

Galerkin method for the numerical solution of boundary value problems using Boubaker wavelets

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Abstract. In this paper, we proposed Galerkin method for the numerical solution of boundary value problems using Boubaker wavelets. Here, we used weight functions as Boubaker wavelets that are assumed bases elements which allow us to obtain the numerical solutions of the boundary value problems. Obtained numerical results using this method are compared with the existing method and exact solution. Some test problems are considered to demonstrate the validity and applicability of the proposed method.

Keywords: Galerkin method; Boubaker wavelets; Boundary value problems..

1. Introduction

Boundary value problems (BVPs) occur frequently in the fields of engineering and science such as gas dynamics, nuclear physics, atomic structures and chemical reactions. In most cases, we do not always find the exact solutions for these equations via analytical methods. In this case, it is very meaningful to give the high precision numerical solutions for this kind of problem by numerical methods. Recently, some of the numerical methods are used for the numerical solutions of such type of problems. For example, Haar wavelet collocation method [1], Legendre wavelet collocation method [2], Series solution [3] etc.

The subject of wavelets has received much attention because of the comprehensive mathematical power and the good application potential of wavelets in many interesting physical problems. Wavelet functions have generated significant interest from both theoretical and applied research over the last few years. The name wavelet comes from the requirement that they should integrate to zero, waving above and below x-axis. The concepts for understanding wavelets were provided recently by Meyer, Mallat, Daubechies, and many others. Since then, the number of applications where wavelets have been used has exploded. Many different types of wavelet functions have been presented over the past few years [4].

In wavelet theory, the contribution of orthogonal bases of compactly supported wavelet by Daubechies (1988) and Multiresolution analysis based fast wavelet transform algorithm by Belkin (1991), wavelet based approximation of ordinary differential equations gained momentum in attractive way. Special interest has been dedicated to the construction of compactly supported smooth wavelet bases. As we have noted earlier that, spectral bases are infinitely differentiable but have global support. On the other side, bases functions used in finite-element methods have small compact support but poor continuity properties. Already we know that, spectral methods have good spectral localization but poor spatial localization, while finite element methods have good spatial localization, but poor spectral localization. Wavelet bases execute to combine the advantages of both spectral and finite element bases. We can look forward to numerical methods based on wavelet bases to be capable to attain good spatial and spectral resolutions. Representation of a smooth function in terms of a series expansion using orthogonal polynomials is a fundamental idea in approximation theory and forms the basis of spectral methods of solution of differential equations with functional arguments. An approach to study differential equations is the use of wavelet function bases in place of other conventional piecewise polynomial trial functions in finite element type methods. Because of its implementation simplicity, the Galerkin method is considered the most widely used in applied mathematics [5-6].

The advantage of wavelet-Galerkin method over finite difference or finite element method has lead to tremendous applications in science and engineering. To a certain extent, the wavelet technique is a strong competitor to the finite element method. Also, the wavelet method provided an efficient alternative technique for solving boundary value problems numerically.

In this paper, we developed Galerkin method for the numerical solution of boundary value problems using Boubaker wavelets. This method is based on expanding the solution by Boubaker wavelets with

unknown coefficients. The properties of Boubaker wavelets together with the Galerkin method are utilized to evaluate the unknown coefficients and then a numerical solution of the boundary value problems is obtained.

The organization of the paper is as follows. Preliminaries of Boubaker wavelets and function approximation are given section 2. Section 3 deals with Boubaker wavelet-Galerkin method for the solution of boundary value problems. Numerical Experiment is given in section 4. Finally, conclusions of the proposed work are discussed in section 5

2. Boubaker wavelets and Function approximation

Boubaker wavelets:

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b varies continuously, we have the following family of continuous wavelets [7]:

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}} \psi\left(\frac{x-b}{a}\right), a, b \in \mathbb{R} \text{ and } a \neq 0$$

If we restrict the parameters a and b to discrete values as

$$a = a_0^{-n}, b = mb_0 a_0^{-n}; a_0 > 1, b_0 > 0$$

we have the following family of discrete wavelets

$$\psi_{n,m}(x) = |a_0|^{\frac{1}{2}} \psi(a_0^n x - mb_0), n, m \in \mathbb{Z}$$

Where $\psi_{n,m}$ form a wavelet basis for a, b . In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{n,m}(x)$ forms an orthonormal basis.

