Journal of Computational Mathematics
Vol.30, No.2, 2012, 101–123.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1109-m3522

WEAK-DUALITY BASED ADAPTIVE FINITE ELEMENT METHODS FOR PDE-CONSTRAINED OPTIMIZATION WITH POINTWISE GRADIENT STATE-CONSTRAINTS*

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Abstract

Adaptive finite element methods for optimization problems for second order linear elliptic partial differential equations subject to pointwise constraints on the ℓ^2-norm of the gradient of the state are considered. In a weak duality setting, i.e. without assuming a constraint qualification such as the existence of a Slater point, residual based a posteriori error estimators are derived. To overcome the lack in constraint qualification on the continuous level, the weak Fenchel dual is utilized. Several numerical tests illustrate the performance of the proposed error estimators.

Mathematics subject classification: 65N30, 90C46, 65N50, 49K20, 49N15, 65K10.
Key words: Adaptive finite element method, A posteriori errors, Dualization, Low regularity, Pointwise gradient constraints, State constraints, Weak solutions.

1. Introduction

In this paper we study the state constrained optimal control problem

$$\left\{\begin{array}{ll} \text{minimize} & J(y,u) := \frac{1}{2}\|y-y_d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u\|_U^2 \quad \text{over } (y,u)\in V\times U \\ \text{subject to} & Ay = u + f \text{ in } V^*, \\ & |\nabla y| \le \psi \text{ a.e. in } \Omega, \end{array}\right. \tag{P}$$

where $\Omega\subset\mathbb{R}^n$, $n\in\{1,2,3\}$, is an open, bounded domain with boundary $\Gamma:=\partial\Omega$, $V:=H_0^1(\Omega)$, $U=L^2(\Omega)$, $y_d\in L^2(\Omega)$, $\alpha>0$, $A:V\to V^*$ denotes a self-adjoint second order linear elliptic partial differential operator, $f\in L^2(\Omega)$, and $\psi\in L^2(\Omega)$ with $\psi\ge\underline{\psi}$ a.e. in Ω for some $\underline{\psi}\in\mathbb{R}_{++}$. Here and below $\|\cdot\|_{0,\Omega}$ refers to the standard $L^2(\Omega)$-norm. In (P) we have $\|\cdot\|_U=\|\cdot\|_{0,\Omega}$. We call y the state and u the control. Of course, more general objective functionals are conceivable, but our choice reflects the often considered tracking-type objective involving a desired state y_d, which may result from measurements, and control costs α. Moreover, convex quadratic

* Received September 15, 2010 / Revised version received April 30, 2011 / Accepted July 29, 2011 / Published online February 24, 2012 /

objectives and affine partial differential equation (PDE) constraints such as those in (P) appear naturally in sequential quadratic programming approaches in optimization.

Pointwise constraints on the gradient of the state, as imposed in (P), are important, e.g., in material science in order to avoid large material stresses. Such stresses may arise from unbalanced cooling regimes in transient phenomena and/or due to the geometric structure of the underlying PDE domain. Usually large stresses cause adverse effects in the material leading to reduced life time or other deterioration. Geometric features of the PDE domain such as cracks or re-entrant corners play also a crucial role in stationary cases in elasticity, where stresses are usually high at the crack tip or the re-entrant corner of, e.g., an L-shaped domain. Thus, it might be desirable to exercise some control in order to reduce these potentially adverse effects.

Even in situations where the PDE domain is smooth, from an optimization theoretic point of view pointwise constraints on the state lead to poor multiplier regularity when characterizing first order optimality by means of a Karush-Kuhn-Tucker (KKT) system. Corresponding theoretical studies can be found in [7–9]. For the derivation of such a KKT-system it is commonly invoked that the feasible set of the optimization problem (like (P)) admits a so-called Slater point. In connection with (P), this requirement results in a function space setting of $V := W^{2,r}(\Omega) \cap H_0^1(\Omega)$, with $r > n$, for the state space and $U := L^r(\Omega)$ for the control space. This yields $\nabla y \in C(\bar{\Omega})^n$ which is needed for the existence of a Slater point.

For pointwise constraints on the gradient and less regular domains (like cracked domains or L-shapes) such a high regularity of the state is out of reach [14]. Hence, the derivation of a primal-dual first order optimality characterization cannot rely on standard tools requiring a constraint qualification such as the existence of a Slater point. As a consequence, one may need to work under a weaker first order condition, i.e., without a bounded set of multipliers associated with $|\nabla y| \leq \psi$ a.e. in Ω. We also note that the domain and the bound ψ have to be compatible in order to yield a non-empty feasible set of (P). For example, requiring $\psi \in L^\infty(\Omega)$ in the presence of a crack, which, however, rules out L^∞-regularity of the gradient of the solution of our PDE with $L^2(\Omega)$-right-hand side, causes an incompatibility and, thus, an empty feasible set. In such cases the optimization problem (P) is void. Hence, throughout this paper we assume that such a data compatibility holds true, i.e., we may assign a well-defined solution operator $G : V^* \to V$ to the PDE in (P).

Adaptive finite element methods have been widely and successfully used for the efficient numerical solution of boundary and initial-boundary value problems for partial differential equations; see, e.g., the monographs [1,3,4,13,23,26] and the many references therein. Recently, residual based a posteriori as well as dual-weighted residual based goal oriented estimators for PDE-constrained optimization problems with pointwise constraints on the control or the state were studied; see, e.g., [5, 17–19, 21, 22, 27]. Concerning constraints on the gradient of the state, however, the present literature is rather scarce; here we refer to recent a priori estimates in [11] based on a certain mixed finite element approach, and to [15]. Compared to pointwise constraints on the state, i.e., $y \leq \Psi$ a.e. in Ω, gradient constraints involve the gradient operator, which has a non-trivial kernel, and require very smooth, i.e. $C^1(\bar{\Omega})$, states in order to guarantee a constraint qualification, such as the existence of a Slater point. Both aspects trouble the existence of a bounded set of Lagrange multiplier with the latter preventing practically relevant, non-smooth PDE domains, as pointed out above. This also has an immediate effect in the a posteriori error analysis as one has to avoid explicit use of a Lagrange multiplier.

In the present paper we are, thus, interested in developing reliable residual based a posteriori error estimators for an adaptive finite element discretization of (P). In particular we study the case of non-smooth domains such that the Slater condition fails to hold due to poor state regularity. In Section 2 we use a state-reduced approach and tools from convex analysis for deriving a first order optimality characterization. Due to the lack of a constraint qualification, the existence and boundedness of Lagrange multipliers cannot be expected in our setting. In order to get some insight into existence and regularity of multipliers we study the Fenchel dual of (P). Then, in the subsequent Section 3, we investigate a discrete version of (P) and its dual. Section 4 is devoted to residual based a posteriori error estimators for adapting our finite element discretization. The final Section 5 contains a selection of numerical tests showing the efficiency of our estimators.

Notation. Throughout this paper we use bold face characters for vectors or vector-valued functions or measures. Similarly, for function spaces such as $L^2(\Omega)^n$ we use $\mathbf{L}^2(\Omega) = L^2(\Omega)^n$. For two Banach spaces B_1 and B_2, $\mathcal{L}(B_1, B_2)$ represents the space of linear, continuous operators from B_1 to B_2. If $B = B_1 = B_2$, then we write $\mathcal{L}(B)$ instead of $\mathcal{L}(B, B)$. Adjoint operators and dual spaces are denoted by superscript '*'. The indicator function of a set S is written as I_S and satisfies $I_S(s) = 0$ if $s \in S$ and $I_S(s) = +\infty$ otherwise. By $\|\cdot\|_{0,\Omega}$ and $(\cdot,\cdot)_{0,\Omega}$ we denote the standard $L^2(\Omega)$-norm and $L^2(\Omega)$-inner product. In a slight misuse of notation we use the same symbols for the norm and inner product in $\mathbf{L}^2(\Omega)$. By $|\cdot|$ we denote the Euclidean norm of a vector in $\mathbb{R}^n$. Weak convergence is denoted by $\rightharpoonup$ and weak* convergence by $\overset{*}{\rightharpoonup}$. Further, 'a.e.' stands for 'almost everywhere'.

2. Optimality Characterizations, Dual Problem, and Weak Solutions

As pointed out in the introduction, without a suitable constraint qualification, such as the existence of a Slater point for the pointwise inequality constraint, Lagrange multipliers for characterizing first order optimality may fail to exist. Here, in our rather weak setting without Slater points, we pursue two directions: on one hand, we study a primal optimality characterization based on a state-reduced problem and the normal cone of the convex feasible set, and on the other hand, utilizing the technique of [12] we study the Fenchel-dual problem of (P).

2.1. Optimality characterization

Let $\hat{V}$ with $\hat{V} \subset V \subset L^2(\Omega)$ be a reflexive Banach space. We assume that $\hat{G} : L^2(\Omega) \to \hat{V}$ is the invertible operator associating with a given $u \in U$ the solution $y = y(u)$ of $Ay = u + f$ in $L^2(\Omega)$. Its existence is guaranteed by a data compatibility assumption. This allows us to express u in terms of $u = \hat{G}^{-1}y - f$. Inserting this relation into (P), we obtain the state-reduced problem

$$\text{minimize} \quad \hat{J}(y) + I_{\hat{K}}(y) \quad \text{over } y \in \hat{V}, \tag{$\hat{P}$}$$

where $\hat{J}(y) = J(y, \hat{G}^{-1}y - f)$ and the closed convex set $\hat{K} \subset \hat{V}$ is given by

$$\hat{K} := \Big\{ v \in \hat{V} \ : \ |\nabla v| \le \psi \text{ a.e. in } \Omega \Big\}.$$

Note $\hat{K}$ is nonempty, and that $\hat{J}(\cdot)$ is a closed convex and proper functional. Further, $J(\cdot, \hat{G}^{-1}\cdot -f)$ is continuous at $y_0 = 0 \in \hat{K}$. Like (P), $(\hat{P})$ admits a unique solution $y \in V$. Hence, [2, Thm. 9.5.5] yields that the optimal solution y satisfies

$$\begin{aligned} 0 &= \hat{J}'(y) + w \\ &= J_y(y, \hat{G}^{-1}y - f) + \hat{G}^{-*} J_u(y, \hat{G}^{-1}y - f) + w \quad \text{for some } w \in N_{\hat{K}}(y). \end{aligned} \tag{2.1}$$

Here, $N_{\hat{K}}(y)$ denotes the normal cone of $\hat{K}$ given by

$$N_{\hat{K}}(y) = \Big\{\xi \in \hat{V}^* \ : \ \langle \xi, z - y\rangle_{\hat{V}^*,\hat{V}} \leq 0 \text{ for all } z \in \hat{K}\Big\}. \tag{2.2}$$

Applying $\hat{G}^*$ to (2.1) we obtain

$$0 = \hat{G}^*\hat{J}'(y) + \hat{G}^*w = \hat{G}^*J_y(y, \hat{G}^{-1}y - f) + J_u(y, \hat{G}^{-1}y - f) + \hat{G}^*w. \tag{2.3}$$

In order to get an amenable representation for (2.3) we define

$$p := \hat{G}^*J_y(y, \hat{G}^{-1}y - f) + \hat{G}^*w \in L^2(\Omega).$$

This yields the adjoint equation

$$Ap - J_y(y, \hat{G}^{-1}y - f) - w = 0 \text{ in } \hat{V}^* \tag{2.4}$$

and (2.3) becomes

$$p + J_u(y, \hat{G}^{-1}y - f) = 0. \tag{2.5}$$

We call p the adjoint state associated with (2.1). Utilizing $Ay = u + f$, we summarize our above findings.

