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Numerical Computations of Eleventh Order Boundary Value Problems with Bezier Polynomials by Galerkin Weighted Residual Method

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Abstract: Some techniques are available to solve numerically higher order boundary value problems. The aim of this paper is to apply Galerkin weighted residual method (GWRM) for solving eleventh order linear and nonlinear boundary value problems. Using GWRM, approximate solutions of eleventh-order boundary value problems are developed. This approach provides the solution in terms of a convergent series. Approximate results are given for several examples to illustrate the implementation and accuracy of the method. The results are depicted both graphically and numerically. All results are compared with the analytical solutions to show the convergence of the proposed algorithm. It is observed that the present method is a more effective tool and yields better results. All problems are computed using the software MATLAB R2017a.

Keywords: Differential Equations; Numerical solutions; Galerkin method; Bezier polynomials.

1. Introduction

Higher order boundary value problems (BVPs) occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydro magnetic stability, astronomy, beam and long wave theory, induction motors, engineering, and applied physics. The boundary value problems of higher order have been examined due to their mathematical importance and applications in diversified applied sciences [1-2]. Twizell et al [3] developed numerical methods for eight, tenth and twelfth order eigenvalue problems arising in thermal instability. Scott and Watts [4] developed a numerical method for the solution of linear BVPs using a combination of superposition and orthonormalization. Siddiqi et al [5] used Variational iteration technique to obtain numerical approximations for eleventh-order BVPs by converting the original problem into a system of integral equations. Very recently Amjad Hussain et al [6] derived the numerical solutions of eleventh-order BVPs using differential transformation method. Siddiqi and Ghazala [7-10] presented the solutions of eight, tenth and twelfth order boundary value problems using spline and Non-polynomial spline.

In the present paper, the eleventh order boundary value problems are solved using the Galerkin weighted residual method. The problem has the following form:

$$c_{11}\frac{d^{11}u}{dx^{11}} + c_{10}\frac{d^{10}u}{dx^{10}} + c_9\frac{d^9u}{dx^9} + c_8\frac{d^8u}{dx^8} + c_7\frac{d^7u}{dx^7} + c_6\frac{d^6u}{dx^6} + c_5\frac{d^5u}{dx^5} + c_4\frac{d^4u}{dx^4} + c_3\frac{d^3u}{dx^3} + c_2\frac{d^2u}{dx^2} + c_1\frac{du}{dx} + c_0u = r,\ a < x < b \quad (1a)$$

subject to the following boundary conditions:

$$u(\mathrm{a}) = \mathrm{A}_0,\ u(b) = \mathrm{B}_0,\ u'(a) = \mathrm{A}_1,\ u'(b) = \mathrm{B}_1,\ u''(a) = \mathrm{A}_2,\ u''(b) = \mathrm{B}_2,\ u'''(a) = \mathrm{A}_3,$$
$$u'''(b) = \mathrm{B}_3,\ u^{(iv)}(a) = \mathrm{A}_4,\ u^{(iv)}(b) = \mathrm{B}_4,\ u^{(v)}(a) = \mathrm{A}_5 \quad (1b)$$

Where $\mathrm{A}_i, i = 0,1,2,3,4,5$ and $\mathrm{B}_j, j = 0,1,2,3,4$ are finite real constants and $\mathrm{c}_i, i = 0,1,\ldots,11$ and r are all continuous and differentiable functions of x defined on the interval [a, b].

The paper is organized in four sections. In section 2, we give a short description on Bezier polynomials. The analysis of Galerkin weighted residual method is discussed in section 3. In section 4, three numerical examples are presented to assess the efficiency of the Galerkin weighted residual technique.

2. Bezier Polynomials

The general form of the Bezier polynomials of nth degree over the interval [0, 1] is defined by

$$B_{j,n}(x) = \sum_{j=0}^{n}\binom{n}{j}x^j(1-x)^{n-j}P_j\ , 0 \le x \le 1$$

Where the binomial coefficients are given by

$$\binom{n}{j} = \frac{n!}{(n-j)!\,j!}$$

The points P_j are called control points for the Bezier curve.

We write first few Bezier polynomials over the interval [0,1]:

$B_0(x) = (1-x)^{19}$, $B_1(x) = 19(1-x)^{18}x$, $B_2(x) = 171(1-x)^{17}x^2$, $B_3(x) = 969(1-x)^{16}x^3$
$B_4(x) = 3876(1-x)^{15}x^4$, $B_5(x) = 11628(1-x)^{14}x^5$, $B_6(x) = 27132(1-x)^{13}x^6$, $B_7(x) = 50388(1-x)^{12}x^7$
$B_8(x) = 75582(1-x)^{11}x^8$, $B_9(x) = 92378(1-x)^{10}x^9$, $B_{10}(x) = 92378(1-x)^9x^{10}$, $B_{11}(x) = 3876(1-x)^8x^{11}$
$B_{12}(x) = 75582(1-x)^7x^{12}$, $B_{13}(x) = 27132(1-x)^6x^{13}$, $B_{14}(x) = 11628(1-x)^5x^{14}$, $B_{15}(x) = 3876(1-x)^4x^{15}$
$B_{16}(x) = 969(1-x)^3x^{16}$, $B_{17}(x) = 171(1-x)^2x^{17}$, $B_{18}(x) = 19(1-x)x^{18}$, $B_{19}(x) = x^{19}$

