed boundary conditions are considered. We illustrate this on the following two point boundary value problem:

$$-u''(x) + cu(x) = f(x), \quad x \in \Omega^M = (0,1), \quad u(0) = u'(1) = 0. \tag{4.1}$$

The standard central difference scheme on a uniform grid with step size $h = 1/n$, leads to the following linear system of algebraic equations with a tri-diagonal matrix:

$$A_h^M \boldsymbol{u} = \boldsymbol{f} \text{ where } A_h^M = T_h^M + cI_n \in \mathbb{R}^{n\times n}, \tag{4.2}$$

where $\boldsymbol{u} = (u_1, \ldots, u_n)^T$, $\boldsymbol{f} = (f_1, \ldots, f_n)^T$, $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix, and

$$T_h^M = \frac{1}{h^2}\left(\text{diag}(-1,2,-1) - e_n^n (e_n^n)^T\right) \in \mathbb{R}^{n\times n}. \tag{4.3}$$

Following the analysis for the discretized Dirichlet problem, we only need to define the extension operators and then consider the corresponding periodic problem (2.7) with adequate N. As is easy to see, because we have both Dirichlet and Neumann conditions, we need to combine the extension operators and consider the composition of an even extension operator and an odd extension operaror, $E_{o,h}E_{e,h}$. Here, $E_{e,h} : \mathbb{R}^n \to \mathbb{R}^{2n}$ is defined as

$$E_{e,h}\boldsymbol{e}_i^n = \boldsymbol{e}_i^{2n} + \boldsymbol{e}_{2n-i}^{2n}, \quad i = 1, \ldots, n. \tag{4.4}$$

The process of extending a function f is depicted in Fig. 4.1.

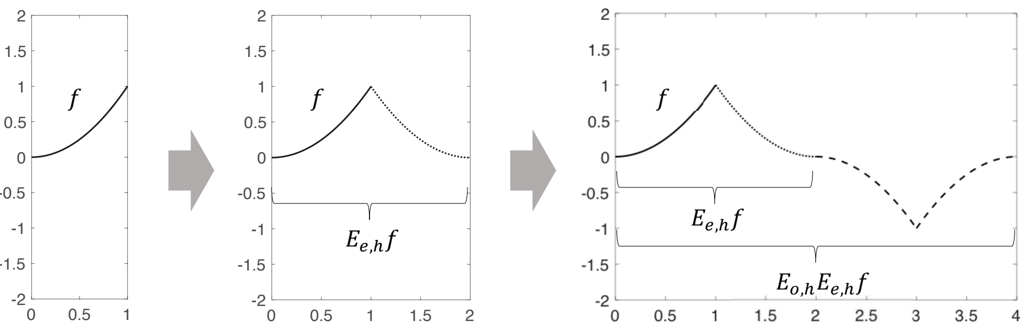

Fig. 4.1. Extension of function f to $E_{o,h}E_{e,h}f$ for the analysis of the one-dimensional problem with mixed boundary conditions.

The resulting periodic problem is discretized on the interval $(0,4)$, because we have two extension operators. The rest of the analysis is analogous to the analysis for the Dirichlet problem considered earlier. Therefore, all constructions for the Dirichet problem also can be carried out to problems with mixed boundary conditions, and in any spatial dimension. For example, let us briefly discuss such a procedure for a two-dimensional reaction-diffusion problem in a square domain $(0, 1)^2$ with mixed boundary conditions. To fix ideas, let us assume that the Dirichlet boundary conditions are imposed on the west and south boundaries, and on the east and north boundaries we have Neumann conditions. An illustration on how to perform the extension of the right-hand side in this case and obtain LFA compatible pairs of operators is shown in Fig. 4.2. Notice that the construction of such extension in 2D (and 3D, etc.) corresponds to applying the extension operator constructed for the 1D problem (4.1) in each of the coordinate directions. Obviously, other combinations of boundary conditions can be handled by using appropriate compositions of the even and odd extensions proposed earlier.

Summarizing, the proposed strategy works if Dirichlet or Neumann boundary conditions are imposed everywhere, and even if Neumann and Dirichlet conditions are combined in any way on the boundary of the domain. All these cases can be dealt with by considering appropriate even, odd or combinations of both extensions, as shown for the mixed boundary conditions.

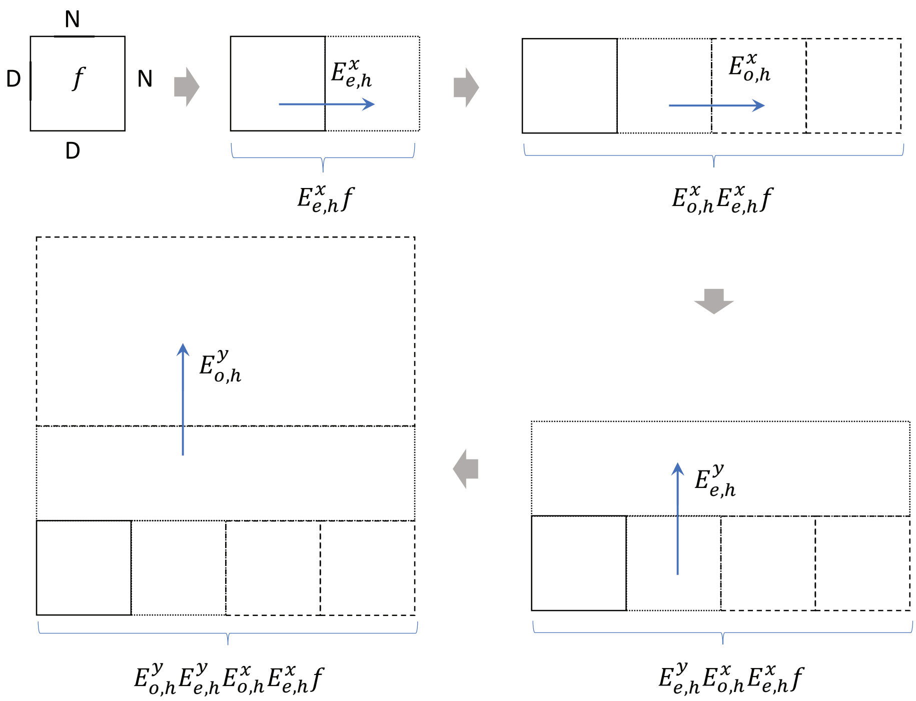

Fig. 4.2. Extension of function f for the analysis of the two-dimensional problem with mixed boundary conditions.

Remark 4.1. An interesting and non-trivial extension of the presented methodology would be to problems defined on more general domains. This case could for instance be handled by using the auxiliary space method introduced by Xu in [8]. This is, however, a topic of our ongoing research and will be reported in our future work.

5. Conclusions

In this work, we studied the accuracy of the predictions of the local Fourier analysis and the novelty of the results is found in several important aspects. Firstly, we have proved that LFA provides exact values of the convergence factors of multigrid methods not only for problems with periodic boundary conditions but also for some problems with Dirichlet, Neumann and mixed boundary conditions. Although, for some of these problems, a rigorous Fourier analysis can also be applied, we remark that the use of local Fourier analysis is more transparent and can be applied to a wider class of problems. The proposed strategy can be used to prove that LFA provides exact convergence rates in cases in which the rigorous Fourier analysis cannot be applied, as when mixed Dirichlet-Neumann boundary conditions are considered.

Finally, the results and techniques reported here show that the LFA provides the exact (rigorous) asymptotic multigrid convergence factors for a wide range of multigrid methods: with different smoothers, different intergrid transfer operators, and when applied to periodic as well as to non-periodic boundary value problems.

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