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Bezier Polynomials and its Applications with the Tenth and Twelfth Order Boundary Value Problems

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Abstract: The aim of this paper is to apply Galerkin weighted residual method for solving tenth and twelfth order linear and nonlinear boundary value problems (BVPs). A trial function is assumed which is made to satisfy the boundary conditions given, and used to generate the residual to be minimized. The method is formulated as a rigorous matrix form. To investigate the effectiveness of the method, numerical examples were considered which were compared with both the analytic solutions and the solutions obtained by our method. It is observed that, the proposed method is very accurate, better, efficient and appropriate. All problems are computed using the software MATLAB.

Keywords: Numerical solutions, Linear and nonlinear tenth and twelfth order BVPs, Galerkin method, Bezier polynomials.

I. Introduction

Tenth and Twelfth order boundary value problems arise in the study of fluid dynamics, hydro magnetic stability, beam and long wave theory, physics, engineering and applied sciences. Owing to their mathematical significance and applications, several methods such as finite difference method, decomposition method and polynomial spline have been used to solve these types of problems. From the literature we observe that, Siddiqi and Twizell [1] solved tenth order BVPs using tenth degree spline where some unforeseen results for the solution and higher order derivatives were acquired near the boundaries of the interval. Siddiqi and Ghazala [2] acquainted the solutions of tenth order BVPs by eleventh degree spline. A reliable algorithm for solving tenth order BVPs using variational iterative method is developed by Muhammad Aslam Noor et al [3]. On the other hand, variational iteration method for the numerical solution of tenth orders BVPs were used by Fazhan and Xiuying [4]. Inayat Ullah et al [5] acquainted the numerical solutions of higher order nonlinear BVPs by new iterative method. Siddiqi and Twizell [6, 7] solved the tenth and twelfth order BVPs using tenth and twelfth degree splines respectively. Siddiqi and Ghazala Akram [8, 9] elaborated the solutions of tenth and twelfth order BVPs applying eleventh and thirteen degree spline respectively. Approximate solutions of twelfth order BVPs were acquainted by Mohy-ud-Din et al [10]. Mirmoradi et al [11] used Homotopy perturbation method to solve Tenth order and Twelfth order boundary value problems.
This article is organized as, in section II, basic concept of Bezier polynomials are introduced. In section III, two formulations for solving linear and nonlinear higher order BVPs including two types of boundary conditions by Galerkin residual method are presented. The proposed formulation is verified on three linear and two nonlinear BVPs in section IV. Finally, in the last section, the conclusion of the paper is inserted.

II. Bezier Polynomials

The Bezier polynomials of nth degree form a complete basis over [0, 1] and they are defined by

$$B_{j,n}(x) = \sum_{j=0}^{n} \binom{n}{j} x^j (1-x)^{n-j} P_j \, , 0 \le x \le 1$$

Where the binomial coefficients are given by

$$\binom{n}{j} = \frac{n!}{(n-j)!\,j!}$$

The points P_j are called control points for the Bezier curve.
We write first 20 Bezier polynomials of degree 19 over the interval [0,1]:

$B_0(x) = (1-x)^{19}$
$B_1(x) = 19(1-x)^{18}x$
$B_2(x) = 171(1-x)^{17}x^2$
$B_3(x) = 969(1-x)^{16}x^3$
$B_4(x) = 3876(1-x)^{15}x^4$
$B_5(x) = 11628(1-x)^{14}x^5$
$B_6(x) = 27132(1-x)^{13}x^6$
$B_7(x) = 50388(1-x)^{12}x^7$
$B_8(x) = 75582(1-x)^{11}x^8$
$B_9(x) = 92378(1-x)^{10}x^9$
$B_{10}(x) = 92378(1-x)^9x^{10}$
$B_{11}(x) = 3876(1-x)^8x^{11}$
$B_{12}(x) = 75582(1-x)^7x^{12}$
$B_{13}(x) = 27132(1-x)^6x^{13}$
$B_{14}(x) = 11628(1-x)^5x^{14}$
$B_{15}(x) = 3876(1-x)^4x^{15}$
$B_{16}(x) = 969(1-x)^3x^{16}$
$B_{17}(x) = 171(1-x)^2x^{17}$
$B_{18}(x) = 19(1-x)x^{18}$
$B_{19}(x) = x^{19}$

Since Bezier polynomials have special properties at $x = 0$ and $x = 1$: $B_{j,n}(0) = 0$ and $B_{j,n}(1) = 0, j = 1,2,\ldots,n-1$ respectively, so that they can be used as set of basis function to satisfy the corresponding homogeneous form of the essential boundary conditions to derive the matrix formulation in the Galerkin method to solve a BVP over the interval [0,1].