Theorem 2.1. *The optimal solution $(y,u) \in V \times L^2(\Omega)$ of (P) is characterized by the existence of a unique pair $(p,w) \in L^2(\Omega) \times \hat{V}^*$ satisfying*

$$Ay - u = f \qquad \text{in } L^2(\Omega), \tag{2.6}$$

$$Ap - y - w = -y_d \qquad \text{in } \hat{V}^*, \tag{2.7}$$

$$p + \alpha u = 0 \qquad \text{in } L^2(\Omega), \tag{2.8}$$

$$w \in N_{\hat{K}}(y) \subset \hat{V}^*, \tag{2.9}$$

with the normal cone $N_{\hat{K}}(y)$ defined by (2.2).

Note that Theorem 2.1 does not provide further insight into structural properties of the Lagrange multiplier associated with the constraints in $\hat{K}$. For this purpose and later use in the discretized setting we switch to the Fenchel dual of (P). Before we commence with studying the dual, we briefly discuss Theorem 2.1 in the smooth setting $\hat{V} = W^{2,r}(\Omega) \cap H_0^1(\Omega)$ with $r > n$ which was considered in [8]. In this case there exists a so-called Slater point, which is a feasible point of (P) such that there exists $\delta > 0$ with $|\nabla(y_0 + y)| \leq \psi$ in Ω for all feasible $y \in B^\infty(0;\delta)$, where $B^\infty(0;\delta)$ denotes the ball of radius δ about 0 in $C^1(\bar{\Omega})$. In fact, $u_0 = -f$ and $y_0 = 0$ with $|\nabla y_0| < \underline{\psi} \leq \psi$ in Ω defines such a Slater point. Under these assumptions, the existence of a Lagrange multiplier $\boldsymbol{\mu}$ in $\mathbf{M}(\bar{\Omega})$, the space of regular Borel vector measures on $\bar{\Omega}$, is deduced in [8] such that w in our setting may be identified with $-\operatorname{div}\boldsymbol{\mu}$.

2.2. Dual problem and weak solution

For the derivation of the dual we remain, for the time being, in the general setting of (P) and use the solution operator $G : V^* \to V$ associated with the PDE in (P). As for $\hat{G}$ before, its existence is guaranteed by a data compatibility assumption. This allows us to rewrite (P) in control-reduced form as

$$\inf\ \mathcal{F}(u) + \mathcal{G}(\Lambda u) \quad \text{over } u \in U \tag{2.10}$$

with

$$\mathcal{F} : U \to \mathbb{R}, \quad \mathcal{F}(u) = J(Gu, u), \tag{2.11}$$

$$\mathcal{G} : \mathbf{L}^2(\Omega) \to \mathbb{R}, \quad \mathcal{G}(\mathbf{q}) = I_{\mathbf{K}}(\mathbf{q}), \tag{2.12}$$

and

$$\Lambda := \nabla G, \quad \Lambda : L^2(\Omega) \to \mathbf{L}^2(\Omega), \tag{2.13}$$

where $I_{\mathbf{K}}(\cdot)$ denotes the indicator function of the convex set

$$\mathbf{K} = \Big\{ \mathbf{w} \in \mathbf{L}^2(\Omega) \ : \ |\mathbf{w}| \le \psi \text{ a.e. in } \Omega \Big\}.$$

The convex conjugate of $\mathcal{F}$, denoted by $\mathcal{F}^*$, is defined by

$$\mathcal{F}^*(u^*) = \sup \Big\{ (u^*, u)_{0,\Omega} - \mathcal{F}(u) \ : \ u \in U \Big\};$$

analogously for $\mathcal{G}^*$. For our particular choice of $\mathcal{F}$ we easily verify

$$\mathcal{F}^*(u^*) = \frac{1}{2} \| u^* + G^* y_d + \alpha f \|^2_{M^{-1}}, \tag{2.14}$$

where $M = G^*G + \alpha \,\mathrm{id}$ with id denoting the identity, and $\| \cdot \|^2_{M^{-1}} = (M^{-1} \cdot, \cdot)_{0,\Omega}$. Observe that $\| \cdot \|^2_{M^{-1}}$, indeed, defines a norm on $L^2(\Omega)$ since the Fredholm-operator $M : L^2(\Omega) \to L^2(\Omega)$ is positive-definite and admits a uniformly bounded inverse, which is positive-definite as well. The convex conjugate of $\mathcal{G}$ is

$$\mathcal{G}^*(\mathbf{q}^*) = \int_\Omega \psi \, |\mathbf{q}^*| \, d\mathbf{x}. \tag{2.15}$$

Indeed, we have $\mathcal{G}^*(\mathbf{q}^*) = \sup\{ (\mathbf{q}^*, \mathbf{q})_{0,\Omega} - I_{\mathbf{K}}(\mathbf{q}) \ : \ \mathbf{q} \in \mathbf{L}^2(\Omega) \}$ which implies $0 \in \mathbf{q}^* - \partial I_{\mathbf{K}}(\mathbf{q})$. This is equivalent to

$$(\mathbf{q}^*, \mathbf{r} - \mathbf{q})_{0,\Omega} \le I_{\mathbf{K}}(\mathbf{r}) - I_{\mathbf{K}}(\mathbf{q}) \quad \text{for all } \mathbf{r} \in \mathbf{K}. \tag{2.16}$$

Since $\mathbf{q} \in \mathbf{K}$ we consider, for all $\mathbf{r} \in \mathbf{K}$, the following two cases:

(i) $|\mathbf{q}^*| = 0$. We immediately obtain $\mathcal{G}^*(\mathbf{q}^*) = 0$.

(ii) $|\mathbf{q}^*| > 0$. In this situation we have $(\mathbf{q}^*, \mathbf{r} - \mathbf{q})_{0,\Omega} \le 0$ for all $\mathbf{r} \in \mathbf{K}$, from which we deduce

$$(\mathbf{q}^*, \mathbf{r})_{0,\Omega} \le (\mathbf{q}^*, \mathbf{q})_{0,\Omega} \le \int_\Omega \psi \, |\mathbf{q}^*| \, d\mathbf{x}.$$

On the other hand, choosing $\mathbf{r}^* = \psi \mathbf{q}^* |\mathbf{q}^*|^{-1}$ we have $|\mathbf{r}^*| \le \psi$ a.e. in Ω and further

$$(\mathbf{q}^*, \mathbf{q})_{0,\Omega} \ge (\mathbf{q}^*, \mathbf{r}^*)_{0,\Omega} = \int_\Omega \psi \, |\mathbf{q}^*| \, d\mathbf{x}.$$

In conclusion, we obtain

$$\mathcal{G}^*(\mathbf{q}^*) = \int_\Omega \psi \, |\mathbf{q}^*| \, d\mathbf{x}.$$

Summarizing both cases (i) and (ii), we infer (2.15). Using [12, Chap. III] and recalling (2.13), the dual of (2.10) is

$$\sup \ -\mathcal{F}^*(\Lambda^* \mathbf{q}^*) - \mathcal{G}^*(-\mathbf{q}^*) \quad \text{over } \mathbf{q}^* \in \mathbf{L}^2(\Omega). \tag{2.17}$$

The above statements prove the following result.

Proposition 2.1. *The Fenchel-dual problem of* (P) *is given by*

$$\inf\ \frac{1}{2}\|G^*(\nabla^*\boldsymbol{\mu}+y_d)+\alpha f\|^2_{M^{-1}}+\int_\Omega \psi\,|\boldsymbol{\mu}|\,d\mathbf{x}\quad \textit{over } \boldsymbol{\mu}\in\mathbf{L}^2(\Omega). \tag{D}$$

Note that compared to (2.17) we write (D) as a minimization problem, we use $G^*\nabla^*=\Lambda^*:\mathbf{L}^2(\Omega)\to L^2(\Omega)$ and $\mathbf{q}^*=\boldsymbol{\mu}$. Below it will turn out that $\boldsymbol{\mu}$ is related to the Lagrange multiplier associated with the pointwise constraint in $\mathbf{K}$.

Next we study stability of (2.10) respectively (D). A typical stability criterion for (2.10) is the existence of a point $u_0\in U$ such that $\mathcal{F}(u_0)<+\infty$, $\mathcal{G}(\Lambda u_0)<+\infty$ and such that $\mathcal{G}$ is continuous at Λu_0. While the first two conditions appear to be unproblematic in our setting, with respect to the latter we have that $\mathcal{G}=I_{\mathbf{K}}$ is only continuous at Λu_0 if $\nabla y_0\in\mathbf{C}(\bar\Omega)$ with $y_0=G(u_0+f)$ and $|\nabla y_0|\le\psi-\tau$ in Ω for some $\tau>0$, i.e., y_0 is a Slater point. Then one deduces from the chain-rule for convex mappings [12, Chap. I, Prop. 5.7] that

$$\partial(I_{\mathbf{K}}\circ(\nabla\circ G))(u)=G^*\nabla^*\partial I_{\mathbf{K}}(G(u))$$

and the existence of a Lagrange multiplier $\boldsymbol{\mu}\in\mathbf{M}(\bar\Omega)$ associated with $|\nabla y|\le\psi$ in Ω; see [8]. Such a (high) regularity requirement for the state space, however, rules out practically relevant domains with, e.g., re-entrant corners, which are admitted in our general problem setting.

Now we are interested in stability of the dual problem (D). In fact, it is straight forward to conclude that $\mathcal{F}^*$ and $\mathcal{G}^*$ are continuous on $\mathbf{L}^2(\Omega)$ and that there exist points where both functionals are finitely valued. Hence, by [12, Chap. III, Thm 4.1] (D) is stable and (with a slight misuse of notation)

$$\inf\ (P)=\sup\ (D). \tag{2.18}$$

Further we conclude that (P) has a solution. Note that this result does not imply a solution of (D).

Let us assume for the moment that (D) would admit a solution $\boldsymbol{\mu}\in\mathbf{L}^2(\Omega)$. Then, by first order optimality, we obtain

$$0\in\nabla GM^{-1}\left(G^*(\nabla^*\boldsymbol{\mu}+y_d)+\alpha f\right)+\psi\partial|\boldsymbol{\mu}|. \tag{2.19}$$

Defining the quantities

$$y:=G(u+f)\in V, \tag{2.20}$$

$$u:=M^{-1}\left(G^*(\nabla^*\boldsymbol{\mu}+y_d)+\alpha f\right)-f\in L^2(\Omega), \tag{2.21}$$

$$p:=G^*(y-y_d-\nabla^*\boldsymbol{\mu})\in V, \tag{2.22}$$

we get the relation

$$\begin{aligned}0&=(G^*G+\alpha\,\mathrm{id})(u+f)-G^*(\nabla^*\boldsymbol{\mu}+y_d)-\alpha f\\&=G^*(y-y_d-\nabla^*\boldsymbol{\mu})+\alpha u=p+\alpha u,\end{aligned} \tag{2.23}$$

which yields a regularity gain of u. From (2.19) we conclude

$$-\nabla y\in\psi\partial|\boldsymbol{\mu}|. \tag{2.24}$$

We recall that $\partial|\boldsymbol{\mu}|=\{\boldsymbol{\xi}\in\mathbf{L}^2(\Omega)\ :\ \boldsymbol{\xi}\cdot(\boldsymbol{\nu}-\boldsymbol{\mu})\le|\boldsymbol{\nu}|-|\boldsymbol{\mu}| \text{ for all } \boldsymbol{\nu}\in\mathbf{L}^2(\Omega)\}$. Thus, choosing first $\boldsymbol{\nu}=0$ and then $\boldsymbol{\nu}=2\boldsymbol{\mu}$, (2.24) yield

$$\nabla y\cdot\boldsymbol{\mu}=\psi\,|\boldsymbol{\mu}|. \tag{2.25}$$

Moreover, from the definition of the subdifferential of $|\cdot|$ we conclude

$$|\nabla y| = \psi \text{ if } |\boldsymbol{\mu}| > 0 \quad \text{and} \quad |\nabla y| \leq \psi \text{ if } |\boldsymbol{\mu}| = 0 \quad \text{a.e. in } \Omega.$$

From this we infer the complementarity relation

$$|\nabla y| \leq \psi \quad \text{and} \quad (|\nabla y| - \psi)\, |\boldsymbol{\mu}| = 0 \text{ a.e. in } \Omega. \tag{2.26}$$

Alternatively, using the definition of $\partial|\cdot|$ and (2.24), (2.26) may be written as

$$(\mathbf{q} - \nabla y, \boldsymbol{\mu})_{0,\Omega} \leq 0 \quad \text{for all } \mathbf{q} \in \mathbf{L}^2(\Omega),\ |\mathbf{q}| \leq \psi \text{ a.e. in } \Omega. \tag{2.27}$$

Observe that (2.20)–(2.23) and (2.27) represent the first order optimality (or Karush-Kuhn-Tucker) system for (D). Moreover, (y, u) be the solution of (P). The uniqueness follows from the coercivity of $\|u\|_U^2$ in U.