Note that each of these $n+1$ polynomials having degree n satisfies the following properties:

(i) $B_{j,n}(x) = 0$ if $j < 0$ or $j > n$

(ii) $\sum_{j=0}^{n} B_{j,n}(x) = 1$

(iii) $B_{j,n}(a) = B_{j,n}(b) = 0,\ j = 1,2,\ldots,n-1$

For these properties, Bezier polynomials are used in the trail functions satisfying the corresponding homogeneous form of the essential boundary conditions in the Galerkin weighted residual method to solve a BVP.

3. Matrix Formulation of Eleventh-order BVPs

In this section, we first derived the matrix formulation for eleventh-order linear BVP and then we extend our idea for solving nonlinear BVP. To solve the boundary value problem (1) by the Galerkin weighted residual method we approximate $\tilde{u}(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} \beta_i B_i(x),\ n \ge 2 \tag{2}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions and $B_i(a) = B_i(b) = 0$, for each $i = 1,2,3,\ldots,n-1$.

Using (2) into (1), the Galerkin weighted residual equations are:

$$\int_a^b \left[c_{11}\frac{d^{11}\tilde{u}}{dx^{11}} + c_{10}\frac{d^{10}\tilde{u}}{dx^{10}} + c_9\frac{d^9\tilde{u}}{dx^9} + c_8\frac{d^8\tilde{u}}{dx^8} + c_7\frac{d^7\tilde{u}}{dx^7} + c_6\frac{d^6\tilde{u}}{dx^6} + c_5\frac{d^5\tilde{u}}{dx^5} + c_4\frac{d^4\tilde{u}}{dx^4} + c_3\frac{d^3\tilde{u}}{dx^3} + c_2\frac{d^2\tilde{u}}{dx^2} + c_1\frac{d\tilde{u}}{dx} + c_0\tilde{u} - r\right]B_j(x)dx = 0,\ j = 1,2,\ldots,n-1 \tag{3}$$

Integrating by parts the terms up to second derivative on the left hand side of (3), we get

$$\begin{aligned}\int_a^b c_{11}\frac{d^{11}\tilde{u}}{dx^{11}}B_j(x)dx = &-\left[\frac{d}{dx}[c_{11}B_j(x)]\frac{d^9\tilde{u}}{dx^9}\right]_a^b + \left[\frac{d^2}{dx^2}[c_{11}B_j(x)]\frac{d^8\tilde{u}}{dx^8}\right]_a^b - \left[\frac{d^3}{dx^3}[c_{11}B_j(x)]\frac{d^7\tilde{u}}{dx^7}\right]_a^b \\ &+ \left[\frac{d^4}{dx^4}[c_{11}B_j(x)]\frac{d^6\tilde{u}}{dx^6}\right]_a^b - \left[\frac{d^5}{dx^5}[c_{11}B_j(x)]\frac{d^5\tilde{u}}{dx^4}\right]_a^b + \left[\frac{d^6}{dx^6}[c_{11}B_j(x)]\frac{d^4\tilde{u}}{dx^4}\right]_a^b - \left[\frac{d^7}{dx^7}[c_{11}B_j(x)]\frac{d^3\tilde{u}}{dx^3}\right]_a^b \\ &+ \left[\frac{d^8}{dx^8}[c_{11}B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_a^b - \left[\frac{d^9}{dx^9}[c_{11}B_j(x)]\frac{d\tilde{u}}{dx}\right]_a^b + \int_a^b \frac{d^{10}}{dx^{10}}[c_{11}B_j(x)]\frac{d\tilde{u}}{dx}dx\end{aligned} \tag{4}$$

$$\begin{aligned}\int_a^b c_{10}\frac{d^{10}\tilde{u}}{dx^{10}}B_j(x)dx = &-\left[\frac{d}{dx}[c_{10}B_j(x)]\frac{d^8\tilde{u}}{dx^8}\right]_a^b + \left[\frac{d^2}{dx^2}[c_{10}B_j(x)]\frac{d^7\tilde{u}}{dx^7}\right]_a^b - \left[\frac{d^3}{dx^3}[c_{10}B_j(x)]\frac{d^6\tilde{u}}{dx^6}\right]_a^b \\ &+ \left[\frac{d^4}{dx^4}[c_{10}B_j(x)]\frac{d^5\tilde{u}}{dx^5}\right]_a^b - \left[\frac{d^5}{dx^5}[c_{10}B_j(x)]\frac{d^4\tilde{u}}{dx^4}\right]_a^b + \left[\frac{d^6}{dx^6}[c_{10}B_j(x)]\frac{d^3\tilde{u}}{dx^3}\right]_a^b - \left[\frac{d^7}{dx^7}[c_{10}B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_a^b \\ &+ \left[\frac{d^8}{dx^8}[c_{10}B_j(x)]\frac{d\tilde{u}}{dx}\right]_a^b - \int_a^b \frac{d^9}{dx^9}[c_{10}B_j(x)]\frac{d\tilde{u}}{dx}dx\end{aligned} \tag{5}$$