III. Formulation of BVPs in Matrix Form

In this section, we first obtain the rigorous matrix formulation for tenth order linear BVP and then we extend our idea for solving nonlinear BVP. For this, we consider a linear tenth order differential equation given by

$$a_{10}\frac{d^{10}u}{dx^{10}} + a_9\frac{d^9u}{dx^9} + a_8\frac{d^8u}{dx^8} + a_7\frac{d^7u}{dx^7} + a_6\frac{d^6u}{dx^6} + a_5\frac{d^5u}{dx^5} + a_4\frac{d^4u}{dx^4} + a_3\frac{d^3u}{dx^3} + a_2\frac{d^2u}{dx^2} + a_1\frac{du}{dx} + a_0u = r,\ a < x < b \tag{1a}$$

subject to the following two types of boundary conditions:

Type 1

$$u(\mathrm{a}) = \mathrm{A}_0,\ u(b) = \mathrm{B}_0,\ u'(a) = \mathrm{A}_1,\ u'(b) = \mathrm{B}_1,\ u''(a) = \mathrm{A}_2,\ u''(b) = \mathrm{B}_2,\ u'''(a) = \mathrm{A}_3,$$
$$u'''(b) = \mathrm{B}_3,\ u^{(iv)}(a) = \mathrm{A}_4,\ u^{(iv)}(b) = \mathrm{B}_4 \tag{1b}$$

Type 2

$$u(\mathrm{a}) = \mathrm{A}_0,\ u(b) = \mathrm{B}_0,\ u''(a) = \mathrm{A}_2,\ u''(b) = \mathrm{B}_2,\ u^{(iv)}(a) = \mathrm{A}_4,\ u^{(iv)}(b) = \mathrm{B}_4,\ u^{(vi)}(a) = \mathrm{A}_6,\ u^{(vi)}(b) = \mathrm{B}_6,$$
$$u^{(viii)}(a) = \mathrm{A}_8,\ u^{(viii)}(b) = \mathrm{B}_8 \tag{1c}$$

Where $\mathrm{A}_i, \mathrm{B}_i, i = 0,1,2,3,4,5,6,8$ are finite real constants and $\mathrm{a}_i, i = 0,1,2,3,4,5,6,7,8,9,10$ and r are all continuous and differentiable functions of x defined on the interval [a, b]. The BVP (1) is solved with both the boundary conditions of type 1 and type 2.

Since we want to use the polynomials, described in section II, as trial functions which are derived over the interval [0, 1], so the BVP (1) is to be converted to an equivalent problem on [0, 1] by replacing x by $(b-a)x + a$, and thus we have:

$$c_{10}\frac{d^{10}u}{dx^{10}} + c_9\frac{d^9u}{dx^9} + c_8\frac{d^8u}{dx^8} + c_7\frac{d^7u}{dx^7} + c_6\frac{d^6u}{dx^6} + c_5\frac{d^5u}{dx^5} + c_4\frac{d^4u}{dx^4} + c_3\frac{d^3u}{dx^3} + c_2\frac{d^2u}{dx^2} + c_1\frac{du}{dx} + c_0u = t,\ 0 < x < 1 \tag{2a}$$

$$u(0) = \mathrm{A}_0,\ u(1) = \mathrm{B}_0,\ \frac{1}{(b-a)}u'(0) = \mathrm{A}_1,\ \frac{1}{(b-a)}u'(1) = \mathrm{B}_1,\ \frac{1}{(b-a)^2}u''(0) = \mathrm{A}_2,$$
$$\frac{1}{(b-a)^2}u''(1) = \mathrm{B}_2,\ \frac{1}{(b-a)^3}u'''(0) = \mathrm{A}_3,\ \frac{1}{(b-a)^3}u'''(1) = \mathrm{B}_3,\ \frac{1}{(b-a)^4}u^{(iv)}(0) = \mathrm{A}_4,\ \frac{1}{(b-a)^4}u^{(iv)}(1) = \mathrm{B}_4 \tag{2b}$$

and

$$u(0) = \mathrm{A}_0,\ u(1) = \mathrm{B}_0,\ \frac{1}{(b-a)^2}u''(0) = \mathrm{A}_1,\ \frac{1}{(b-a)^2}u''(1) = \mathrm{B}_1,\ \frac{1}{(b-a)^4}u^{(iv)}(0) = \mathrm{A}_2,$$
$$\frac{1}{(b-a)^4}u^{(iv)}(1) = \mathrm{B}_2,\ \frac{1}{(b-a)^6}u^{(vi)}(0) = \mathrm{A}_3,\ \frac{1}{(b-a)^6}u^{(vi)}(1) = \mathrm{B}_3,\ \frac{1}{(b-a)^8}u^{(viii)}(0) = \mathrm{A}_4,\ \frac{1}{(b-a)^8}u^{(viii)}(1) = \mathrm{B}_4 \tag{2c}$$

where

$c_{10}=\frac{1}{(b-a)^{10}}\,a_{10}\big((b-a)x+a\big),\ c_9=\frac{1}{(b-a)^9}\,a_9\big((b-a)x+a\big),\ c_8=\frac{1}{(b-a)^8}\,a_8\big((b-a)x+a\big),\ c_7=\frac{1}{(b-a)^7}\,a_7\big((b-a)x+a\big),\ c_6=\frac{1}{(b-a)^6}\,a_6\big((b-a)x+a\big),\ c_5=\frac{1}{(b-a)^5}\,a_5\big((b-a)x+a\big),\ c_4=\frac{1}{(b-a)^4}\,a_4\big((b-a)x+a\big), c_3=\frac{1}{(b-a)^3}\,a_3\big((b-a)x+a\big), c_2=\frac{1}{(b-a)^2}\,a_2\big((b-a)x+a\big),\ c_1=\frac{1}{(b-a)}\,a_1\big((b-a)x+a\big),\ c_0=a_0\big((b-a)x+a\big),\ t=r\big((b-a)x+a\big)$ (2d)

We approximate the solution of the differential equation (2a) as

$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n-1}\beta_i\,B_i(x)\,,n\geq 2$ (3)

Here $\theta_0(x)$ is specified by the essential boundary conditions, $B_i(x)$ are the Bezier polynomials which must satisfy the corresponding homogeneous boundary conditions such that $B_i\,(0)=B_i(1)=0$, for each $i=1,2,3,\dots,n-1$.