A close inspection of the objective functional of (D), however, reveals that it lacks coercivity in $\mathbf{L}^2(\Omega)$ such that existence of a solution cannot be guaranteed. But due to the structure of the objective we still have the following result.

Theorem 2.2. *Let $(\boldsymbol{\mu}_k) \subset \mathbf{L}^2(\Omega)$ be a minimizing sequence for (D). Then there exist $\boldsymbol{\mu} \in \mathbf{M}(\Omega)$ and a subsequence $(\boldsymbol{\mu}_{k_l})$ such that $\boldsymbol{\mu}_{k_l} \overset{*}{\rightharpoonup} \boldsymbol{\mu}$ in $\mathbf{M}(\Omega)$.*

Proof. Let $(\boldsymbol{\mu}_k) \subset \mathbf{L}^2(\Omega)$ be a minimizing sequence, and note that we have

$$\int_\Omega \psi |\boldsymbol{\mu}| d\mathbf{x} \geq \underline{\psi} \int_\Omega |\boldsymbol{\mu}| d\mathbf{x}$$

with $\underline{\psi} > 0$. From this and since $\boldsymbol{\mu} = 0$ is feasible for (D), we have that $(\boldsymbol{\mu}_k)$ is bounded in $\mathbf{L^1}(\Omega)$. Hence, there exist $\boldsymbol{\mu} \in \mathbf{M}(\Omega)$ and a subsequence (k_l) such that $\boldsymbol{\mu}_{k_l} \overset{*}{\rightharpoonup} \boldsymbol{\mu}$ in $\mathbf{M}(\Omega)$. $\square$

Finally we note that the dualization process may be performed for a tightened version of (P) given by

$$\begin{cases} \text{minimize} \quad J(y,u) := \frac{1}{2}\|y - y_d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u\|_{0,\Omega}^2 \quad \text{over } (y,u) \in \hat{V} \times L^2(\Omega) \\ \text{subject to} \quad Ay = u + f \text{ in } L^2(\Omega), \\ \qquad\qquad\quad |\nabla y| \leq \psi \text{ a.e. in } \Omega, \end{cases} \tag{$\hat{P}$}$$

Now we assume $\nabla \in \mathcal{L}(\hat{V}, \mathbf{L}^2(\Omega))$. The Fenchel dual associated with $(\hat{P})$ reads

$$\inf \frac{1}{2}\|\hat{G}^*(\nabla^* \boldsymbol{\mu} + y_d) + \alpha f\|_{M^{-1}}^2 + \int_\Omega \psi\, |\boldsymbol{\mu}|\, d\mathbf{x} \quad \text{over } \boldsymbol{\mu} \in \mathbf{L}^2(\Omega). \tag{$\hat{D}$}$$

Let $(\boldsymbol{\mu}_k)$ denote a minimizing sequence for $(\hat{D})$. Then the invertibility of $\hat{G} : L^2(\Omega) \to \hat{V}$ yields the weak convergence of $\nabla^* \boldsymbol{\mu}_{k_l} \rightharpoonup w$ in $\hat{V}^*$ along a subsequence. This proves the following result.

Theorem 2.3. *Let $(\boldsymbol{\mu}_k) \subset \mathbf{L}^2(\Omega)$ be a minimizing sequence for $(\hat{D})$. Then there exists $\boldsymbol{\mu} \in \mathbf{M}(\Omega)$ and a subsequence $(\boldsymbol{\mu}_{k_l})$ such that $\boldsymbol{\mu}_{k_l} \overset{*}{\rightharpoonup} \boldsymbol{\mu}$ in $\mathbf{M}(\Omega)$ and $\nabla^* \boldsymbol{\mu}_{k_l} \rightharpoonup w$ in $\hat{V}^*$. Moreover, the limit element $w \in \hat{V}^*$ satisfies*

$$Ay - u = f \qquad \text{in } L^2(\Omega), \tag{2.28}$$

$$Ap + w - y = -y_d, \quad \text{in } \hat{V}^*, \tag{2.29}$$

$$p + \alpha u = 0 \qquad \text{in } L^2(\Omega). \tag{2.30}$$

Note that (2.28)–(2.30) corresponds to (2.20)–(2.22) with $\nabla^*\boldsymbol{\mu}$ replaced by w. However, the limit process in Theorem 2.2 does not provide a limit version of the complementarity system (2.26). On the other hand, in view of (2.27) and $\boldsymbol{\mu} \in \mathbf{M}(\Omega)$, we expect the extension

$$\langle \mathbf{q} - \nabla y, \boldsymbol{\mu}\rangle_{\mathbf{M}(\Omega)^*,\mathbf{M}(\Omega)} \le 0 \quad \text{for all } \mathbf{q} \in \mathbf{M}(\Omega)^*,\ |\mathbf{q}| \le \psi \text{ a.e. in } \Omega. \tag{2.31}$$

As we pointed out above, $\hat{V} = W^{2,r}(\Omega) \cap H_0^1(\Omega)$ allows for Slater points. Since $C(\bar{\Omega}) \supset W^{1,r}(\Omega) \cap H_0^1(\Omega)$ we have $\nabla y \in \mathbf{C}(\bar{\Omega})$ and obtain a multiplier $\boldsymbol{\mu} \in \mathbf{M}(\bar{\Omega}) = \mathbf{C}(\bar{\Omega})^*$ for $|\nabla y| \le \psi$ in Ω. As $\mathbf{M}(\bar{\Omega})$ is not reflexive, a solution $y \in V$ satisfying $\nabla y \in \mathbf{M}(\bar{\Omega})^*$ (but $\nabla y \notin \mathbf{C}(\bar{\Omega})$) and (2.28)–(2.31) is called a *weak solution* of (P).

3. Discrete Problem and Duality

From here onwards we assume that $A = -\operatorname{div}(a\nabla\cdot) + d\cdot$, and that due to a non-smooth domain Ω we have $\hat{V} = W_0^{1,r}(\Omega)$ for some $r > 1$. The dual of $\hat{V}$ is denoted by $\hat{V}^* = W^{-1,s}(\Omega)$ with $r^{-1} + s^{-1} = 1$. Such an assumption is supported by the results in [14]. For convenience we repeat the first order optimality system of Theorem 2.1 in a variational form. Below, $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ denotes the bounded and V-elliptic bilinear form induced by A.

$$a(y,v) \ - \ (f+u,v)_{0,\Omega} = \ 0 \qquad \text{for all } v \in W_0^{1,s}(\Omega), \tag{3.1a}$$

$$a(z,p) \ - \ (y-y^d,z)_{0,\Omega} \ - \ \langle w,z\rangle_{W^{-1,s},W_0^{1,r}} = \ 0 \qquad \text{for all } z \in W_0^{1,r}(\Omega), \tag{3.1b}$$

$$p \ + \ \alpha u = \ 0, \tag{3.1c}$$

$$w \in N_{\hat{K}}(y) \subset W^{-1,s}(\Omega). \tag{3.1d}$$

We proceed by discretizing $(\hat{P})$, respectively (P). For this purpose we introduce the space of linear finite elements

$$V_h := \Big\{ v_h \in C^0(\bar{\Omega}) \,|\, v_h \in H_0^1(\Omega) \text{ is a linear polynomial on each } T \in \mathcal{T}_h \Big\}$$

with the appropriate modification for boundary elements. Here $\mathcal{T}_h \equiv \mathcal{T}_h(\Omega)$ denotes a quasi-uniform triangulation of Ω with $\mathcal{E}_h(D)$ denoting the set of element faces contained in $D \subseteq \Omega$, and $\mathcal{N}_h(D)$ the set of nodal points contained in D. For $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$ we set $h_T := \operatorname{diam}(T)$ and $h_E := \operatorname{diam}(E)$. The maximum mesh size is $h := \max\{h_T | T \in \mathcal{T}_h\}$. For $v_h \in V_h$ the quantity

$$\left[\nabla v_h \cdot \eta\right]_E := \nabla v_{h_T} \cdot \eta_T + \nabla v_{h_{T'}} \cdot \eta_{T'}$$

denotes the jump of the normal fluxes of v_h along the inter-element face E joining the elements T and T', where η_T denotes the unit outward normal on ∂T. Throughout we suppose that $\bar{\Omega}$ is the union of the elements of $\mathcal{T}_h$ so that element faces lying on the boundary might be curved.

The discrete approximation of the operator G is considered next. In fact, for given $v \in L^2(\Omega)$ we denote by $z_h = G_h(v) \in V_h$ the solution of

$$a(z_h, v_h) \quad = \quad (v, v_h)_{0,\Omega} \qquad \text{for all } v_h \in V_h.$$

Note that $G_h : L^2(\Omega) \to V_h$ is surjective. Moreover, by well known inverse estimates we have for every $v_h^* \in V_h^*$ and $s \ge 1$

$$\|G_h^* v_h^*\|_{0,\Omega}^2 \ge C \|v_h^*\|_{W^{-1,s}}^2 \tag{3.2}$$

with a constant $C > 0$ independent of h. Observe that $\hat{V}_h = V_h$ and, thus, $\hat{G}_h = G_h$. For each $T \in \mathcal{T}_h$ let $z_T \in \mathbb{R}^n$ denote a constant vector. We define

$$\mathbf{C}_h := \{\mathbf{z}_h : \Omega \to \mathbb{R}^n \,:\, \mathbf{z}_{h|T} = \mathbf{z}_T \text{ on } T\}, \text{ and } \mathbf{K}_h := \{\mathbf{z}_h \in \mathbf{C}_h : |\mathbf{z}_{h|T}| \le \psi_h,\, T \in \mathcal{T}_h\},$$

where we set $\psi_h := \frac{1}{|T|}\int_T \psi$ on $T \in \mathcal{T}_h$. Observe that $\psi_h \ge \underline{\psi} > 0$ in Ω.

Employing the variational discretization scheme of [20] when discretizing $(\hat{P})$, we obtain

$$\begin{cases} \text{minimize} & J_h(u) := \frac{1}{2}\|y_h - y_d\|^2_{0,\Omega} + \frac{\alpha}{2}\|u\|^2_{0,\Omega} \quad \text{over } u \in L^2(\Omega) \\ \text{subject to} & y_h = G_h(u+f) \text{ and } \nabla y_h \in \mathbf{K}_h. \end{cases} \tag{$\hat{P}_h$}$$

Due to the convexity of $(\hat{P}_h)$ and the existence of a Slater point for $\nabla y_h \in \mathbf{K}_h$ we obtain the following result, which can readily be deduced from [28].