$$\begin{aligned}\int_a^b c_9\frac{d^9\tilde{u}}{dx^9}B_j(x)dx = &-\left[\frac{d}{dx}[c_9B_j(x)]\frac{d^7\tilde{u}}{dx^7}\right]_a^b + \left[\frac{d^2}{dx^2}[c_9B_j(x)]\frac{d^6\tilde{u}}{dx^6}\right]_a^b - \left[\frac{d^3}{dx^3}[c_9B_j(x)]\frac{d^5\tilde{u}}{dx^5}\right]_a^b \\ &+ \left[\frac{d^4}{dx^4}[c_9B_j(x)]\frac{d^4\tilde{u}}{dx^4}\right]_a^b - \left[\frac{d^5}{dx^5}[c_9B_j(x)]\frac{d^3\tilde{u}}{dx^3}\right]_a^b + \left[\frac{d^6}{dx^6}[c_9B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_a^b - \left[\frac{d^7}{dx^7}[c_9B_j(x)]\frac{d\tilde{u}}{dx}\right]_a^b \\ &+ \int_a^b \frac{d^8}{dx^8}[c_9B_j(x)]\frac{d\tilde{u}}{dx}dx\end{aligned} \tag{6}$$

$$\begin{aligned}\int_a^b c_8\frac{d^8\tilde{u}}{dx^8}B_j(x)dx = &-\left[\frac{d}{dx}[c_8B_j(x)]\frac{d^6\tilde{u}}{dx^6}\right]_a^b + \left[\frac{d^2}{dx^2}[c_8B_j(x)]\frac{d^5\tilde{u}}{dx^5}\right]_a^b - \left[\frac{d^3}{dx^3}[c_8B_j(x)]\frac{d^4\tilde{u}}{dx^4}\right]_a^b \\ &+ \left[\frac{d^4}{dx^4}[c_8B_j(x)]\frac{d^3\tilde{u}}{dx^3}\right]_a^b - \left[\frac{d^5}{dx^5}[c_8B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_a^b + \left[\frac{d^6}{dx^6}[c_8B_j(x)]\frac{d\tilde{u}}{dx}\right]_a^b - \int_a^b \frac{d^7}{dx^7}[c_8B_j(x)]\frac{d\tilde{u}}{dx}dx\end{aligned} \tag{7}$$

$$\begin{aligned}\int_a^b c_7\frac{d^7\tilde{u}}{dx^7}B_j(x)dx = &-\left[\frac{d}{dx}[c_7B_j(x)]\frac{d^5\tilde{u}}{dx^5}\right]_a^b + \left[\frac{d^2}{dx^2}[c_7B_j(x)]\frac{d^4\tilde{u}}{dx^4}\right]_a^b - \left[\frac{d^3}{dx^3}[c_7B_j(x)]\frac{d^3\tilde{u}}{dx^3}\right]_a^b \\ &+ \left[\frac{d^4}{dx^4}[c_7B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_a^b - \left[\frac{d^5}{dx^5}[c_7B_j(x)]\frac{d\tilde{u}}{dx}\right]_a^b + \int_a^b \frac{d^6}{dx^6}[c_7B_j(x)]\frac{d\tilde{u}}{dx}dx\end{aligned} \tag{8}$$

$$\int_a^b c_6 \frac{d^6\tilde{u}}{dx^6} B_j(x)dx = -\left[\frac{d}{dx}\left[c_6B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_a^b + \left[\frac{d^2}{dx^2}\left[c_6B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b - \left[\frac{d^3}{dx^3}\left[c_6B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b$$
$$+ \left[\frac{d^4}{dx^4}\left[c_6B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b - \int_a^b \frac{d^5}{dx^5}\left[c_6B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (9)$$

$$\int_a^b c_5 \frac{d^5\tilde{u}}{dx^5} B_j(x)dx = -\left[\frac{d}{dx}\left[c_5B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b + \left[\frac{d^2}{dx^2}\left[c_5B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b - \left[\frac{d^3}{dx^3}\left[c_5B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b + \int_a^b \frac{d^4}{dx^4}\left[c_5B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (10)$$

$$\int_a^b c_4 \frac{d^4\tilde{u}}{dx^4} B_j(x)dx = -\left[\frac{d}{dx}\left[c_4B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b + \left[\frac{d^2}{dx^2}\left[c_4B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b - \int_a^b \frac{d^3}{dx^3}\left[c_4B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (11)$$

$$\int_a^b c_3 \frac{d^3\tilde{u}}{dx^3} B_j(x)dx = -\left[\frac{d}{dx}\left[c_3B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b + \int_a^b \frac{d^2}{dx^2}\left[c_3B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (12)$$

$$\int_a^b c_2 \frac{d^2\tilde{u}}{dx^2} B_j(x)dx = -\int_a^b \frac{d}{dx}\left[c_2B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (13)$$