Using (3) into (2a), the Galerkin weighted residual equations are:

$\int_0^1\left[c_{10}\frac{d^{10}\tilde{u}}{dx^{10}}+c_9\frac{d^9\tilde{u}}{dx^9}+c_8\frac{d^8\tilde{u}}{dx^8}+c_7\frac{d^7\tilde{u}}{dx^7}+c_6\frac{d^6\tilde{u}}{dx^6}+c_5\frac{d^5\tilde{u}}{dx^5}+c_4\frac{d^4\tilde{u}}{dx^4}+c_3\frac{d^3\tilde{u}}{dx^3}+c_2\frac{d^2\tilde{u}}{dx^2}+c_1\frac{d\tilde{u}}{dx}+c_0\tilde{u}-t\right]B_j(x)dx=0,\ j=1,2,\dots,n-1$ (4)

Formulation 1

In this portion, we have derived the matrix formulation by applying the boundary conditions of type 1. Integrating by parts the terms up to second derivative on the left hand side of (4), we have

$$\int_0^1 c_{10}\frac{d^{10}\tilde{u}}{dx^{10}}B_j(x)dx=\left[c_{10}\frac{d^9\tilde{u}}{dx^9}B_j(x)\right]_0^1-\int_0^1\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^9\tilde{u}}{dx^9}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^8\tilde{u}}{dx^8}dx\quad [\text{Since } B_i\,(0)=B_i(1)=0]$$

$$=-\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^7\tilde{u}}{dx^7}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1+\int_0^1\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^6\tilde{u}}{dx^6}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$
$$+\left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[c_{10}B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$
$$+\left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{10}B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\int_0^1\frac{d^6}{dx^6}\left[c_{10}B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$
$$+\left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{10}B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{10}B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$-\int_0^1\frac{d^7}{dx^7}\left[c_{10}B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$
$$+\left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{10}B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{10}B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$-\left[\frac{d^7}{dx^7}\left[c_{10}B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\int_0^1\frac{d^8}{dx^8}\left[c_{10}B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\tilde{u}}{dx^8}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1-\left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$
$$+\left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_{10}B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^6}{dx^6}\left[c_{10}B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$-\left[\frac{d^7}{dx^7}\left[c_{10}B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\left[\frac{d^8}{dx^8}\left[c_{10}B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^9}{dx^9}\left[c_{10}B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (5)$$

$$\int_0^1 c_9\frac{d^9\tilde{u}}{dx^9}B_j(x)dx=-\left[\frac{d}{dx}\left[c_9B_j(x)\right]\frac{d^7\tilde{u}}{dx^7}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_9B_j(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1$$
$$-\left[\frac{d^3}{dx^3}\left[c_9B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1+\left[\frac{d^4}{dx^4}\left[c_9B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_9B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$+\left[\frac{d^6}{dx^6}\left[c_9B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\left[\frac{d^7}{dx^7}\left[c_9B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1+\int_0^1\frac{d^8}{dx^8}\left[c_9B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (6)$$

$$\int_0^1 c_8\frac{d^8\tilde{u}}{dx^8}B_j(x)dx=-\left[\frac{d}{dx}\left[c_8B_j(x)\right]\frac{d^6\tilde{u}}{dx^6}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_8B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1$$

$$-\left[\frac{d^3}{dx^3}\left[c_8B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^4}{dx^4}\left[c_8B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_8B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$
$$+\left[\frac{d^6}{dx^6}\left[c_8B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^7}{dx^7}\left[c_8B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (7)$$

$$\int_0^1 c_7\frac{d^7\tilde{u}}{dx^7}B_j(x)dx=-\left[\frac{d}{dx}\left[c_7B_j(x)\right]\frac{d^5\tilde{u}}{dx^5}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_7B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1$$
$$-\left[\frac{d^3}{dx^3}\left[c_7B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1+\left[\frac{d^4}{dx^4}\left[c_7B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1-\left[\frac{d^5}{dx^5}\left[c_7B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1$$
$$+\int_0^1\frac{d^6}{dx^6}\left[c_7B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (8)$$

$$\int_0^1 c_6\frac{d^6\tilde{u}}{dx^6}B_j(x)dx=-\left[\frac{d}{dx}\left[c_6B_j(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_6B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1$$
$$-\left[\frac{d^3}{dx^3}\left[c_6B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\left[\frac{d^4}{dx^4}\left[c_6B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1-\int_0^1\frac{d^5}{dx^5}\left[c_6B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (9)$$

$$\int_0^1 c_5\frac{d^5\tilde{u}}{dx^5}B_j(x)dx=-\left[\frac{d}{dx}\left[c_5B_j(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_5B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1$$
$$-\left[\frac{d^3}{dx^3}\left[c_5B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1+\int_0^1\frac{d^4}{dx^4}\left[c_5B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (10)$$

$$\int_0^1 c_4\frac{d^4\tilde{u}}{dx^4}B_j(x)dx=-\left[\frac{d}{dx}\left[c_4B_j(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_0^1+\left[\frac{d^2}{dx^2}\left[c_4B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1$$
$$-\int_0^1\frac{d^3}{dx^3}\left[c_4B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (11)$$

$$\int_0^1 c_3\frac{d^3\tilde{u}}{dx^3}B_j(x)dx=-\left[\frac{d}{dx}\left[c_3B_j(x)\right]\frac{d\tilde{u}}{dx}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[c_3B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (12)$$

$$\int_0^1 c_2\frac{d^2\tilde{u}}{dx^2}B_j(x)dx=-\int_0^1\frac{d}{dx}\left[c_2B_j(x)\right]\frac{d\tilde{u}}{dx}dx \quad (13)$$