Lemma 3.1. *Problem $(\hat{P}_h)$ admits a unique solution $u_h \in L^2(\Omega)$. Moreover, there exist Lagrange multipliers $\boldsymbol{\mu}_T \in \mathbb{R}^n, T \in \mathcal{T}_h$ and $p_h \in V_h$ such that $y_h = G_h(u_h + f)$ is characterized by*

$$a(v_h, p_h) - (y_h - y_d, v_h)_{0,\Omega} - \sum_{T\in\mathcal{T}_h} \nabla v_{h|T} \cdot \boldsymbol{\mu}_T = 0 \qquad \forall v_h \in V_h, \tag{3.3}$$

$$p_h + \alpha u_h = 0 \qquad \text{in } \Omega, \tag{3.4}$$

$$\sum_{T\in\mathcal{T}_h} \big(\mathbf{q}_T - \nabla y_{h|T}\big) \cdot \boldsymbol{\mu}_T \le 0 \qquad \forall \mathbf{q}_h \in \mathbf{K}_h. \tag{3.5}$$

Next we associate with $\boldsymbol{\mu}$ the function $\boldsymbol{\mu}_h \in \mathbf{L}^2(\Omega)$ through

$$(f, \boldsymbol{\mu}_h)_{0,\Omega} := \sum_{T\in\mathcal{T}_h} \oint_T f\, dx \cdot \boldsymbol{\mu}_T \text{ for all } f \in \mathbf{L}^2(\Omega).$$

Then, $\boldsymbol{\mu}_{h|T} = \boldsymbol{\mu}_T$, and since $\boldsymbol{\mu}_h \in \mathbf{L}^2(\Omega)$ the negative weak divergence of $\boldsymbol{\mu}_h$ - as an element of $W^{-1,s}(\Omega)$ - is given by

$$\langle -\nabla^* \boldsymbol{\mu}_h, v\rangle_{W^{-1,s},W_0^{1,r}} = (\nabla v, \boldsymbol{\mu}_h)_{0,\Omega} \text{ for all } v \in W_0^{1,r}(\Omega).$$

Using this definition, the system (3.3)-(3.5) may be reformulated as

$$a(y_h, v_h) \;-\; (f + u_h, v_h)_{0,\Omega} \;=\; 0 \qquad \forall v_h \in V_h, \tag{3.6a}$$

$$a(v_h, p_h) \;-\; (y_h - y^d, v_h)_{0,\Omega} \;+\; \langle \nabla^* \cdot \boldsymbol{\mu}_h, v_h\rangle_{W^{-1,s},W_0^{1,r}} \;=\; 0 \qquad \forall v_h \in V_h\ , \tag{3.6b}$$

$$p_h \;+\; \alpha u_h \;=\; 0, \tag{3.6c}$$

$$\sum_{T\in\mathcal{T}_h(\Omega)} (\mathbf{q}_h|_T - \nabla y_h|_T) \cdot \boldsymbol{\mu}_h|_T \;\le\; 0 \qquad \forall \mathbf{q}_h \in \mathbf{K}_h. \tag{3.6d}$$

The Fenchel dual associated with $(\hat{P}_h)$ reads

$$\inf \frac{1}{2}\|G_h^*(\nabla^* \boldsymbol{\mu}_h + y_d) + \alpha f\|^2_{M_h^{-1}} + \int_\Omega \psi_h |\boldsymbol{\mu}_h| d\mathbf{x} \quad \text{ over } \boldsymbol{\mu}_h \in \mathbf{C}_h, \tag{$\hat{D}_h$}$$

where $M_h := G_h G_h^* + \alpha\, \mathrm{id}$. For every $h > 0$, $(\hat{D}_h)$ admits a solution $\boldsymbol{\mu}_h$, which together with u_h, y_h, p_h satisfies (3.6). Furthermore, in analogy to (2.18) we have

$$\min(\hat{P}_h) = \max(\hat{D}_h).$$

Moreover, since $\mathbf{0}$ is feasible for every $(\hat{D}_h)$ we have

$$\max\{\|\boldsymbol{\mu}_h\|_{\mathbf{L}^1}, \|\nabla^*\boldsymbol{\mu}_h\|_{W^{-1,s}}\} \leq C,$$

with a constant $C > 0$ independent of the mesh size h. The latter estimate follows from (3.2). Furthermore, we have the following representation of the discrete multipliers.

Lemma 3.2. *Let u_h denote the unique solution of $(\hat{P}_h)$ with corresponding state $y_h = G_h(u_h + f)$ and multiplier $(\boldsymbol{\mu}_T)_{T\in\mathcal{T}_h}$. Then we have*

$$\boldsymbol{\mu}_T = |\boldsymbol{\mu}_T|\frac{1}{\psi_h}\nabla y_{h|T} \text{ for all } T \in \mathcal{T}_h. \tag{3.7}$$

Proof. Fix $T \in \mathcal{T}_h$. The assertion is immediate if $\boldsymbol{\mu}_T = 0$. Suppose that $\boldsymbol{\mu}_T \neq 0$, and define $\mathbf{q}_h : \bar{\Omega} \to \mathbb{R}^n$ by

$$\mathbf{q}_{h|\tilde{T}} := \begin{cases} \nabla y_{h|T}, & \tilde{T} \neq T, \\ \psi_h \frac{\boldsymbol{\mu}_T}{|\boldsymbol{\mu}_T|}, & \tilde{T} = T. \end{cases}$$

Clearly, $\mathbf{q}_h \in \mathbf{K}_h$ so that (3.5) implies

$$\boldsymbol{\mu}_T \cdot \big(\psi_h \frac{\boldsymbol{\mu}_T}{|\boldsymbol{\mu}_T|} - \nabla y_{h|T}\big) \leq 0.$$

Thus, since $\Big(\nabla y_{h|T}\Big)_{T\in\mathcal{T}_h} \in \mathbf{K}_h$, we find

$$\psi_h|\boldsymbol{\mu}_T| \leq \boldsymbol{\mu}_T \cdot \nabla y_{h|T} \leq \psi_h|\boldsymbol{\mu}_T|.$$

Hence, we obtain $\dfrac{\boldsymbol{\mu}_T}{|\boldsymbol{\mu}_T|} = \dfrac{1}{\psi_h}\nabla y_{h|T}$ which yields the assertion. $\square$

As a consequence of Lemma 3.2 we immediately infer that

$$|\boldsymbol{\mu}_T| = \frac{1}{\psi_h}\boldsymbol{\mu}_T \cdot \nabla y_{h|T} \text{ for all } T \in \mathcal{T}_h. \tag{3.8}$$

4. Residual Based a Posteriori Error Estimator

We continue with an a posteriori analysis of our discrete problem. In order to prepare the main result, our residual-type a posteriori error estimator is introduced next:

$$\eta \;:=\; \eta_y \;+\; \eta_p \;, \tag{4.1a}$$

$$\eta_y \;:=\; \Big(\sum_{T\in\mathcal{T}_h(\Omega)} \eta_{y,T}^r + \sum_{E\in\mathcal{E}_h(\Omega)} \eta_{y,E}^r\Big)^{1/r} \;, \tag{4.1b}$$

$$\eta_p \;:=\; \Big(\sum_{T\in\mathcal{T}_h(\Omega)} \eta_{p,T}^s\Big)^{1/s} + \Big(\sum_{E\in\mathcal{E}_h(\Omega)} \eta_{p,E}^s\Big)^{1/s} \;. \tag{4.1c}$$

Here, $\eta_{y,T}, \eta_{p,T}, T \in \mathcal{T}_h(\Omega)$, are the element residuals associated with the strong form of the state and the adjoint state equations

$$\eta_{y,T}^r \;:= h_T^r \;\|f + u_h + \nabla\cdot a\nabla y_h - dy_h\|_{0,T}^r \;, \tag{4.2a}$$

$$\eta_{p,T}^s \;:= h_T^s \;\|y_h - y^d + \nabla\cdot a\nabla p_h - dp_h\|_{0,T}^s \;, \tag{4.2b}$$

whereas $\eta_{y,E}, \eta_{y,E}$ stand for the face residuals

$$\eta^r_{y,E} := h_E^{r/2}\ \|\boldsymbol{\nu}_E \cdot [a\nabla y_h]_E\|^r_{0,E}\ , \tag{4.3a}$$
$$\eta^s_{p,E} := h_E^{s/2}\ \|\boldsymbol{\nu}_E \cdot ([\boldsymbol{\mu}_h]_E - [a\nabla p_h]_E)\|^s_{0,E}\ . \tag{4.3b}$$

Our main result is stated next.

Theorem 4.1. *Let* $(y,u,p,w) \in W_0^{1,r}(\Omega) \times L^2(\Omega) \times W_0^{1,s}(\Omega) \times W^{-1,s}(\Omega)$ *and* $(y_h,u_h,p_h,\boldsymbol{\mu}_h) \in V_h \times V_h \times V_h \times \mathbf{C}_h$ *be the solutions of* (3.1a)-(3.1d) *and* (3.6a)-(3.6d). *Moreover, let* η *be the estimator according to* (4.1a). *Then it holds that*

$$\|y-y_h\|^2_{W_0^{1,r}} \ +\ \|u-u_h\|^2_{0,\Omega} \ \lesssim\ \eta_y^2 \ +\ \eta_p^2 \ - \langle w, y-y_h\rangle_{W^{-1,s},W_0^{1,r}} + \eta_y. \tag{4.4}$$
$$\|y-y_h\|^2_{0,\Omega} \ +\ \|u-u_h\|^2_{0,\Omega} \ \lesssim\ \eta_y \ +\ \eta_p^2 \ - \langle w, y-y_h\rangle_{W^{-1,s},W_0^{1,r}}. \tag{4.5}$$

The proof of Theorem 4.1 will be split into a series of lemmas. For the discrete control $u_h \in V_h$ and the discrete multiplier $\boldsymbol{\mu}_h \in \mathbf{C}_h$, we introduce an intermediate state $y(u_h) \in W_0^{1,r}(\Omega)$ and an intermediate adjoint state $p(\boldsymbol{\mu}_h) \in W_0^{1,s}(\Omega)$ as the solution of

$$a(y(u_h),v) \ -\ (f+u_h,v)_{0,\Omega} \ =\ 0 \qquad \forall v \in W_0^{1,s}(\Omega), \tag{4.6}$$
$$a(z,p(\boldsymbol{\mu}_h)) \ -\ (y_h - y^d, z)_{0,\Omega} \ +\ \langle \nabla^* \boldsymbol{\mu}_h, z\rangle_{W^{-1,s},W_0^{1,r}} \ =\ 0 \qquad \forall z \in W_0^{1,r}(\Omega). \tag{4.7}$$

We start by estimating the error between the optimal state and its adjoint and the intermediate state and adjoint, respectively.