Putting equations (4) to (13) into equation (3) and using approximation for $\tilde{u}(x)$ given in equation (2) and after applying the boundary conditions given in equation (1b) and rearranging the terms for the resulting equations we get a system of equations in matrix form as

$$\sum_{i=1}^{n-1} M_{i.j}\,\beta_i = N_j\,,\ j = 1,2,\ldots,n-1 \quad (14a)$$

Where

$$M_{i.j} = \int_a^b \Big\{\Big[\frac{d^{10}}{dx^{10}}\left[c_{11}B_j(x)\right] - \frac{d^9}{dx^9}\left[c_{10}B_j(x)\right] + \frac{d^8}{dx^8}\left[c_9B_j(x)\right] - \frac{d^7}{dx^7}\left[c_8B_j(x)\right] + \frac{d^6}{dx^6}\left[c_7B_j(x)\right] - \frac{d^5}{dx^5}\left[c_6B_j(x)\right] +$$
$$\frac{d^4}{dx^4}\left[c_5B_j(x)\right] - \frac{d^3}{dx^3}\left[c_4B_j(x)\right] + \frac{d^2}{dx^2}\left[c_3B_j(x)\right] - \frac{d}{dx}\left[c_2B_j(x)\right] + c_1B_j(x)\Big]\frac{d}{dx}\left[B_j(x)\right] + c_0B_i(x)B_j(x)\Big\}dx -$$
$$\left[\frac{d}{dx}\left[c_{11}B_j(x)\right]\frac{d^9}{dx^9}\left[B_i(x)\right]\right]_{x=b} + \left[\frac{d}{dx}\left[c_{11}B_j(x)\right]\frac{d^9}{dx^9}\left[B_i(x)\right]\right]_{x=a}$$
$$+ \left[\frac{d^2}{dx^2}\left[c_{11}B_j(x)\right]\frac{d^8}{dx^8}\left[B_i(x)\right]\right]_{x=b} - \left[\frac{d^2}{dx^2}\left[c_{11}B_j(x)\right]\frac{d^8}{dx^8}\left[B_i(x)\right]\right]_{x=a} - \left[\frac{d^3}{dx^3}\left[c_{11}B_j(x)\right]\frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=b}$$
$$+ \left[\frac{d^3}{dx^3}\left[c_{11}B_j(x)\right]\frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=a} + \left[\frac{d^4}{dx^4}\left[c_{11}B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=b} - \left[\frac{d^4}{dx^4}\left[c_{11}B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=a}$$
$$- \left[\frac{d^5}{dx^5}\left[c_{11}B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=b} - \left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8}{dx^8}\left[B_i(x)\right]\right]_{x=b} + \left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8}{dx^8}\left[B_i(x)\right]\right]_{x=a}$$
$$+ \left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=b} - \left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=a} - \left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=b}$$
$$+ \left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=a} + \left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=b} - \left[\frac{d}{dx}\left[c_9B_j(x)\right]\frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=b}$$
$$+ \left[\frac{d}{dx}\left[c_9B_j(x)\right]\frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=a} + \left[\frac{d^2}{dx^2}\left[c_9B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=b} - \left[\frac{d^2}{dx^2}\left[c_9B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=a}$$
$$- \left[\frac{d^3}{dx^3}\left[c_9B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=b} - \left[\frac{d}{dx}\left[c_8B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=b} + \left[\frac{d}{dx}\left[c_8B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=a}$$
$$+ \left[\frac{d^2}{dx^2}\left[c_8B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=b} - \left[\frac{d}{dx}\left[c_7B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=b} \quad (14b)$$