Boubaker wavelets are defined as follows:

$$\psi_{n,m}(x) = \begin{cases} \sqrt{2m+1} \frac{(2m!)}{(m!)^2} 2^{\frac{k+1}{2}} B_m(2^{k+1}x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & \text{Otherwise} \end{cases} \quad (2.1)$$

where, k is any positive integer, $n = 1, 2, 3, \dots, 2^{k-1}$ is an argument and $m = 0, 1, 2, 3, \dots, M-1$ is the order of Boubaker functions

$$B_0(x) = 1, \quad B_1(x) = \frac{1}{2}(2x - 1).$$

$$B_2(x) = \frac{1}{6}(6x^2 - 6x + 1), \quad B_3(x) = \frac{1}{20}(20x^3 - 30x^2 + 12x - 1)$$

and so on. For instance, for $k = 1$ and $M = 4$, we get the Boubaker wavelet bases as follows:

$$\psi_{1,0}(x) = 2,$$

$$\psi_{1,1}(x) = 2\sqrt{3}(8x - 3),$$

$$\psi_{1,2}(x) = 2\sqrt{5}(96x^2 - 72x + 13),$$

$$\psi_{1,3}(x) = 2\sqrt{7}(1280x^3 - 1440x^2 + 528x - 63) \text{ and so on.}$$

Function approximation:

Suppose $y(x) \in L^2(0,1)$ is expanded in terms of Boubaker wavelets as:

$$y(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x) \quad (2.2)$$

Truncating the above infinite series, we get

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \psi_{n,m}(x) \quad (2.3)$$

3. Method of solution

Consider the boundary value of the problem is of the form,

$$y'' + \alpha y' + \beta y = f(x) \quad (3.1)$$

With boundary conditions $y(0) = a, y(1) = b$ (3.2)

Where α, β are constant and $f(x)$ be a continuous function.

Write the Eq. (3.1) as $R(x) = y'' + \alpha y' + \beta y - f(x)$ (3.3)

where $R(x)$ is the residual of the Eq. (3.1) When $R(x) = 0$ for the exact solution, $u(x)$ only which will satisfy the boundary conditions.

Consider the trail series solution of the Eq. (3.1), $y(x)$ defined over $[0, 1)$ can be expanded as a modified Boubaker, satisfying the given boundary conditions which is involving unknown parameter as follows,

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \psi_{n,m}(x) \quad (3.4)$$

where $c_{i,j}$'s are unknown coefficients to be determined.

Accuracy in the solution is increased by choosing higher degree Laguerre wavelet polynomials.

Differentiating Eq. (3.4) twice with respect to x and substitute the values of y, y', y'' in Eq. (4.3). To find $c_{n,m}$'s we choose weight functions as assumed bases elements and integrate on boundary values together with the residual to zero [8].

$$\text{i.e.} \quad \int_0^1 \psi_{1,m}(x) R(x) dx = 0, m = 0, 1, 2, \dots$$

then we obtain a system of linear algebraic equations, on solving this system, we get unknown parameters. Then substitute these unknowns in the trail solution i.e. Eq. (3.4), we obtained numerical solution of Eq. (3.1).

In order to know the accuracy of BWGM for the test problems, we use the error measure i.e. maximum absolute error. The maximum absolute error will be calculated by

$$E_{max} = \max |y(x, t)_e - u(x, t)_a|,$$

where $y(x, t)_e$ and $u(x, t)_a$ are exact and approximate solutions respectively.