Lemma 4.1. *Let* $(y_h,u_h,p_h,\boldsymbol{\mu}_h) \in V_h \times V_h \times V_h \times \mathbf{C}_h$ *be the solution of* (3.6a)-(3.6d) *and let* $(y(u_h),p(\boldsymbol{\mu}_h)) \in W_0^{1,r}(\Omega) \times W_0^{1,s}(\Omega)$ *be the intermediate states as given by* (4.6), (4.7). *Then it holds that*

$$\|y(u_h)-y_h\|_{1,r} \ \lesssim \Big(\sum_{T\in\mathcal{T}_h} \eta^r_{y,T}\Big)^{1/r} + \Big(\sum_{E\in\mathcal{E}_h(\Omega)} \eta^r_{y,E}\Big)^{1/r}, \tag{4.8a}$$
$$\|p(\boldsymbol{\mu}_h)-p_h\|_{1,s} \ \lesssim \Big(\sum_{T\in\mathcal{T}_h(\Omega)} \eta^s_{p,T})\Big)^{1/s} + \Big(\sum_{E\in b\mathcal{E}_h(\Omega} \eta^s_{p,E}\Big)^{1/s}\ . \tag{4.8b}$$

Proof. We denote by $\Pi_h^{(t)} : W^{1,t} \to V_h$ for $2 \le t < \infty$ the Scott-Zhang interpolation operator satisfying

$$\|v - \Pi_h^{(t)} v\|_{0,t,T} \ \lesssim h_T\ \|v\|_{1,t,D_T}, \qquad T \in \mathcal{T}_h(\Omega), \tag{4.9a}$$
$$\|v - \Pi_h^{(t)} v\|_{0,p,E} \ \lesssim h_E^{1/2}\ \|v\|_{1,t,D_E}, \qquad E \in \mathcal{E}_h(\Omega), \tag{4.9b}$$

where $D_T := \bigcup\{T' \in \mathcal{T}_h(\Omega) \mid \mathcal{N}_h(T') \cap \mathcal{N}_h(T) \neq \emptyset\}$ and $D_E := \bigcup\{T' \in \mathcal{T}_h(\Omega) \mid \mathcal{N}_h(T') \cap \mathcal{N}_h(E) \neq \emptyset\}$ (cf., e.g., [24]). The generalization of the Lax-Milgram Lemma (cf. Theorem 5.4 in [25]) gives

$$\|y(u_h)-y_h\|_{1,r} \ \le \sup_{\|v\|_{1,s}\le 1} |a(y(u_h)-y_h,v)|\ . \tag{4.10}$$

Since $\Pi_h^{(s)} v \in V_h$ is an admissible test function in (3.6a),(4.6), we get

$$\begin{aligned} a(y(u_h)-y_h,v) \ &=\ a(y(u_h)-y_h, v-\Pi_h^{(s)}v) \\ &= (f+u_h, v-\Pi_h^{(s)}v)_{0,\Omega} - a(y_h, v-\Pi_h^{(s)}v)\ . \end{aligned} \tag{4.11}$$

Green's formula and a straightforward estimation yield

$$\begin{aligned} & a(y(u_h) - y_h, v - \Pi_h^{(s)} v) \\ = & \sum_{T \in \mathcal{T}_h} (f + u_h + \nabla \cdot a \nabla y_h - d y_h, v - \Pi_h^{(s)} v)_{0,T} \\ & \quad - \sum_{E \in \mathcal{E}_h(\Omega)} (\boldsymbol{\nu}_E \cdot [a \nabla y_h]_E, v - \Pi_h^{(s)} v)_{0,E} \\ \leq & \sum_{T \in \mathcal{T}_h} \| f + u_h + \nabla \cdot a \nabla y_h - d y_h \|_{0,r,T} \| v - \Pi_h^{(s)} v \|_{0,s,T} \\ & \quad + \sum_{E \in \mathcal{E}_h(\Omega)} \| \boldsymbol{\nu}_E \cdot [a \nabla y_h]_E \|_{0,r,E} \ \| v - \Pi_h^{(s)} v \|_{0,s,E} \ . \end{aligned} \tag{4.12}$$

The local approximation properties (4.9a),(4.9b) of $\Pi_h^{(s)}$ give

$$\begin{aligned} & a(y(u_h) - y_h, v - \Pi_h^{(s)} v) \\ \leq & \sum_{T \in \mathcal{T}_h} h_T \| f + u_h + \nabla \cdot a \nabla y_h - d y_h \|_{0,r,T} \| v \|_{1,s,D_T} \\ & \quad + \sum_{E \in \mathcal{E}_h(\Omega)} h_E^{1/2} \ \| \boldsymbol{\nu}_E \cdot [a \nabla y_h]_E \|_{0,r,E} \ \| v \|_{1,s,D_E} \\ \leq & \Big(\sum_{T \in \mathcal{T}_h} h_T^r \| f + u_h + \nabla \cdot a \nabla y_h - d y_h \|_{0,r,T}^r \Big)^{1/r} \ \Big(\sum_{T \in \mathcal{T}_h} \| v \|_{1,s,T}^s \Big)^{1/s} \\ & \quad + \Big(\sum_{E \in \mathcal{E}_h(\Omega)} h_E^{r/2} \ \| \boldsymbol{\nu}_E \cdot [a \nabla y_h]_E \|_{0,r,E}^r \Big)^{1/r} \ \Big(\sum_{E \in \mathcal{E}_h(\Omega)} \| v \|_{1,s,D_E}^s \Big)^{1/s} \\ \leq & \| v \|_{1,s} \ \left(\Big(\sum_{T \in \mathcal{T}_h} \eta_{y,T}^r \Big)^{1/r} + \Big(\sum_{E \in \mathcal{E}_h(\Omega)} \eta_{y,E}^r \Big)^{1/r} \right) \ . \end{aligned} \tag{4.13}$$

Now, (4.8) follows readily from (4.10)-(4.13). On the other hand, the generalization of the Lax-Milgram theorem (cf. Theorem 5.4 in [25]) yields

$$\| p(\boldsymbol{\mu}_h) - p_h \|_{1,s} \leq \sup_{\|z\|_{1,r} \leq 1} | a(z, p(\boldsymbol{\mu}_h) - p_h) | \ . \tag{4.14}$$

Since $\Pi_h^{(r)} z \in V_h$ is an admissible test function in (3.6b) and (4.7), we find

$$a(z, p(\boldsymbol{\mu}_h) - p_h) = a(z - \Pi_h^{(r)} z, p(\boldsymbol{\mu}_h) - p_h) \ . \tag{4.15}$$

Using (4.7), we obtain

$$\begin{aligned} & a(z - \Pi_h^{(r)} z, p(\boldsymbol{\mu}_h) - p_h) \\ = & (z - \Pi_h^{(r)} z, y_h - y^d) - \langle \nabla^* \boldsymbol{\mu}_h, z - \Pi_h^{(r)} z \rangle_{W^{-1,s}, W_0^{1,r}} - a(z - \Pi_h^{(r)} z, p_h) \\ = & \sum_{T \in \mathcal{T}_h} (z - \Pi_h^{(r)} z, y_h - y^d + \nabla \cdot a \nabla p_h - d p_h)_{0,\Omega} \\ & - \sum_{E \in \mathcal{E}_h(\Omega)} (z - \Pi_h^{(r)} z, \boldsymbol{\nu}_E \cdot ([a \nabla p_h]_E - [\boldsymbol{\mu}_h]_E))_{0,E}. \end{aligned} \tag{4.16}$$

Here we have used the fact that

$$\langle \nabla^* \boldsymbol{\mu}_h, v_h \rangle_{W^{-1,s}, W_0^{1,r}} \;=\; - \sum_{E \in \mathcal{E}_h(\Omega)} (\boldsymbol{\nu}_E \cdot [\boldsymbol{\mu}_h]_E, v_h)_{0,E},$$

which follows from the definition of $\nabla^* \boldsymbol{\mu}_h$, $\boldsymbol{\mu}_h|_T \in P_0(T)^2$, and integration by parts.

Taking advantage of the local approximation properties (4.9a),(4.9b) of $\Pi_h^{(r)}$, straightforward estimation yields

$$\begin{aligned}
&|a(z - \Pi_h^{(r)} z, p(\boldsymbol{\mu}_h) - p_h)| \\
\le\; & \sum_{T \in \mathcal{T}_h} \|z - \Pi_h^{(r)} z\|_{0,r,T} \; \|y_h - y^d + \nabla \cdot a \nabla p_h - d p_h\|_{0,s,T} \\
&+ \sum_{E \in \mathcal{E}_h(\Omega)} \|z - \Pi_h^{(r)} z\|_{0,r,E} \; \|\boldsymbol{\nu}_E \cdot ([a \nabla p_h]_E - [\boldsymbol{\mu}_h]_E)\|_{0,s,E} \\
\le\; & \sum_{T \in \mathcal{T}_h} h_T \; \|y_h - y^d + \nabla \cdot a \nabla p_h - d p_h\|_{0,s,T} \; \|z\|_{1,r,D_T} \\
&+ \sum_{E \in \mathcal{E}_h(\Omega)} h_E^{1/2} \; \|\boldsymbol{\nu}_E \cdot ([a \nabla p_h]_E - [\boldsymbol{\mu}_h]_E)\|_{0,s,E} \; \|z\|_{1,r,D_E} \\
\le\; & \Big(\sum_{T \in \mathcal{T}_h} h_T^s \; \|y_h - y^d + \nabla \cdot a \nabla p_h - d p_h\|_{0,s,T}^s \Big)^{1/s} \; \Big(\sum_{T \in \mathcal{T}_h} \|z\|_{1,r,D_T}^r \Big)^{1/r} \\
&+ \Big(\sum_{E \in \mathcal{E}_h(\Omega)} h_E^{s/2} \; \|\boldsymbol{\nu}_E \cdot ([a \nabla p_h]_E - [\boldsymbol{\mu}_h]_E)\|_{0,s,E}^s \Big)^{1/s} \; \Big(\sum_{E \in \mathcal{E}_h(\Omega)} \|z\|_{1,r,D_E}^r \Big)^{1/r} \\
\le\; & \|z\|_{1,r,\Omega} \; \Big(\Big(\sum_{T \in \mathcal{T}_h} \eta_{p,T}^s \Big)^{1/s} + \Big(\sum_{E \in \mathcal{E}_h(\Omega)} \eta_{p,E}^s \Big)^{1/s} \Big) \, .
\end{aligned} \tag{4.17}$$

Then, (4.8b) is an immediate consequence of (4.14) and (4.17). □

The error between the optimal control and its finite element approximation, and between the optimal state and the discrete optimal state is estimated next.

Lemma 4.2. *Let* $(y, u, p, \boldsymbol{\mu}, w) \in W_0^{1,r}(\Omega) \times L^2(\Omega) \times W_0^{1,s}(\Omega) \times \mathbf{M} \times W^{-1,s}(\Omega)$ *and* $(y_h, u_h, p_h, \boldsymbol{\mu}_h) \in V_h \times V_h \times V_h \times \mathbf{C}_h$ *be the solutions of* (3.1a)-(3.1d) *and* (3.6a)-(3.6d)*, respectively, and assume that* $(y(u_h), p(\boldsymbol{\mu}_h)) \in W_0^{1,r} \times W_0^{1,s}$ *are the intermediate states as given by* (4.6) *and* (4.7)*. Then, it holds that*

$$\begin{aligned}
&\|y - y_h\|_{0,\Omega}^2 + \|u - u_h\|_{0,\Omega}^2 \\
\lesssim\; & \|y(u_h) - y_h\|_{W_0^{1,r}} + \|p(\boldsymbol{\mu}_h) - p_h\|_{W_0^{1,s}}^2 - \langle w, y - y_h \rangle_{W^{-1,s}, W_0^{1,r}} \, .
\end{aligned} \tag{4.18}$$

Proof. In view of (3.1c) and (3.6c) we obtain

$$\alpha \|u - u_h\|_{0,\Omega}^2 \;=\; (u - u_h, p(\boldsymbol{\mu}_h) - p)_{0,\Omega} + (u - u_h, p_h - p(\boldsymbol{\mu}_h))_{0,\Omega} =: \mathrm{(I)} + \mathrm{(II)} \, . \tag{4.19}$$

Using the complementarity condition (3.6d) together with uniform boundedness of $\|\nabla^*\boldsymbol{\mu}_h\|_{W^{-1,s}}$ in h we have

$$\begin{aligned}
(\mathrm{I}) &= a(y-y(u_h),p(\boldsymbol{\mu}_h)-p) = (y_h-y,y-y(u_h))_{0,\Omega} - \langle w+\nabla^*\boldsymbol{\mu}_h, y-y(u_h)\rangle_{W^{-1,s},W_0^{1,r}}\\
&= -\|y_h-y\|_{0,\Omega}^2 + (y_h-y,y_h-y(u_h))_{0,\Omega} - \langle w+\nabla^*\boldsymbol{\mu}_h, y-y(u_h)\rangle_{W^{-1,s},W_0^{1,r}}\\
&\le -\frac{1}{2}\|y_h-y\|_{0,\Omega}^2 + \frac{1}{2}\|y_h-y(u_h)\|_{0,\Omega}^2\\
&\quad - \langle w+\nabla^*\boldsymbol{\mu}_h, y-y_h\rangle_{W^{-1,s},W_0^{1,r}} - \langle w+\nabla^*\boldsymbol{\mu}_h, y_h-y(u_h)\rangle_{W^{-1,s},W_0^{1,r}}\\
&\le -\frac{1}{2}\|y_h-y\|_{0,\Omega}^2 + \frac{1}{2}\|y_h-y(u_h)\|_{0,\Omega}^2\\
&\quad - \langle w, y-y_h\rangle_{W^{-1,s},W_0^{1,r}} - \langle w+\nabla^*\boldsymbol{\mu}_h, y_h-y(u_h)\rangle_{W^{-1,s},W_0^{1,r}}\\
&\le -\frac{1}{2}\|y_h-y\|_{0,\Omega}^2 + \frac{1}{2}\|y_h-y(u_h)\|_{0,\Omega}^2 - \langle w, y-y_h\rangle_{W^{-1,s},W_0^{1,r}} + C\|y_h-y(u_h)\|_{W_0^{1,r}}.
\end{aligned}$$

Finally, for the second addend in (4.19) an application of Young's inequality gives

$$(\mathrm{II}) \;\le\; \frac{\alpha}{2}\;\|u-u_h\|_{0,\Omega}^2 \;+\; 2\alpha^{-1}\;\|p(\boldsymbol{\mu}_h)-p_h\|_{W_0^{1,s}}^2, \tag{4.20}$$

which implies the assertion. □

Combining our preparatory results we prove our main theorem.