Substituting equations (5) to (13) into equation (4) and using approximation for $\tilde{u}(x)$ given in equation (3) and after imposing the boundary conditions given in equation (2b) and rearranging the terms for the resulting equations we get a system of equations in matrix form as

$$\sum_{i=1}^{n-1}E_{i.j}\,\beta_i=G_j\,,\ j=1,2,\dots,n-1 \quad (14a)$$

Where

$$E_{i.j}=\int_0^1\left\{\left[-\frac{d^9}{dx^9}\left[c_{10}B_j(x)\right]+\frac{d^8}{dx^8}\left[c_9B_j(x)\right]-\frac{d^7}{dx^7}\left[c_8B_j(x)\right]+\frac{d^6}{dx^6}\left[c_7B_j(x)\right]-\frac{d^5}{dx^5}\left[c_6B_j(x)\right]+\frac{d^4}{dx^4}\left[c_5B_j(x)\right]-\right.\right.$$
$$\left.\left.\frac{d^3}{dx^3}\left[c_4B_j(x)\right]+\frac{d^2}{dx^2}\left[c_3B_j(x)\right]-\frac{d}{dx}\left[c_2B_j(x)\right]+c_1B_j(x)\right]\frac{d}{dx}\left[B_j(x)\right]+c_0B_i(x)B_j(x)\right\}dx-$$
$$\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8}{dx^8}\left[B_i(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8}{dx^8}\left[B_i(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=0}$$
$$-\left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=1}+\left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=1}-\left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=0}$$
$$-\left[\frac{d}{dx}\left[c_9B_j(x)\right]\frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_9B_j(x)\right]\frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^2}{dx^2}\left[c_9B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_9B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=0}$$
$$-\left[\frac{d^3}{dx^5}\left[c_9B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=1}+\left[\frac{d^3}{dx^5}\left[c_9B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=0}$$
$$-\left[\frac{d}{dx}\left[c_8B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_8B_j(x)\right]\frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^2}{dx^2}\left[c_8B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_8B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=0}$$
$$-\left[\frac{d}{dx}\left[c_7B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_7B_j(x)\right]\frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=0} \quad (14b)$$

$$G_j=\int_0^1\left\{tB_j(x)+\left[\frac{d^9}{dx^9}\left[c_{10}B_j(x)\right]-\frac{d^8}{dx^8}\left[c_9B_j(x)\right]+\frac{d^7}{dx^7}\left[c_8B_j(x)\right]-\frac{d^6}{dx^6}\left[c_7B_j(x)\right]+\frac{d^5}{dx^5}\left[c_6B_j(x)\right]-\frac{d^4}{dx^4}\left[c_5B_j(x)\right]+\right.\right.$$
$$\left.\left.\frac{d^3}{dx^3}\left[c_4B_j(x)\right]-\frac{d^2}{dx^2}\left[c_3B_j(x)\right]+\frac{d}{dx}\left[c_2B_j(x)\right]-c_1B_j(x)\right]\frac{d\theta_0}{dx}-c_0\theta_0B_j(x)\right\}dx+\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=1}-$$