4. Numerical Experiment

Problem 4.1 First, consider the boundary value problem [9],

$$y'' + y = -x, 0 \leq x \leq 1 \quad (4.1)$$

With boundary conditions: $y(0) = 0, y(1) = 0$ (4.2)

The implementation of the eq. (4.1) as per the method explained in section 3 is as follows:

The residual of eq. (4.1) can be written as: $R(x) = y'' + y + x$ (4.3)

Now, choosing the weight function $w(x) = x(1-x)$ for Boubaker wavelet bases to satisfy the given boundary conditions Eq. (4.2), i.e. $\psi(x) = w(x) \times \psi(x)$

$$\psi_{1,0}(x) = \psi_{1,0}(x) \times x(1-x) = 2x(1-x)$$

$$\psi_{1,1}(x) = \psi_{1,1}(x) \times x(1-x) = 2\sqrt{3}(8x-3)x(1-x)$$

$$\psi_{1,2}(x) = \psi_{1,2}(x) \times x(1-x) = 2\sqrt{5}(96x^2-72x+13)x(1-x)$$

Assuming the trail solution of Eq. (4.1) for $k = 1$ and $m = 2$ is given by

$$y(x) = c_{1,0}\psi_{1,0}(x) + c_{1,1}\psi_{1,1}(x) + c_{1,2}\psi_{1,2}(x) \quad (4.4)$$

Then the Eq. (4.4) becomes

$$y(x) = c_{1,0}\{2x(1-x)\} + c_{1,1}\{2\sqrt{3}(8x-3)x(1-x)\} + c_{1,2}\{2\sqrt{5}(96x^2-72x+13)x(1-x)\} \quad (4.5)$$

Differentiating Eq. (4.5) twice w.r.t. x and substitute the values of y, y', y'' in Eq. (4.3), we get the residual of Eq. (4.1).

The "weight functions" are the same as the bases functions. Then by the weighted Galerkin method, we consider the following:

$$\int_0^1 \psi_{1,j}(x) R(x) dx = 0, j = 0, 1, 2 \quad (4.6)$$

For $j = 0, 1, 2$ in Eq. (4.6),

$$\text{i.e.} \quad \left. \begin{aligned} \int_0^1 \psi_{1,0}(x) R(x) dx &= 0 \\ \int_0^1 \psi_{1,1}(x) R(x) dx &= 0 \\ \int_0^1 \psi_{1,2}(x) R(x) dx &= 0 \end{aligned} \right\} \quad (4.7)$$

From Eq. (4.7), we have system of algebraic equations with unknown coefficients i.e. $c_{1,0}, c_{1,1}$ and $c_{1,2}$. Solving this by Gauss elimination method, we obtain the values of $c_{1,0} = 0.1286, c_{1,1} = 0.0064$ and $c_{1,2} =$

−0.0001. Substituting these values in Eq. (4.5), we get the numerical solution. The comparison of numerical solution and the absolute errors are presented in table 1 and numerical solution with exact solution of Eq. (4.1) is $y(x) = \frac{\sin(x)}{\sin(1)} - x$ in figure 1.

Table 1. Comparison of numerical solution and absolute error with exact solution of the problem 4.1.

x	Numerical solution		Exact solution	Absolute error	
	LWGM*	BWGM		LWGM*	BWGM
0.1	0.018976	0.018486	0.018642	3.34e-04	1.56e-04
0.2	0.036319	0.036011	0.036098	2.21e-04	8.70e-05
0.3	0.051077	0.051215	0.051195	1.18e-04	2.00e-05
0.4	0.062611	0.062839	0.062783	1.72e-04	5.60e-05
0.5	0.069621	0.069731	0.069747	1.26e-04	1.60e-05
0.6	0.071254	0.070838	0.071018	2.36e-04	1.80e-04
0.7	0.065973	0.065212	0.065585	3.88e-04	3.73e-04
0.8	0.051651	0.052008	0.052502	8.51e-04	4.94e-04
0.9	0.031515	0.030483	0.030902	6.13e-04	4.19e-04