Proof of Theorem 4.1. Taking into account that

$$\|y-y(u_h)\|_{W_0^{1,r}}^2 \;\le\; C\;\|u-u_h\|_{0,\Omega}^2$$

for some $C>0$, we get

$$\begin{aligned}
&\|y-y_h\|_{W_0^{1,r}}^2 + \|u-u_h\|_{0,\Omega}^2 \qquad (4.21)\\
&\le 2\;\|y-y(u_h)\|_{W_0^{1,r}}^2 + 2\;\|y(u_h)-y_h\|_{W_0^{1,r}}^2 + \|u-u_h\|_{0,\Omega}^2\\
&\le 2\;\|y(u_h)-y_h\|_{W_0^{1,r}}^2 + (1+2C)\;\|u-u_h\|_{0,\Omega}^2\;.
\end{aligned}$$

Now the assertions (4.5) and (4.4) follow from an application of the results of Lemma 4.1 and Lemma 4.2 in (4.21). □

5. Numerical Tests

Finally we present some numerical tests related to problem (P) for $n=2$. From here onwards element faces are called element edges. We set $A := -\Delta$, $f=0$ and consider the two-dimensional domain

$$\Omega := \{(x_1,x_2) = (r\cos\phi, r\sin\phi) \in \mathbb{R}^2 \,:\, r\in(0,1), \phi\in(0,\omega)\}$$

with boundary parts $\Gamma_1 := \{(\cos\phi,\sin\phi)\in\mathbb{R}^2 \,:\, \phi\in(0,\omega)\}$ and $\Gamma_2 := ([0,1]\times\{0\}) \cup \{(r\cos\omega, r\sin\omega) \,:\, r\in(0,1)\}$. We choose $\alpha=1$, $y_d = r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\phi$ and $\psi\in L^r(\Omega)$ with appropriate $r>1$. The optimization problem considered here for the numerical experiments reads

$$\left\{\begin{array}{l}
\min J(y_h,u) = \frac{1}{2}\|y_h-y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 \text{ over } (y_h,u)\in V_h\times L^2(\Omega)\\
\text{ s.t. } \int_\Omega \nabla y_h\nabla v_h dx = (u,v_h)_{0,\Omega}\;\; \forall v_h\in V_h,\;\; y_h = \sin\frac{\pi}{\omega}\phi \text{ on } \Gamma_1,\;\; y_h=0 \text{ on } \Gamma_2,\\
\qquad |\nabla y_h|^2 \le \psi_h^2 \text{ in } \bar\Omega.
\end{array}\right. \tag{P_2}$$

We note that for the solution of the problem

$$-\Delta y = 0 \text{ in } \Omega,\ y = \sin\frac{\pi}{\omega}\phi \text{ on } \Gamma_1,\ y = 0 \text{ on } \Gamma_2,$$

we have $y \in W^{1,r}(\Omega)$ and

$$|\nabla y(x)| = \frac{\pi}{\omega}|x|^{\frac{\pi-\omega}{\omega}} \in L^r(\Omega) \text{ with } r := \frac{2\omega}{\omega-\pi}$$

.

The numerical solution of (P_2) is obtained by the semi-smooth Newton method applied to the optimality system of (P_2), see e.g. [16]. For the description of the method we follow [10]. Let $c > 0$ be fixed and let us split the index set $\bullet$ of all nodes into Γ, the index set of all boundary nodes, and I, the index set of all interior nodes. In order to cope with the non-homogenous Dirichlet boundary conditions we introduce the modified stiffness matrix

$$\tilde{K} := \begin{bmatrix} I_\Gamma & 0_{\Gamma I} \\ K_{I\Gamma} & K_I \end{bmatrix} \in \mathbb{R}^{np\times np},$$

where np denotes the number of vertices of the triangulation of Ω. Then the optimality system of (P_2) can be expressed in the form

$$F(y_h, u_h, \tilde{\mu}_h) = \begin{bmatrix} \alpha\tilde{K}u_h + \begin{bmatrix} 0_\Gamma \\ M_{I\bullet}(y_h - y_d) + \Big[\sum_T 2\tilde{\mu}_{h|T}\nabla y_{h|T}\cdot\nabla b_{j|T}\Big]_{j\in I} \end{bmatrix} \\ \tilde{K}y_h - \begin{bmatrix} y_\Gamma \\ M_{I\bullet}u_h \end{bmatrix} \\ \Big[-\tilde{\mu}_{h|T} + \max(0, \tilde{\mu}_{h|T} + c(|\nabla y_{h|T}|^2 - \psi^2_{h|T}))\Big]_{T=1}^{nt} \end{bmatrix} = 0,$$

which is amenable to the semi-smooth Newton method. We note that the scalar multiplier μ_h of problem $(\hat{P}_h)$ is coupled to $\tilde{\mu}_h$ through the relation $\mu_{h|T} = 2\psi_{h|T}\tilde{\mu}_{h|T}$.

In the following numerical examples mesh-refinement is based on the residual estimators defined in (4.1a) and on the right-hand side (4.5). For the refinement the following bulk- and/or max-marking procedure is employed. For this purpose let

$$\eta_{1,T} = h_T^r\|f + u_h + \nabla\cdot a\nabla y_h - dy_h\|_{0,T}^r, \qquad \eta_{2,T} = h_T^s\|y_h - y_d + \nabla\cdot a\nabla p_h - dp_h\|_{0,T}^s,$$
$$\eta_{3,E} = \frac{1}{2}\sum_{E\subset\partial T} h_E^{r/2}\|\boldsymbol{\nu}_E\cdot[a\nabla y_h]_E\|_{0,E}^r, \qquad \eta_{4,E} = \frac{1}{2}\sum_{E\subset\partial T} h_E^{s/2}\|\boldsymbol{\nu}_E\cdot([a\nabla p_h]_E - [\mu_h]_E)\|_{0,E}^s.$$

For $\vartheta_1, \vartheta_2 \in [0,1]$ the bulk criterion chooses minimal subsets M_1, M_2 of triangles such that

$$\sum_{T\in M_1}\eta_{1,T} + \eta_{3,E} \geq \vartheta_1\sum_T \eta_{1,T} + \eta_{3,E},$$
$$\sum_{T\in M_2}\eta_{2,T} + \eta_{4,E} \geq \vartheta_2\sum_T \eta_{2,T} + \eta_{4,E}.$$

The max-marking criterion for $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in [0,1]$ chooses these subsets according to

$$M_i := \Big\{T : \eta_{i,T} \geq \vartheta_i \max_T \eta_{i,T}\Big\}, \qquad i = 1, \cdots, 4.$$

Consequently, set of triangles marked for refinement is given by

$$M := \cup_i M_i.$$

In our numerical experiments we choose $\vartheta_i = 0.7$ for bulk marking, and $\vartheta_i = 0.5$ for max-marking. We use local congruent refinement with conforming closure, where a new node is projected onto the circle containing his mother nodes. This procedure produces nice meshes for the domains which we consider in our numerical experiments. The parameters r and s are associated with the regularity of the involved variables and satisfy $\frac{1}{r} + \frac{1}{s} = 1$.

In the following numerical solutions and the contributions to the error indicators are displayed on uniformly refined meshes $\mathcal{T}_l$, whose mesh parameters for $\omega = \frac{3}{2}\pi$ are displayed in Tab. 5.1.

Table 5.1: Mesh parameters for $\omega = \frac{3}{2}\pi$: l refinement level, np number of vertices, nt number of triangles, h grid size.

l	np	nt	h
1	35	48	0.4203
2	117	192	0.2219
3	425	768	0.1138
4	1617	3072	0.0578
5	6305	12288	0.0292
6	24897	49152	0.0146
7	98945	196608	0.0073

5.1. Example 1 with inactive origin

In this example we use $\omega = \frac{5}{4}\pi, r = 10$, and $\psi(x) = 2|x|^{-\frac{1}{5}} + |x| - 1.9 \in L^{10}(\Omega)$. The gradient constraints are not active in the origin, but in the crescent-shaped black area, see Fig. 5.1 where the numerical solution u_h and y_h together with the active set (black area) is presented.

Fig. 5.2 shows the adaptively refined meshes obtained by bulk-marking (left) and max-marking. Refinement due to the error indicators associated to the dual variables (i.e. p and μ)

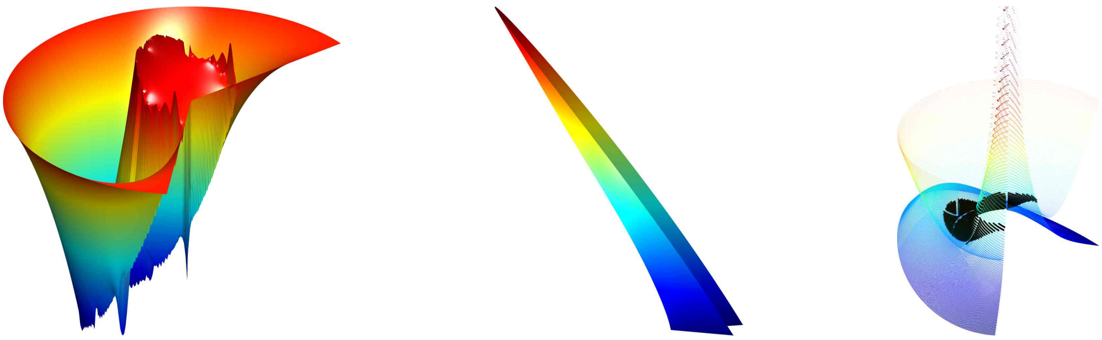

Fig. 5.1. Example 1: u_h (left), y_h (middle), and active set of all T with $|\nabla y_{h|T}| = \psi(x_T)$ (right) for $l = 6$.