$$\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=0} - \left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1} + \left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0} + \left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1} -$$
$$\left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0} - \left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1} + \left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0} +$$
$$\left[\frac{d}{dx}\left[c_9B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1} - \left[\frac{d}{dx}\left[c_9B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0} - \left[\frac{d^2}{dx^2}\left[c_9B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1} + \left[\frac{d^2}{dx^2}\left[c_9B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$
$$+ \left[\frac{d^3}{dx^3}\left[c_9B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1} - \left[\frac{d^3}{dx^3}\left[c_9B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0} +$$
$$\left[\frac{d}{dx}\left[c_8B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=1} - \left[\frac{d}{dx}\left[c_8B_j(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$
$$- \left[\frac{d^2}{dx^2}\left[c_8B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1} + \left[\frac{d^2}{dx^2}\left[c_8B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0} +$$
$$\left[\frac{d}{dx}\left[c_7B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1} - \left[\frac{d}{dx}\left[c_7B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0} + \left[\frac{d^5}{dx^5}\left[c_{10}B_j(x)\right]\right]_{x=1} \times (b-a)^4\, B_4 -$$
$$\left[\frac{d^5}{dx^5}\left[c_{10}B_j(x)\right]\right]_{x=0} \times (b-a)^4\, A_4 - \left[\frac{d^6}{dx^6}\left[c_{10}B_j(x)\right]\right]_{x=1} \times (b-a)^3\, B_3 +$$
$$\left[\frac{d^6}{dx^6}\left[c_{10}B_j(x)\right]\right]_{x=0} \times (b-a)^3\, A_3 + \left[\frac{d^7}{dx^7}\left[c_{10}B_j(x)\right]\right]_{x=1} \times (b-a)^2\, B_2 -$$
$$\left[\frac{d^7}{dx^7}\left[c_{10}B_j(x)\right]\right]_{x=0} \times (b-a)^2\, A_2 - \left[\frac{d^8}{dx^8}\left[c_{10}B_j(x)\right]\right]_{x=1} \times (b-a)\, B_1 +$$
$$\left[\frac{d^8}{dx^8}\left[c_{10}B_j(x)\right]\right]_{x=0} \times (b-a)\, A_1 - \left[\frac{d^4}{dx^4}\left[c_9B_j(x)\right]\right]_{x=1} \times (b-a)^4\, B_4 +$$
$$\left[\frac{d^4}{dx^4}\left[c_9B_j(x)\right]\right]_{x=0} \times (b-a)^4\, A_4 + \left[\frac{d^5}{dx^5}\left[c_9B_j(x)\right]\right]_{x=1} \times (b-a)^3\, B_3 -$$
$$\left[\frac{d^5}{dx^5}\left[c_9B_j(x)\right]\right]_{x=0} \times (b-a)^3\, A_3 - \left[\frac{d^6}{dx^6}\left[c_9B_j(x)\right]\right]_{x=1} \times (b-a)^2\, B_2 +$$
$$\left[\frac{d^6}{dx^6}\left[c_9B_j(x)\right]\right]_{x=0} \times (b-a)^2\, A_2 + \left[\frac{d^7}{dx^7}\left[c_9B_j(x)\right]\right]_{x=1} \times (b-a)\, B_1 -$$
$$\left[\frac{d^7}{dx^7}\left[c_9B_j(x)\right]\right]_{x=0} \times (b-a)\, A_1 + \left[\frac{d^3}{dx^3}\left[c_8B_j(x)\right]\right]_{x=1} \times (b-a)^4\, B_4 -$$
$$\left[\frac{d^3}{dx^3}\left[c_8B_j(x)\right]\right]_{x=0} \times (b-a)^4\, A_4 - \left[\frac{d^4}{dx^4}\left[c_8B_j(x)\right]\right]_{x=1} \times (b-a)^3\, B_3 +$$
$$\left[\frac{d^4}{dx^4}\left[c_8B_j(x)\right]\right]_{x=0} \times (b-a)^3 A_3 + \left[\frac{d^5}{dx^5}\left[c_8B_j(x)\right]\right]_{x=1} \times (b-a)^2\, B_2 -$$
$$\left[\frac{d^5}{dx^5}\left[c_8B_j(x)\right]\right]_{x=0} \times (b-a)^2\, A_2 - \left[\frac{d^6}{dx^6}\left[c_8B_j(x)\right]\right]_{x=1} \times (b-a)\, B_1 +$$
$$\left[\frac{d^6}{dx^6}\left[c_8B_j(x)\right]\right]_{x=0} \times (b-a)\, A_1 - \left[\frac{d^2}{dx^2}\left[c_7B_j(x)\right]\right]_{x=1} \times (b-a)^4\, B_4 +$$
$$\left[\frac{d^2}{dx^2}\left[c_7B_j(x)\right]\right]_{x=0} \times (b-a)^4\, A_4 + \left[\frac{d^3}{dx^3}\left[c_7B_j(x)\right]\right]_{x=1} \times (b-a)^3\, B_3 -$$
$$\left[\frac{d^3}{dx^3}\left[c_7B_j(x)\right]\right]_{x=0} \times (b-a)^3\, A_3 - \left[\frac{d^4}{dx^4}\left[c_7B_j(x)\right]\right]_{x=1} \times (b-a)^2\, B_2 +$$
$$\left[\frac{d^4}{dx^4}\left[c_7B_j(x)\right]\right]_{x=0} \times (b-a)^2\, A_2 + \left[\frac{d^5}{dx^5}\left[c_7B_j(x)\right]\right]_{x=1} \times (b-a)\, B_1 -$$
$$\left[\frac{d^5}{dx^5}\left[c_7B_j(x)\right]\right]_{x=0} \times (b-a)\, A_1 + \left[\frac{d}{dx}\left[c_6B_j(x)\right]\right]_{x=1} \times (b-a)^4\, B_4 -$$
$$\left[\frac{d}{dx}\left[c_6B_j(x)\right]\right]_{x=0} \times (b-a)^4\, A_4 - \left[\frac{d^2}{dx^2}\left[c_6B_j(x)\right]\right]_{x=1} \times (b-a)^3\, B_3 +$$
$$\left[\frac{d^2}{dx^2}\left[c_6B_j(x)\right]\right]_{x=0} \times (b-a)^3\, A_3 + \left[\frac{d^3}{dx^3}\left[c_6B_j(x)\right]\right]_{x=1} \times (b-a)^2\, B_2 -$$
$$\left[\frac{d^3}{dx^3}\left[c_6B_j(x)\right]\right]_{x=0} \times (b-a)^2\, A_2 - \left[\frac{d^4}{dx^4}\left[c_6B_j(x)\right]\right]_{x=1} \times (b-a)\, B_1 +$$