$$N_j = \int_a^b \Big\{rB_j(x) - \Big[\frac{d^{10}}{dx^{10}}\left[c_{11}B_j(x)\right] - \frac{d^9}{dx^9}\left[c_{10}B_j(x)\right] + \frac{d^8}{dx^8}\left[c_9B_j(x)\right] - \frac{d^7}{dx^7}\left[c_8B_j(x)\right] + \frac{d^6}{dx^6}\left[c_7B_j(x)\right] -$$
$$\frac{d^5}{dx^5}\left[c_6B_j(x)\right] + \frac{d^4}{dx^4}\left[c_5B_j(x)\right] - \frac{d^3}{dx^3}\left[c_4B_j(x)\right] - \frac{d^2}{dx^2}\left[c_3B_j(x)\right] - \frac{d}{dx}\left[c_2B_j(x)\right] + c_1B_j(x)\Big]\frac{d\theta_0}{dx} - c_0\theta_0B_j(x)\Big\}dx$$
$$+ \left[\frac{d}{dx}\left[c_{11}B_j(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=b} - \left[\frac{d}{dx}\left[c_{11}B_j(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=a} - \left[\frac{d^2}{dx^2}\left[c_{11}B_j(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=b} + \left[\frac{d^2}{dx^2}\left[c_{11}B_j(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=a} +$$
$$\left[\frac{d^3}{dx^3}\left[c_{11}B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=b} - \left[\frac{d^3}{dx^3}\left[c_{11}B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=a} - \left[\frac{d^4}{dx^4}\left[c_{11}B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=b} + \left[\frac{d^4}{dx^4}\left[c_{11}B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=a} +$$
$$\left[\frac{d^5}{dx^5}\left[c_{11}B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b} + \left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=b} - \left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=a} - \left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=b} +$$
$$\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=a} + \left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=b} - \left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=a} - \left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b} +$$
$$\left[\frac{d}{dx}\left[c_9B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=b} - \left[\frac{d}{dx}\left[c_9B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=a} - \left[\frac{d^2}{dx^2}\left[c_9B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=b} +$$
$$\left[\frac{d^2}{dx^2}\left[c_9B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=a} + \left[\frac{d^3}{dx^3}\left[c_9B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b} + \left[\frac{d}{dx}\left[c_8B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=b} -$$