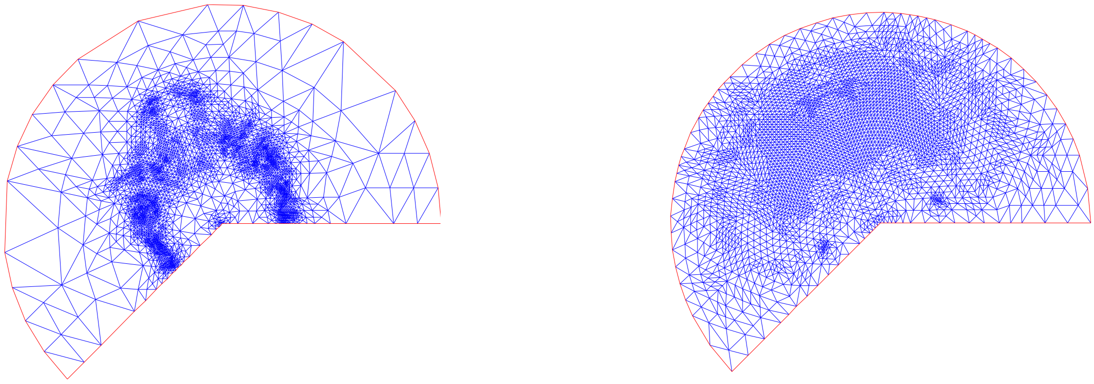

Fig. 5.2. Example 1: Mesh with bulk-marking ($np = 4020$, $nt = 7976$) (left), mesh with max-marking ($np = 3417$, $nt = 6744$) (right).

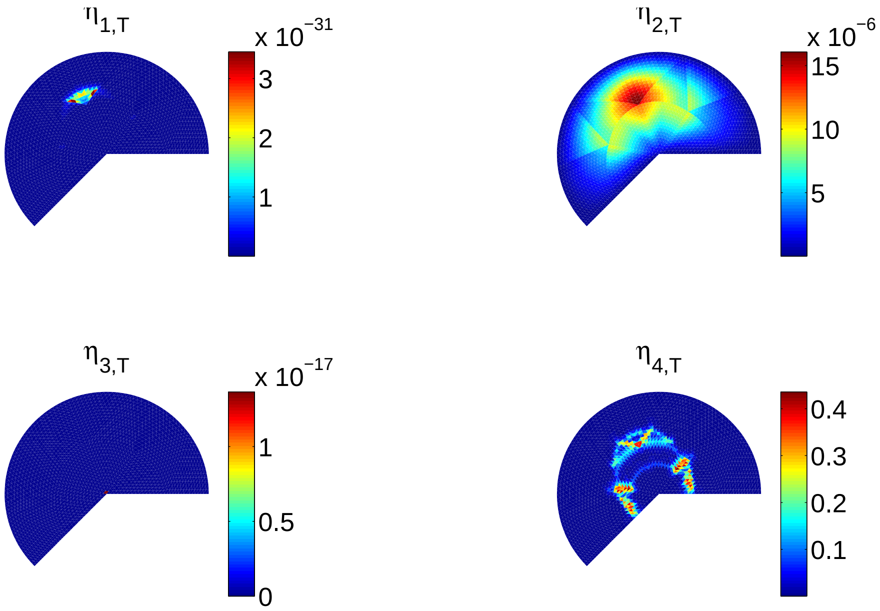

Fig. 5.3. Example 1: η for $l = 4$.

is enforced in the crescent-shaped black area, whereas refinement caused by the primal variable (i.e. y and u) is observed in the origin.

In Fig. 5.3 the different contributions to the total error estimator defined in (4.2a)-(4.3b) are displayed on a uniform mesh for refinement level $l = 4$. One clearly observes that the edge-residual of the dual variables delivers by far the largest contribution. In Fig. 5.4 the contributions of p_h and μ_h to (4.3b) are displayed separately. One finds that the contribution of p_h is dominant.

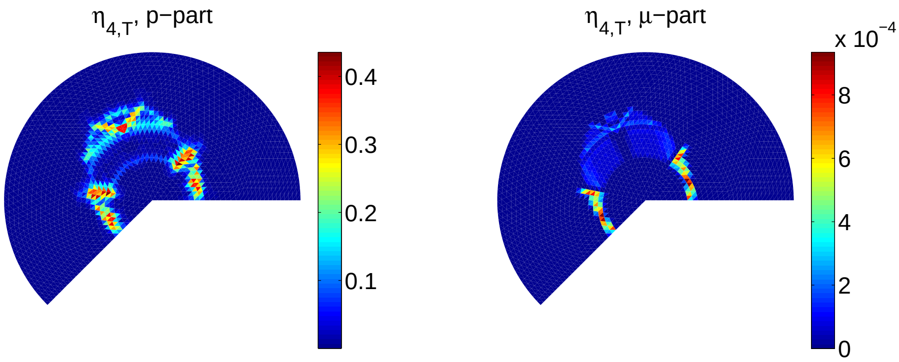

Fig. 5.4. $\eta_{4,E}$-parts for $l = 4$.

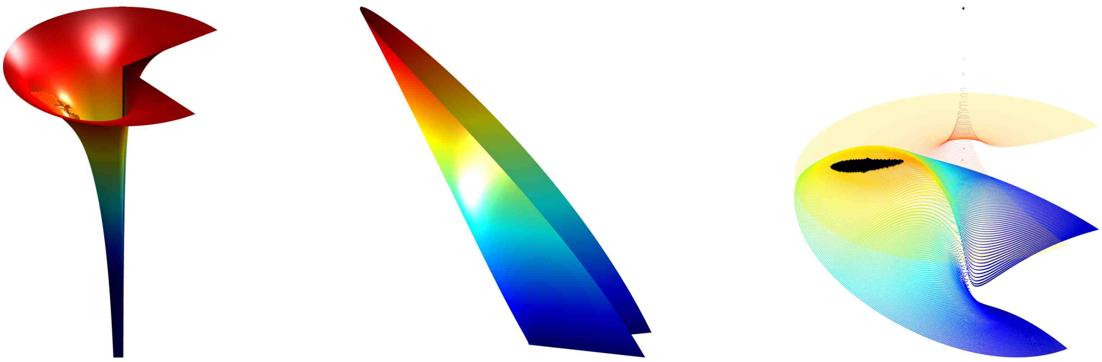

Fig. 5.5. u_h (left), y_h (middle), and active set of all T with $|\nabla y_{h|T}| = \psi(x_T)$ (right) for $l = 6$.

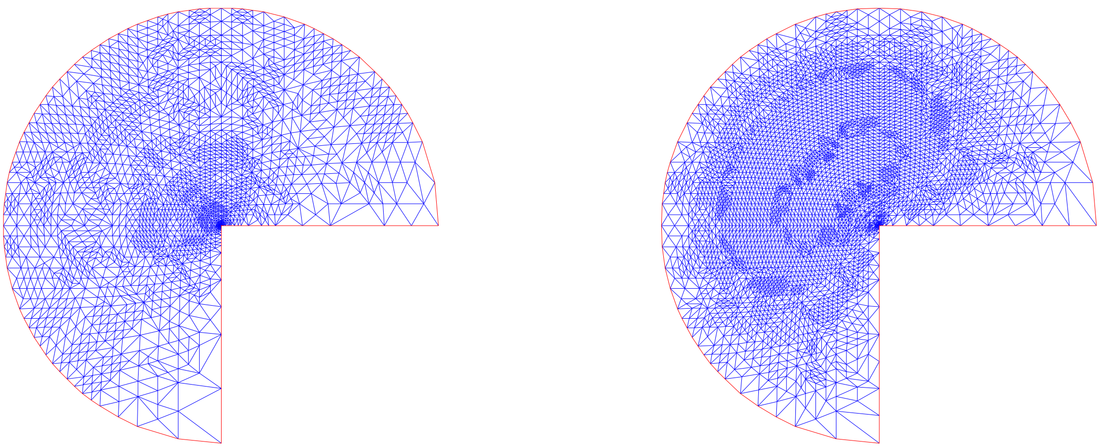

Fig. 5.6. Mesh with bulk-marking ($np = 2345$, $nt = 4559$) (left), mesh with max-marking ($np = 4392$, $nt = 8692$) (right)

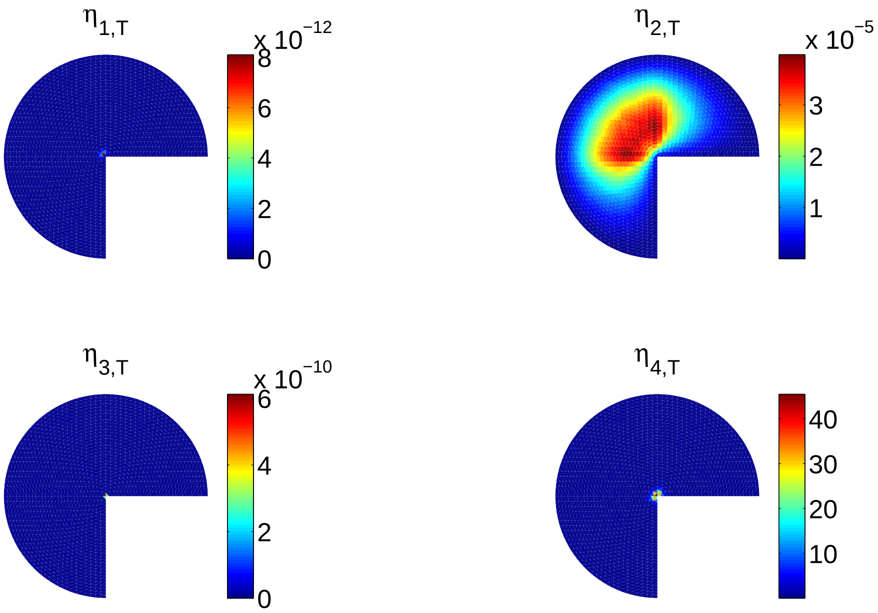

Fig. 5.7. η for $l = 4$.

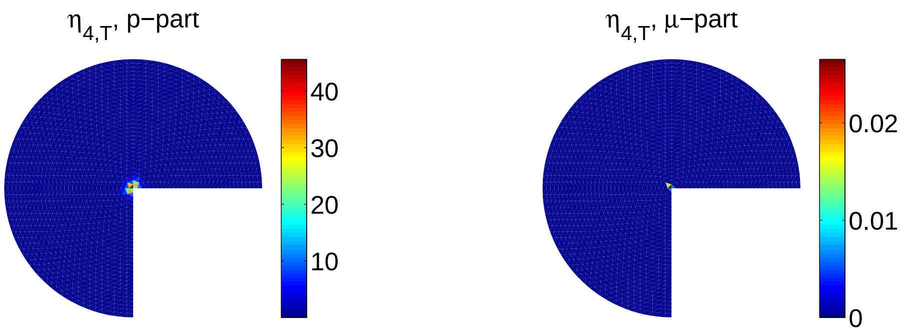

Fig. 5.8. $\eta_{4,E}$-parts for $l = 4$.

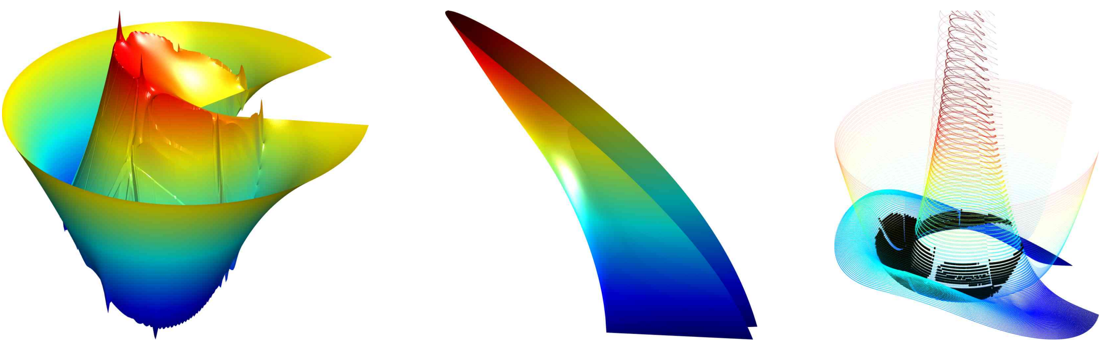

Fig. 5.9. u_h (left), y_h (middle), and active set of all T with $|\nabla y_{h|T}| = \psi(x_T)$ (right) for $l = 6$.