$$\left[\frac{d^4}{dx^4}\left[c_6 B_j(x)\right]\right]_{x=0} \times (b-a)\, A_1 + \left[\frac{d}{dx}\left[c_5 B_j(x)\right]\right]_{x=1} \times (b-a)^3\, B_3 - \left[\frac{d}{dx}\left[c_5 B_j(x)\right]\right]_{x=0} \times (b-a)^3\, A_3 - \left[\frac{d^2}{dx^2}\left[c_5 B_j(x)\right]\right]_{x=1} \times (b-a)^2\, B_2 + \left[\frac{d^2}{dx^2}\left[c_5 B_j(x)\right]\right]_{x=0} \times (b-a)^2\, A_2 + \left[\frac{d^3}{dx^3}\left[c_5 B_j(x)\right]\right]_{x=1} \times (b-a)\, B_1 - \left[\frac{d^3}{dx^3}\left[c_5 B_j(x)\right]\right]_{x=0} \times (b-a)\, A_1 + \left[\frac{d}{dx}\left[c_4 B_j(x)\right]\right]_{x=1} \times (b-a)^2\, B_2 - \left[\frac{d}{dx}\left[c_4 B_j(x)\right]\right]_{x=0} \times (b-a)^2\, A_2 - \left[\frac{d^2}{dx^2}\left[c_4 B_j(x)\right]\right]_{x=1} \times (b-a)\, B_1 + \left[\frac{d^2}{dx^2}\left[c_4 B_j(x)\right]\right]_{x=0} \times (b-a)\, A_1 + \left[\frac{d}{dx}\left[c_3 B_j(x)\right]\right]_{x=1} \times (b-a)\, B_1 - \left[\frac{d}{dx}\left[c_3 B_j(x)\right]\right]_{x=0} \times (b-a)\, A_1, \quad j = 1,2,\dots,n-1 \tag{14c}$$

Solving the system (14a), we find the values of the parameters β_i, and then substituting into (3), we get the approximate solution of the BVP (2). If we replace x by $\frac{x-a}{x+a}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (1).

Formulation 2

In this portion, we have derived the matrix formulation by applying the boundary conditions of type 2. In the same way of formulation 1, integrating by parts the terms up to second derivative on the left hand side of (4), and after applying the boundary conditions prescribed in type 2, equation (2c), we get a system of equations in matrix form as

$$\sum_{i=1}^{n-1} E_{i.j}\, \beta_i = G_j\,,\ j = 1,2,\dots,n-1 \tag{15a}$$

Where

$$E_{i.j} = \int_0^1 \Big\{\Big[-\frac{d^9}{dx^9}\left[c_{10} B_j(x)\right] + \frac{d^8}{dx^8}\left[c_9 B_j(x)\right] - \frac{d^7}{dx^7}\left[c_8 B_j(x)\right] + \frac{d^6}{dx^6}\left[c_7 B_j(x)\right] - \frac{d^5}{dx^5}\left[c_6 B_j(x)\right] + \frac{d^4}{dx^4}\left[c_5 B_j(x)\right] - \frac{d^3}{dx^3}\left[c_4 B_j(x)\right] + \frac{d^2}{dx^2}\left[c_3 B_j(x)\right] - \frac{d}{dx}\left[c_2 B_j(x)\right] + c_1 B_j(x)\Big] \frac{d}{dx}\left[B_j(x)\right] + c_0 B_i(x) B_j(x)\Big\}\, dx$$
$$+ \left[\frac{d^2}{dx^2}\left[c_{10} B_j(x)\right] \frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[c_{10} B_j(x)\right] \frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=0}$$
$$+ \left[\frac{d^4}{dx^4}\left[c_{10} B_j(x)\right] \frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=1} - \left[\frac{d^4}{dx^4}\left[c_{10} B_j(x)\right] \frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=0}$$
$$+ \left[\frac{d^6}{dx^6}\left[c_{10} B_j(x)\right] \frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=1} - \left[\frac{d^6}{dx^6}\left[c_{10} B_j(x)\right] \frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=0}$$
$$+ \left[\frac{d^8}{dx^8}\left[c_{10} B_j(x)\right] \frac{d}{dx}\left[B_i(x)\right]\right]_{x=1} - \left[\frac{d^8}{dx^8}\left[c_{10} B_j(x)\right] \frac{d}{dx}\left[B_i(x)\right]\right]_{x=0}$$
$$- \left[\frac{d}{dx}\left[c_9 B_j(x)\right] \frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=1} + \left[\frac{d}{dx}\left[c_9 B_j(x)\right] \frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=0}$$
$$- \left[\frac{d^3}{dx^3}\left[c_9 B_j(x)\right] \frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=1} + \left[\frac{d^3}{dx^3}\left[c_9 B_j(x)\right] \frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=0}$$
$$- \left[\frac{d^5}{dx^5}\left[c_9 B_j(x)\right] \frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=1} + \left[\frac{d^5}{dx^5}\left[c_9 B_j(x)\right] \frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=0}$$
$$- \left[\frac{d^7}{dx^7}\left[c_9 B_j(x)\right] \frac{d}{dx}\left[B_i(x)\right]\right]_{x=1} + \left[\frac{d^7}{dx^7}\left[c_9 B_j(x)\right] \frac{d}{dx}\left[B_i(x)\right]\right]_{x=0}$$
$$+ \left[\frac{d^2}{dx^2}\left[c_8 B_j(x)\right] \frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[c_8 B_j(x)\right] \frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=0}$$
$$+ \left[\frac{d^4}{dx^4}\left[c_8 B_j(x)\right] \frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=1} - \left[\frac{d^4}{dx^4}\left[c_8 B_j(x)\right] \frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=0}$$
$$+ \left[\frac{d^6}{dx^6}\left[c_8 B_j(x)\right] \frac{d}{dx}\left[B_i(x)\right]\right]_{x=1} - \left[\frac{d^6}{dx^6}\left[c_8 B_j(x)\right] \frac{d}{dx}\left[B_i(x)\right]\right]_{x=0}$$
$$- \left[\frac{d}{dx}\left[c_7 B_j(x)\right] \frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=1} + \left[\frac{d}{dx}\left[c_7 B_j(x)\right] \frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=0}$$
$$- \left[\frac{d^3}{dx^3}\left[c_7 B_j(x)\right] \frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=1} + \left[\frac{d^3}{dx^3}\left[c_7 B_j(x)\right] \frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=0}$$
$$- \left[\frac{d^5}{dx^5}\left[c_7 B_j(x)\right] \frac{d}{dx}\left[B_i(x)\right]\right]_{x=1} + \left[\frac{d^5}{dx^5}\left[c_7 B_j(x)\right] \frac{d}{dx}\left[B_i(x)\right]\right]_{x=0}$$