$$\left[\frac{d}{dx}\left[c_8B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=a}-\left[\frac{d^2}{dx^2}\left[c_8B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}+\left[\frac{d}{dx}\left[c_7B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}-\left[\frac{d^5}{dx^5}\left[c_{11}B_j(x)\right]\right]_{x=a}\times A_5-$$
$$\left[\frac{d^6}{dx^6}\left[c_{11}B_j(x)\right]\right]_{x=b}\times B_4+\left[\frac{d^6}{dx^6}\left[c_{11}B_j(x)\right]\right]_{x=a}\times A_4+\left[\frac{d^7}{dx^7}\left[c_{11}B_j(x)\right]\right]_{x=b}\times B_3-\left[\frac{d^7}{dx^7}\left[c_{11}B_j(x)\right]\right]_{x=a}\times A_3-$$
$$\left[\frac{d^8}{dx^8}\left[c_{11}B_j(x)\right]\right]_{x=b}\times B_2+\left[\frac{d^8}{dx^8}\left[c_{11}B_j(x)\right]\right]_{x=a}\times A_2+\left[\frac{d^9}{dx^9}\left[c_{11}B_j(x)\right]\right]_{x=b}\times B_1-\left[\frac{d^9}{dx^9}\left[c_{11}B_j(x)\right]\right]_{x=a}\times A_1+$$
$$\left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\right]_{x=a}\times A_5+\left[\frac{d^5}{dx^5}\left[c_{10}B_j(x)\right]\right]_{x=b}\times B_4-\left[\frac{d^5}{dx^5}\left[c_{10}B_j(x)\right]\right]_{x=a}\times A_4-\left[\frac{d^6}{dx^6}\left[c_{10}B_j(x)\right]\right]_{x=b}\times B_3+$$
$$\left[\frac{d^6}{dx^6}\left[c_{10}B_j(x)\right]\right]_{x=a}\times A_3+\left[\frac{d^7}{dx^7}\left[c_{10}B_j(x)\right]\right]_{x=b}\times B_2-\left[\frac{d^7}{dx^7}\left[c_{10}B_j(x)\right]\right]_{x=a}\times A_2-\left[\frac{d^8}{dx^8}\left[c_{10}B_j(x)\right]\right]_{x=b}\times B_1+$$
$$\left[\frac{d^8}{dx^8}\left[c_{10}B_j(x)\right]\right]_{x=a}\times A_1-\left[\frac{d^3}{dx^3}\left[c_9B_j(x)\right]\right]_{x=a}\times A_5-\left[\frac{d^4}{dx^4}\left[c_9B_j(x)\right]\right]_{x=b}\times B_4+\left[\frac{d^4}{dx^4}\left[c_9B_j(x)\right]\right]_{x=a}\times A_4+$$
$$\left[\frac{d^5}{dx^5}\left[c_9B_j(x)\right]\right]_{x=b}\times B_3-\left[\frac{d^5}{dx^5}\left[c_9B_j(x)\right]\right]_{x=a}\times A_3-\left[\frac{d^6}{dx^6}\left[c_9B_j(x)\right]\right]_{x=b}\times B_2+\left[\frac{d^6}{dx^6}\left[c_9B_j(x)\right]\right]_{x=a}\times A_2+$$
$$\left[\frac{d^7}{dx^7}\left[c_9B_j(x)\right]\right]_{x=b}\times B_1-\left[\frac{d^7}{dx^7}\left[c_9B_j(x)\right]\right]_{x=a}\times A_1-\left[\frac{d^2}{dx^2}\left[c_8B_j(x)\right]\right]_{x=a}\times A_5+\left[\frac{d^3}{dx^3}\left[c_8B_j(x)\right]\right]_{x=b}\times B_4-$$
$$\left[\frac{d^3}{dx^3}\left[c_8B_j(x)\right]\right]_{x=a}\times A_4-\left[\frac{d^4}{dx^4}\left[c_8B_j(x)\right]\right]_{x=b}\times(b-a)^3B_3+\left[\frac{d^4}{dx^4}\left[c_8B_j(x)\right]\right]_{x=a}\times A_3+\left[\frac{d^5}{dx^5}\left[c_8B_j(x)\right]\right]_{x=b}\times B_2-$$
$$\left[\frac{d^5}{dx^5}\left[c_8B_j(x)\right]\right]_{x=a}\times A_2-\left[\frac{d^6}{dx^6}\left[c_8B_j(x)\right]\right]_{x=b}\times(b-a)\,B_1+\left[\frac{d^6}{dx^6}\left[c_8B_j(x)\right]\right]_{x=a}\times A_1-\left[\frac{d}{dx}\left[c_7B_j(x)\right]\right]_{x=a}\times A_5-$$
$$\left[\frac{d^2}{dx^2}\left[c_7B_j(x)\right]\right]_{x=b}\times B_4+\left[\frac{d^2}{dx^2}\left[c_7B_j(x)\right]\right]_{x=a}\times A_4+\left[\frac{d^3}{dx^3}\left[c_7B_j(x)\right]\right]_{x=b}\times B_3-\left[\frac{d^3}{dx^3}\left[c_7B_j(x)\right]\right]_{x=a}\times A_3-$$
$$\left[\frac{d^4}{dx^4}\left[c_7B_j(x)\right]\right]_{x=b}\times B_2+\left[\frac{d^4}{dx^4}\left[c_7B_j(x)\right]\right]_{x=a}\times A_2+\left[\frac{d^5}{dx^5}\left[c_7B_j(x)\right]\right]_{x=b}\times B_1-\left[\frac{d^5}{dx^5}\left[c_7B_j(x)\right]\right]_{x=a}\times A_1+$$
$$\left[\frac{d}{dx}\left[c_6B_j(x)\right]\right]_{x=b}\times B_4-\left[\frac{d}{dx}\left[c_6B_j(x)\right]\right]_{x=a}\times A_4-\left[\frac{d^2}{dx^2}\left[c_6B_j(x)\right]\right]_{x=b}\times B_3+\left[\frac{d^2}{dx^2}\left[c_6B_j(x)\right]\right]_{x=a}\times A_3+$$
$$\left[\frac{d^3}{dx^3}\left[c_6B_j(x)\right]\right]_{x=b}\times B_2-\left[\frac{d^3}{dx^3}\left[c_6B_j(x)\right]\right]_{x=a}\times A_2-\left[\frac{d^4}{dx^4}\left[c_6B_j(x)\right]\right]_{x=b}\times B_1+\left[\frac{d^4}{dx^4}\left[c_6B_j(x)\right]\right]_{x=a}\times A_1+$$
$$\left[\frac{d}{dx}\left[c_5B_j(x)\right]\right]_{x=b}\times B_3-\left[\frac{d}{dx}\left[c_5B_j(x)\right]\right]_{x=a}\times A_3-\left[\frac{d^2}{dx^2}\left[c_5B_j(x)\right]\right]_{x=b}\times B_2+\left[\frac{d^2}{dx^2}\left[c_5B_j(x)\right]\right]_{x=a}\times A_2+$$
$$\left[\frac{d^3}{dx^3}\left[c_5B_j(x)\right]\right]_{x=b}\times B_1-\left[\frac{d^3}{dx^3}\left[c_5B_j(x)\right]\right]_{x=a}\times A_1+\left[\frac{d}{dx}\left[c_4B_j(x)\right]\right]_{x=b}\times B_2-\left[\frac{d}{dx}\left[c_4B_j(x)\right]\right]_{x=a}\times A_2-$$
$$\left[\frac{d^2}{dx^2}\left[c_4B_j(x)\right]\right]_{x=b}\times B_1+\left[\frac{d^2}{dx^2}\left[c_4B_j(x)\right]\right]_{x=a}\times A_1+\left[\frac{d}{dx}\left[c_3B_j(x)\right]\right]_{x=b}\times B_1-\left[\frac{d}{dx}\left[c_3B_j(x)\right]\right]_{x=a}\times A_1,\quad j=1,2,\ldots,n-1 \tag{14c}$$

Solving the system (14a), we find the values of the parameters β_i, and then substituting these parameters into equation (2), we get the approximate solution of the BVP (1).

For nonlinear eleventh-order BVP, we first compute the initial values on neglecting the nonlinear terms and using the systems (14). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP.

4. Numerical Examples

To implement the method, three examples are considered.