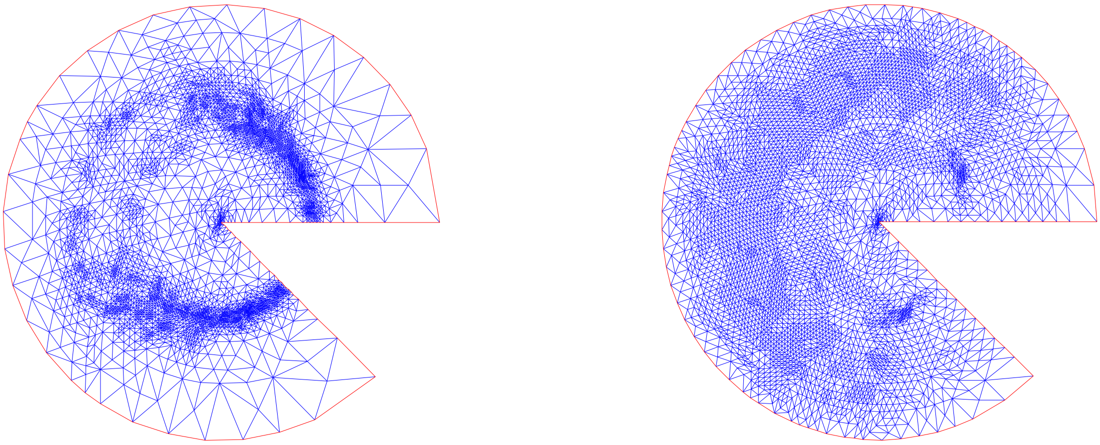

Fig. 5.10. Mesh with bulk-marking ($np = 4549$, $nt = 9019$) (left), mesh with max-marking ($np = 4560$, $nt = 9008$) (right).

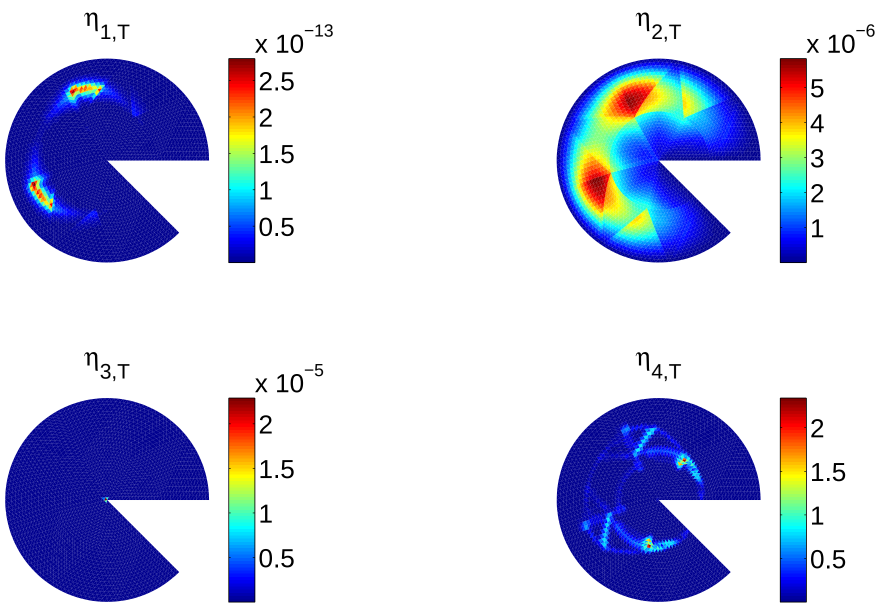

Fig. 5.11. η for $l = 4$.

5.2. Example 2 with active origin

In this example we choose $\omega = \frac{3}{2}\pi, r = 6$, and $\psi(x) = 0.1|x|^{-\frac{1}{3}} + 0.9 \in L^6(\Omega)$. The gradient constraints are active in the origin, see Fig. 5.5, where the numerical solution u_h and y_h together with the active set (black area) is presented. The control now is pronounced in a neighborhood of the origin, which causes refinement there.

Fig. 5.6 presents the adaptively refined meshes obtained by bulk-marking (left) and max-marking (right). Refinement due to the error indicators associated to the primal and dual variables is enforced near the origin, and due to the dual variables in the black area of active

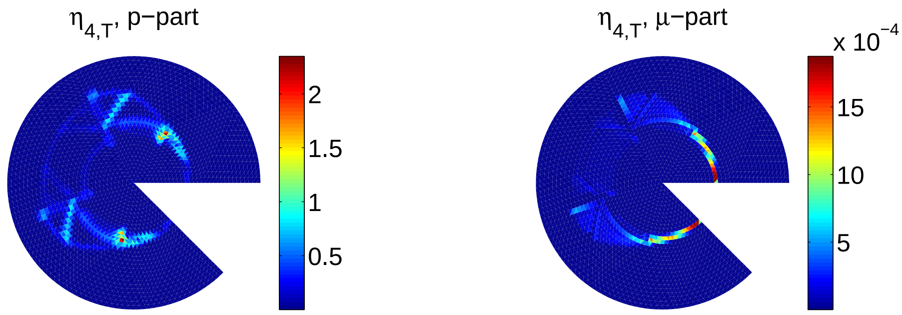

Fig. 5.12. $\eta_{4,E}$-parts for $l = 4$.

constraints.

In Fig. 5.7 the different contributions to the error estimator defined in (4.2a)-(4.3b) are displayed on a uniform mesh for refinement level $l = 4$. One clearly sees that the edge-residual of the dual variables delivers by far the largest contribution in a neighborhood of the origin.

In Fig. 5.8 the contributions of p_h and μ_h to (4.3b) are displayed separately. It is clearly shown that the contribution of p_h is dominant.

5.3. Example 3 with inactive origin

Now we have $\omega = \frac{7}{4}\pi, r = \frac{14}{3}$, and $\psi(x) = 2|x|^{-\frac{3}{7}} + 2|x| - 3 \in L^{\frac{14}{3}}(\Omega)$. The numerical results are very similar to those of Example 1, where in the present example the crescent-shaped area is even more pronounced, see Fig. 5.9. The meshes obtained by the different marking strategies under investigation are shown in Fig. 5.10. As before, we in Fig. 5.11 and 5.12 depict the various contributions to the error estimator for mesh refinement. As announced earlier, the conclusions are similar as in Example 1.

Acknowledgement. Parts of this work were obtained during a stay of the authors at the Mathematisches Forschungsinstitut Oberwolfach within the Research in Pairs programme in March 2009. The authors greatfully acknowledge the support and the hospitality of institute. Furthermore, the authors would like to thank Andreas Günther for his support in providing the numerical computations.

References

[1] M. Ainsworth and J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis, Pure and Applied Mathematics, New York, Wiley-Interscience, John Wiley & Sons, New York, 2000.

[2] H. Attouch, G. Buttazzo, and G. Michaille, Variational Analysis in Sobolev and BV Spaces, volume 6 of *MPS/SIAM Series on Optimization*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2006.

[3] I. Babuška and T. Strouboulis, The Finite Element Method and its Reliability, Numerical Mathematics and Scientific Computation, The Clarendon Press Oxford University Press, New York, 2001.

[4] W. Bangerth and R. Rannacher, Adaptive Finite Element Methods for Differential Equations, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2003.

[5] R. Becker, H. Kapp, and R. Rannacher, Adaptive finite element methods for optimal control of partial differential equations: basic concept, *SIAM J. Control Optim.*, **39**:1 (2000), 113–132 (electronic).

[6] D. Braess, R.H.W. Hoppe, and J. Schöberl, A posteriori estimators for obstacle problems by the hypercircle method, *Comput. Vis. Sci.*, **11**:4-6 (2008), 351–362.

[7] E. Casas, Control of an elliptic problem with pointwise state constraints, *SIAM J. Control Optim.*, **24**:6 (1986), 1309–1318.

[8] E. Casas and L.A. Fernandez, Optimal control of semilinear equations with pointwise constraints on the gradient of the state, *Appl. Math. Opti.*, **27** (1993), 35–56.

[9] E. Casas, F. Tröltzsch, and A. Unger, Second order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations, *SIAM J. Control Optim.*, **38**:5 (2000), 1369–1391 (electronic).

[10] K. Deckelnick, M. Hinze, A Finite Element Approximation to Elliptic Control Problems in the Presence of Control and State Constraints, Hamburger Beiträge zur Angewandten Mathematik HBAM2007-01 (2007).

[11] K. Deckelnick, A. Günther, and M. Hinze, Finite element approximation of elliptic control problems with constraints on the gradient, *Numer. Math.*, (2009), 335–350.

[12] I. Ekeland and R. Témam, Convex Analysis and Variational Problems, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[13] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson, Computational Differential Equations, Cambridge University Press, Cambridge, 1996.

[14] P. Grisvard. Elliptic Problems in Nonsmooth Domains, volume 24 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[15] A. Günther, and M. Hinze, Elliptic Control Problems with Gradient Constraints - Variational Discrete Versus Piecewise Constant Controls, Comput. Optim. Appl., DOI: 10.1007/s10589-009-9308-8 (2009).

[16] M. Hintermüller, K. Ito, K. Kunisch, The primal-dual active set strategy as a semi-smooth Newton method, *SIAM J. Optimiz.*, **13** (2003), 865–888.

[17] M. Hintermüller and R. Hoppe, Adaptive finite element methods for control constrained distributed and boundary optimal control problems. In M. Heinkenschloss and L. Vicente, eds., *Lecture Notes in Computational Science and Engineering*, No. 67, Springer-Verlag, Berlin-Heidelberg-New York, 2008.

[18] M. Hintermüller and R. Hoppe, Goal-oriented adaptivity in control constrained optimal control of partial differential equations, *SIAM J. Control Optim.*, **47** (2008), 1721–1743.

[19] M. Hintermüller, R. Hoppe, Y. Iliash, and M. Kieweg, An a posteriori error analysis of adaptive finite element methods for distributed elliptic control problems with control constraints, *ESAIM: Control, Optimisation and Calculus of Variations*, **14**:3 (2008), 540–560.

[20] M. Hinze, A variational discretization concept in control constrained optimization: the linear-quadratic case, *Comput. Optim. Appl.*, **30**:1 (2005), 45–61.

[21] R. Li, W. Liu, H. Ma, and T. Tang, Adaptive finite element approximation for distributed elliptic optimal control problems, *SIAM J. Control Optim.*, **41**:5 (2002), 1321–1349 (electronic).

[22] R. Li, W. Liu, and N. Yan, A posteriori error estimates of recovery type for distributed convex optimal control problems, *J. Sci. Comput.*, **33**:2 (2007), 155–182.

[23] P. Neittaanmäki and S. Repin, Reliable methods for computer simulation, volume 33 of *Studies in Mathematics and its Applications*, Elsevier Science B.V., Amsterdam, 2004, Error control and a posteriori estimates.

[24] L.R. Scott and S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, *Math. Comp.*, **54**:190 (1990), 483–493.

[25] C.G. Simader, On Dirichlet's boundary value problem, Lecture Notes in Mathematics, Vol. 268. Springer-Verlag, Berlin, 1972.

[26] R. Verführt, A Review of A Posteriori Estimation and Adaptive Mesh- Renement Techniques, Wiley-Teubner, New York, Stuttgart, 1996.

[27] B. Vexler and W. Wollner, Adaptive finite elements for elliptic optimization problems with control constraints, *SIAM J. Control Optim.*, **47**:1 (2008), 509–534.

[28] J. Zowe, S. Kurcyusz, Regularity and stability for the mathematical programming problem in Banach spaces, *Appl. Math. Opt.*, **5** (1979), 49–62.