$$+\left[\frac{d^2}{dx^2}\left[c_6B_j(x)\right]\frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_6B_j(x)\right]\frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^4}{dx^4}\left[c_6B_j(x)\right]\frac{d}{dx}\left[B_i(x)\right]\right]_{x=1}-\left[\frac{d^4}{dx^4}\left[c_6B_j(x)\right]\frac{d}{dx}\left[B_i(x)\right]\right]_{x=0}$$
$$-\left[\frac{d}{dx}\left[c_5B_j(x)\right]\frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_5B_j(x)\right]\frac{d^3}{dx^3}\left[B_i(x)\right]\right]_{x=0}$$
$$-\left[\frac{d^3}{dx^3}\left[c_5B_j(x)\right]\frac{d}{dx}\left[B_i(x)\right]\right]_{x=1}+\left[\frac{d^3}{dx^3}\left[c_5B_j(x)\right]\frac{d}{dx}\left[B_i(x)\right]\right]_{x=0}$$
$$+\left[\frac{d^2}{dx^2}\left[c_4B_j(x)\right]\frac{d}{dx}\left[B_i(x)\right]\right]_{x=1}-\left[\frac{d^2}{dx^2}\left[c_4B_j(x)\right]\frac{d}{dx}\left[B_i(x)\right]\right]_{x=0}$$
$$-\left[\frac{d}{dx}\left[c_3B_j(x)\right]\frac{d}{dx}\left[B_i(x)\right]\right]_{x=1}+\left[\frac{d}{dx}\left[c_3B_j(x)\right]\frac{d}{dx}\left[B_i(x)\right]\right]_{x=0} \quad (15b)$$