Example 1: Consider the following eleventh order linear boundary value problem [5, 6]

$$\frac{d^{11}u}{dx^{11}}-u=-22(5+x)e^x,\ 0\le x\le 1 \tag{15a}$$

subject to the boundary conditions:

$$u(0)=1,\ u(1)=\ 0,\ u'(0)=1,\ u'(1)=-2e,\ u''(0)=-1,\ u''(1)=-6e,\ u'''(0)=-5,$$
$$u'''(1)=-12e,\ u^{(iv)}(0)=-11,\ u^{(iv)}(1)=-20e,\ u^{(v)}(0)=-19 \tag{15b}$$

The analytic solution of the above problem is, $u(x)=(1-x^2)e^x$.

The comparison of the exact solution with the approximate solution, of the example 1, obtained using the GWRM, is shown in Table 1.

On the contrary it is observed that the accuracy is found nearly the order 10^{-16} in [5] by Siddiqi et al and nearly the order 10^{-13} in [6] by Amjad Hussain et al. We have shown the exact and approximate solutions in figure 1 of example 1 for $n = 15$.

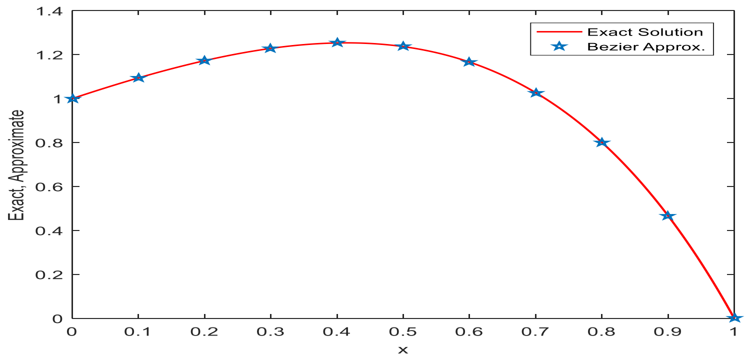

Figure 1: Numerical vs analytical solutions for example 1.

Table 1: Comparison between approximate and exact solutions for example 1 in $\boldsymbol{u_i}$

$\boldsymbol{x}$	**Exact Solutions**	**15, Bezier polynomials**	
		Approx. Solutions	**Absolute Error**
0.0	1.0000000000	1.0000000000	0.000000000
0.1	1.0941192089	1.0941192089	3.89×10^{-16}
0.2	1.1725466478	1.1725466478	1.65×10^{-16}
0.3	1.2283715149	1.2283715149	4.88×10^{-16}
0.4	1.2531327460	1.2531327460	2.38×10^{-16}
0.5	1.2365409530	1.2365409530	1.01×10^{-16}
0.6	1.1661560322	1.1661560322	3.29×10^{-16}
0.7	1.0270138808	1.0270138808	1.65×10^{-15}
0.8	0.8011947343	0.8011947343	5.39×10^{-16}
0.9	0.4673245911	0.4673245911	2.17×10^{-16}
1.0	0.0000000000	0.0000000000	0.000000000

Example 2: Consider the following eleventh order linear boundary value problem [5, 6]

$$\frac{d^{11}u}{dx^{11}} + u = 22(x\cos x + 5\sin x) + (1 - x^2)(\cos x + \sin x),\ 0 \le x \le 1 \quad (16a)$$

subject to the boundary conditions:

$$u(0) = 1,\ \ u(1) = 0,\ \ u'(0) = 0,\ \ u'(1) = -2\cos 1,\ \ u''(0) = -3,\ \ u''(1) = 4\sin 1 - 2\cos 1,\ \ u'''(0) = 0,\ \ u'''(1) = 6(\sin 1 + \cos 1),\ \ u^{(iv)}(0) = 13,\ \ u^{(iv)}(1) = -8\sin 1 + 12\cos 1,\ \ u^{(v)}(0) = 0\ . \quad (16b)$$

The analytic solution of the above problem is, $u(x) = (1 - x^2)\cos x$.

The comparison of the exact solution with the approximate solution, of the example 2, obtained using the GWRM, is shown in Table 2.

On the contrary it is observed that the maximum absolute errors were found by Siddiqi et al [5] is 5.148×10^{-12} in [6] and by Amjad Hussain et al [8] is 9.560×10^{-13}. Now the exact and approximate solutions are depicted in figure 2 of example 2 for $n = 16$.

Example 3: Consider the following eleventh order non-linear boundary value problem [5]

$$\frac{d^{11}u}{dx^{11}} = 11(\cos x - \sin x) - x(\cos x + \sin x) - u^2 + x^2(1 - 2\cos x \sin x),\ 0 \le x \le 1 \quad (17a)$$

subject to the boundary conditions:

$$u(0) = 0,\ \ u(1) = \sin 1 - \cos 1,\ \ u'(0) = -1,\ \ u'(1) = 2\sin 1,\ \ u''(0) = 2,\ \ u''(1) = \sin 1 + 3\cos 1,\ \ u'''(0) = 3,\ \ u'''(1) = -4\sin 1 + 2\cos 1,\ \ u^{(iv)}(0) = -4,\ \ u^{(iv)}(1) = -3\sin 1 - 5\cos 1,\ \ u^{(v)}(0) = -5. \quad (17b)$$

The analytic solution of the above problem is, $u(x) = x(\sin x - \cos x)$.