* LWGM: Laguerre wavelet-Galerkin method

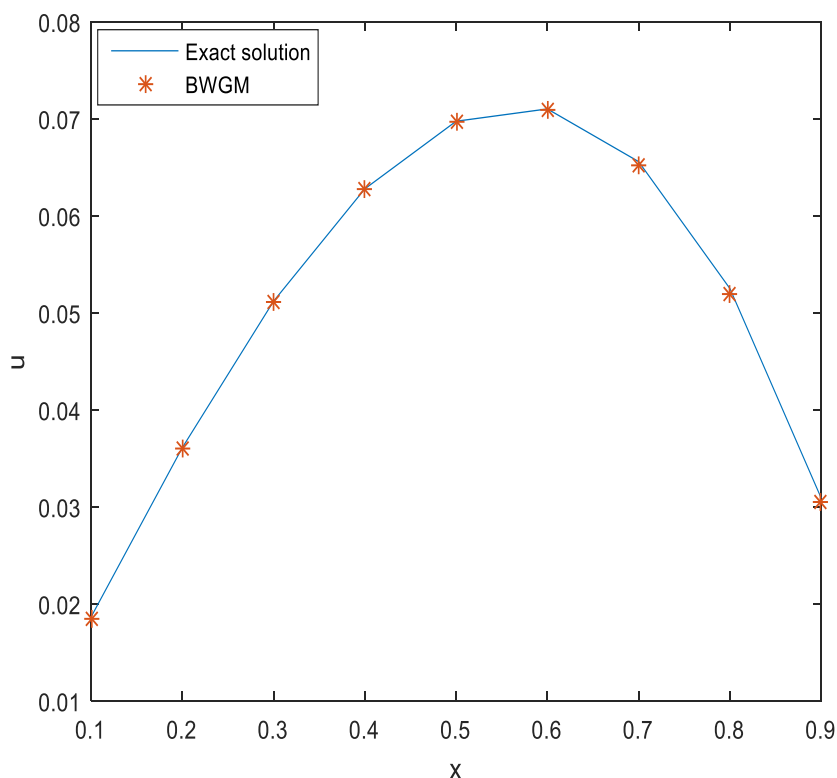


Fig. 1. Comparison of numerical solution with exact solution of the problem 4.1.

Problem 4.2 Now, consider the another boundary value problem [10],

$$y'' + \frac{16}{9}\pi^2 y = \frac{7}{9}\pi^2 \sin \pi x, 0 \leq x \leq 1 \quad (4.8)$$

With boundary conditions: $y(0) = 0, y(1) = 0$ (4.9)

As explained in section 3 and in the previous problem, we obtain the values of $c_{1,0} = 1.9614$, $c_{1,1} = 0.0318$ and $c_{1,2} = -0.0082$. Substituting these values in Eq. (4.5), we get the numerical solution. The comparison of numerical solution and the absolute errors are presented in table 2 and numerical solution with exact solution of Eq. (4.8) is $y(x) = \sin(\pi x)$ in figure 2.

Table 2 Comparison of numerical solution and absolute error with exact solution of the problem 4.2.

x	Numerical solution		Exact solution	Absolute error	
	LWGM ([10])	BWGM		LWGM ([10])	BWGM
0.1	0.3087468	0.308930	0.3090169	2.70e-04	8.69e-05
0.2	0.5925196	0.588656	0.5877852	4.73e-03	8.71e-04
0.3	0.8151813	0.809599	0.8090169	6.16e-03	5.82e-04
0.4	0.9540854	0.950632	0.9510565	3.03e-03	4.25e-04
0.5	0.9982500	0.999072	1.0000000	1.75e-03	9.28e-04
0.6	0.9465312	0.950687	0.9510565	4.53e-03	3.69e-04
0.7	0.7952968	0.809697	0.8090169	1.37e-02	6.80e-04
0.8	0.5811001	0.588766	0.5877852	6.69e-03	9.81e-04
0.9	0.3093530	0.309013	0.3090169	3.36e-04	3.90e-06

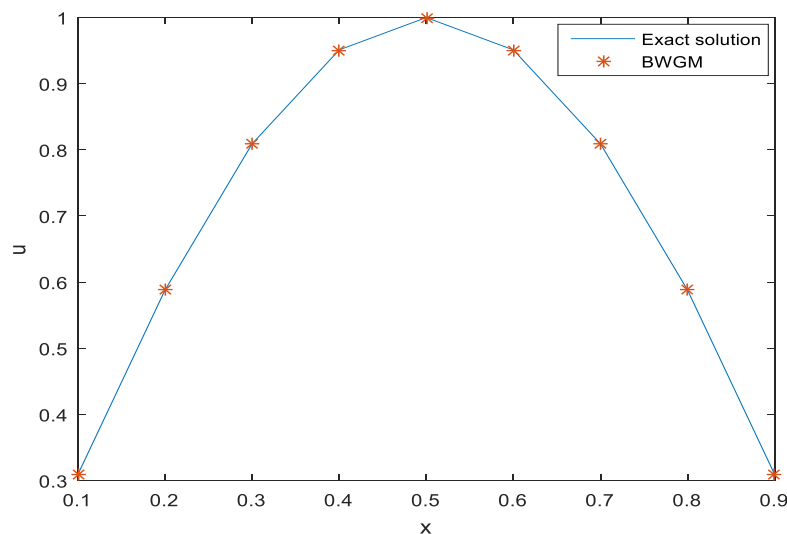


Fig. 2 Comparison of numerical solution with exact solution of the problem 4.2.

Problem 4.3 Finally, consider the boundary value problem [11],

$$y'' - y' = -1, 0 \leq x \leq 1 \quad (4.10)$$

With boundary conditions: $y(0) = 0, y(1) = 0$ (4.11)

As explained in section 3 and in the problem 4.1, we obtain the values of $c_{1,0} = 0.2351$, $c_{1,1} = 0.0055$ and $c_{1,2} = 0.0001$. Substituting these values in Eq. (4.5), we get the numerical solution. The comparison of numerical solution and the absolute errors are presented in table 3 and numerical solution with exact solution of Eq. (4.1) is $y(x) = x - \left(\frac{e^x - 1}{e - 1}\right)$ in figure 3.

Table 3. Comparison of numerical solution and absolute error with exact solution of the problem 4.3.

x	Numerical solution		Exact solution	Absolute error	
	LWGM*	BWGM		LWGM*	BWGM
0.1	0.038684	0.038796	0.038793	1.09e-04	3.00e-06
0.2	0.071099	0.071131	0.071149	5.00e-05	1.80e-05
0.3	0.096232	0.096357	0.096390	1.58e-04	3.30e-05
0.4	0.113656	0.113734	0.113769	1.13e-04	3.50e-05
0.5	0.122420	0.122429	0.122459	3.90e-05	3.00e-05
0.6	0.121367	0.121514	0.121546	1.79e-04	3.20e-05
0.7	0.109825	0.109972	0.110020	1.95e-04	4.80e-05
0.8	0.086853	0.086651	0.086764	8.90e-05	7.30e-05
0.9	0.050414	0.050466	0.050545	1.31e-04	7.90e-05

* LWGM: Laguerre wavelet-Galerkin method

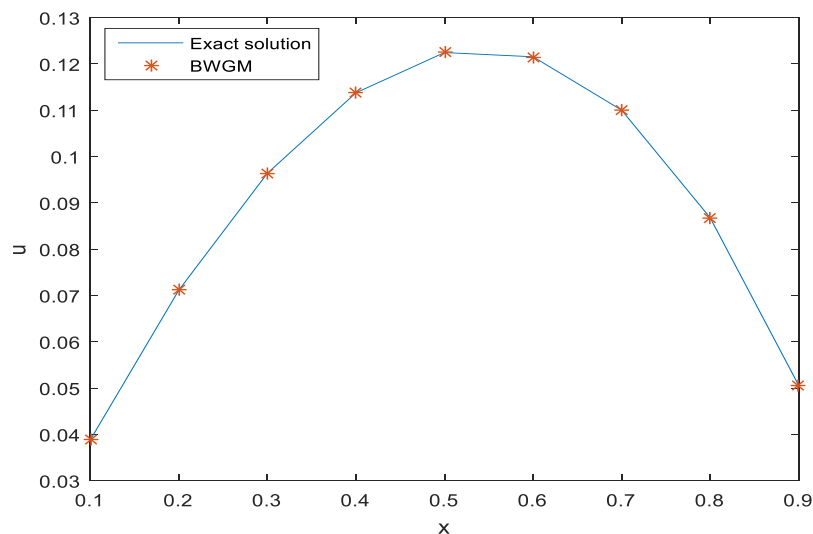


Fig. 3. Comparison of numerical solution with exact solution of the problem 4.3.

5. Conclusions

In this paper, we proposed the Galerkin method for the numerical solution of boundary value problems using Boubaker wavelets. From the tables and figures, we observe that the numerical solutions obtained by the proposed method are better than existing method i.e. LWGM, and nearer to the exact solution. Hence the Galerkin method using Boubaker wavelets is effective for solving boundary value problems.

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