The comparison of the exact solution with the approximate solution, of the example 3, obtained using the GWRM, is shown in Table 3.

Table 2: Comparison between approximate and exact solutions for example 2 in $\boldsymbol{u_i}$

x	Exact Solutions	16, Bezier polynomials	
		Approx. Solutions	Absolute Error
0.0	1.0000000000	1.0000000000	0.000000000
0.1	0.9850541236	0.9850541236	6.93×10^{-17}
0.2	0.9408639147	0.9408639147	1.11×10^{-16}
0.3	0.8693562051	0.8693562051	4.69×10^{-16}
0.4	0.7736912350	0.7736912350	8.92×10^{-16}
0.5	0.6581869214	0.6581869214	2.13×10^{-16}
0.6	0.5282147935	0.5282147935	5.71×10^{-16}
0.7	0.3900695155	0.3900695155	7.98×10^{-16}
0.8	0.2508144154	0.2508144154	6.25×10^{-16}
0.9	0.1181058940	0.1181058940	1.20×10^{-15}
1.0	0.0000000000	0.0000000000	0.000000000

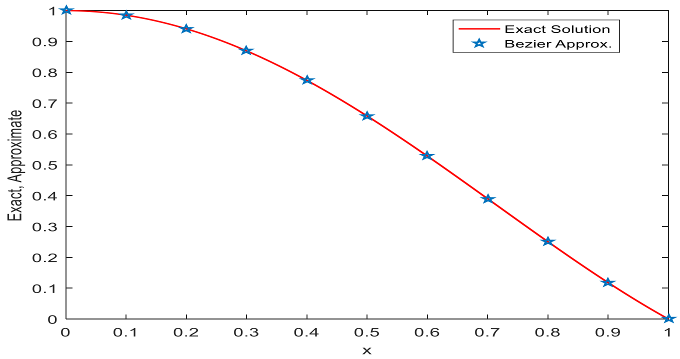

Figure 2: Numerical vs analytical solutions for example 2.

On the contrary it is observed that the maximum absolute errors were found by Siddiqi et al [5] is 4.415×10^{-10}. Now the exact and approximate solutions are depicted in figure 3 of example 3 for $n = 14$.

Table 3: Comparison between approximate and exact solutions for example 3 in $\boldsymbol{u_i}$

x	Exact Solutions	14, Bezier polynomials	
		Approx. Solutions	Absolute Error
0.0	0.0000000000	0.0000000000	0.000000000
0.1	-0.0895170749	-0.0895170749	3.96×10^{-14}
0.2	-0.1562794494	-0.1562794494	1.68×10^{-14}
0.3	-0.1979448847	-0.1979448847	6.64×10^{-14}
0.4	-0.2126570607	-0.2126570607	6.94×10^{-14}
0.5	-0.1990785116	-0.1990785116	9.23×10^{-15}
0.6	-0.1564158849	-0.1564158849	5.35×10^{-15}
0.7	-0.0844371500	-0.0844371500	1.32×10^{-14}
0.8	0.0165195052	0.0165195052	2.61×10^{-15}
0.9	0.1455452472	0.1455452472	3.48×10^{-15}
1.0	0.3011686789	0.3011686789	0.000000000

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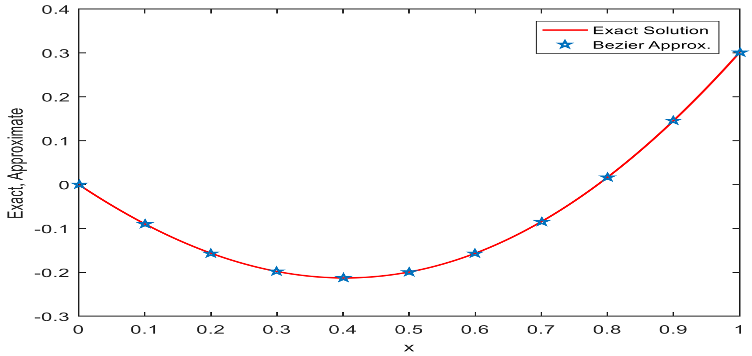

Figure 3: Numerical vs analytical solutions for example 3.

5. Conclusions

In this paper, we derived the complete formulation of Galerkin weighted residual method for eleventh-order linear and non-linear boundary value problems. The results are presented in a data structured table and sketching graphically By observing all those figures and table, it is clear that the presented outcome exhibits the higher estimated order of convergence of this method. So, we can conclude that the present method is an accurate and reliable analytical technique for boundary value problems.

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