$$G_j=\int_0^1\left\{tB_j(x)+\left[\frac{d^9}{dx^9}\left[c_{10}B_j(x)\right]-\frac{d^8}{dx^8}\left[c_9B_j(x)\right]+\frac{d^7}{dx^7}\left[c_8B_j(x)\right]-\frac{d^6}{dx^6}\left[c_7B_j(x)\right]+\frac{d^5}{dx^5}\left[c_6B_j(x)\right]-\frac{d^4}{dx^4}\left[c_5B_j(x)\right]+\frac{d^3}{dx^3}\left[c_4B_j(x)\right]-\frac{d^2}{dx^2}\left[c_3B_j(x)\right]+\frac{d}{dx}\left[c_2B_j(x)\right]-c_1B_j(x)\right]\frac{d\theta_0}{dx}-c_0\theta_0B_j(x)\right\}dx-\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}+$$
$$\left[\frac{d^2}{dx^2}\left[c_{10}B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0}-\left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}+\left[\frac{d^4}{dx^4}\left[c_{10}B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}-\left[\frac{d^6}{dx^6}\left[c_{10}B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}+$$
$$\left[\frac{d^6}{dx^6}\left[c_{10}B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}-$$
$$\left[\frac{d^8}{dx^8}\left[c_{10}B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^8}{dx^8}\left[c_{10}B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}+\left[\frac{d}{dx}\left[c_9B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=1}-$$
$$\left[\frac{d}{dx}\left[c_9B_j(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=0}+\left[\frac{d^3}{dx^3}\left[c_9B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d^3}{dx^3}\left[c_9B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}+$$
$$\left[\frac{d^5}{dx^5}\left[c_9B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}-\left[\frac{d^5}{dx^5}\left[c_9B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}+\left[\frac{d^7}{dx^7}\left[c_9B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}-$$
$$\left[\frac{d^7}{dx^7}\left[c_9B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_8B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}+\left[\frac{d^2}{dx^2}\left[c_8B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}-$$
$$\left[\frac{d^4}{dx^4}\left[c_8B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}+\left[\frac{d^4}{dx^4}\left[c_8B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}-\left[\frac{d^6}{dx^6}\left[c_8B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}+$$
$$\left[\frac{d^6}{dx^6}\left[c_8B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}+\left[\frac{d}{dx}\left[c_7B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d}{dx}\left[c_7B_j(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=0}+$$
$$\left[\frac{d^3}{dx^3}\left[c_7B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}-\left[\frac{d^3}{dx^3}\left[c_7B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}+\left[\frac{d^5}{dx^5}\left[c_7B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}-$$
$$\left[\frac{d^5}{dx^5}\left[c_7B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[c_6B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}+\left[\frac{d^2}{dx^2}\left[c_6B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}-$$
$$\left[\frac{d^4}{dx^4}\left[c_6B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^4}{dx^4}\left[c_6B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}+\left[\frac{d}{dx}\left[c_5B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=1}-$$
$$\left[\frac{d}{dx}\left[c_5B_j(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=0}+\left[\frac{d^3}{dx^3}\left[c_5B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}-\left[\frac{d^3}{dx^3}\left[c_5B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}-$$
$$\left[\frac{d^2}{dx^2}\left[c_4B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}+\left[\frac{d^2}{dx^2}\left[c_4B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}+\left[\frac{d}{dx}\left[c_3B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=1}-$$
$$\left[\frac{d}{dx}\left[c_3B_j(x)\right]\frac{d\theta_0}{dx}\right]_{x=0}+\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\right]_{x=1}\times(b-a)^8\,B_8-$$
$$\left[\frac{d}{dx}\left[c_{10}B_j(x)\right]\right]_{x=0}\times(b-a)^8\,A_8+\left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\right]_{x=1}\times(b-a)^6\,B_6-$$
$$\left[\frac{d^3}{dx^3}\left[c_{10}B_j(x)\right]\right]_{x=0}\times(b-a)^6\,A_6+\left[\frac{d^5}{dx^5}\left[c_{10}B_j(x)\right]\right]_{x=1}\times(b-a)^4\,B_4-$$
$$\left[\frac{d^5}{dx^5}\left[c_{10}B_j(x)\right]\right]_{x=0}\times(b-a)^4\,A_4+\left[\frac{d^7}{dx^7}\left[c_{10}B_j(x)\right]\right]_{x=1}\times(b-a)^2\,B_2-$$
$$\left[\frac{d^7}{dx^7}\left[c_{10}B_j(x)\right]\right]_{x=0}\times(b-a)^2\,A_2-\left[\frac{d^2}{dx^2}\left[c_9B_j(x)\right]\right]_{x=1}\times(b-a)^6\,B_6+$$
$$\left[\frac{d^2}{dx^2}\left[c_9B_j(x)\right]\right]_{x=0}\times(b-a)^6\,A_6-\left[\frac{d^4}{dx^4}\left[c_9B_j(x)\right]\right]_{x=1}\times(b-a)^4\,B_4+$$
$$\left[\frac{d^4}{dx^4}\left[c_9B_j(x)\right]\right]_{x=0}\times(b-a)^4\,A_4-\left[\frac{d^6}{dx^6}\left[c_9B_j(x)\right]\right]_{x=1}\times(b-a)^2\,B_2+$$
$$\left[\frac{d^6}{dx^6}\left[c_9B_j(x)\right]\right]_{x=0}\times(b-a)^2\,A_2+\left[\frac{d}{dx}\left[c_8B_j(x)\right]\right]_{x=1}\times(b-a)^6\,B_6-$$
$$\left[\frac{d}{dx}\left[c_8B_j(x)\right]\right]_{x=0}\times(b-a)^6A_6+\left[\frac{d^3}{dx^3}\left[c_8B_j(x)\right]\right]_{x=1}\times(b-a)^4\,B_4-$$

$$\left[\frac{d^3}{dx^3}\left[c_8 B_j(x)\right]\right]_{x=0} \times (b-a)^4 A_4 + \left[\frac{d^5}{dx^5}\left[c_8 B_j(x)\right]\right]_{x=1} \times (b-a)^2 B_2 -$$
$$\left[\frac{d^5}{dx^5}\left[c_8 B_j(x)\right]\right]_{x=0} \times (b-a)^2 A_2 - \left[\frac{d^2}{dx^2}\left[c_7 B_j(x)\right]\right]_{x=1} \times (b-a)^4 B_4 +$$
$$\left[\frac{d^2}{dx^2}\left[c_7 B_j(x)\right]\right]_{x=0} \times (b-a)^4 A_4 - \left[\frac{d^4}{dx^4}\left[c_7 B_j(x)\right]\right]_{x=1} \times (b-a)^2 B_2 +$$
$$\left[\frac{d^4}{dx^4}\left[c_7 B_j(x)\right]\right]_{x=0} \times (b-a)^2 A_2 + \left[\frac{d}{dx}\left[c_6 B_j(x)\right]\right]_{x=1} \times (b-a)^4 B_4 -$$
$$\left[\frac{d}{dx}\left[c_6 B_j(x)\right]\right]_{x=0} \times (b-a)^4 A_4 + \left[\frac{d^3}{dx^3}\left[c_6 B_j(x)\right]\right]_{x=1} \times (b-a)^2 B_2 -$$
$$\left[\frac{d^3}{dx^3}\left[c_6 B_j(x)\right]\right]_{x=0} \times (b-a)^2 A_2 - \left[\frac{d^2}{dx^2}\left[c_5 B_j(x)\right]\right]_{x=1} \times (b-a)^2 B_2 +$$
$$\left[\frac{d^2}{dx^2}\left[c_5 B_j(x)\right]\right]_{x=0} \times (b-a)^2 A_2 + \left[\frac{d}{dx}\left[c_4 B_j(x)\right]\right]_{x=1} \times (b-a)^2 B_2 -$$
$$\left[\frac{d}{dx}\left[c_4 B_j(x)\right]\right]_{x=0} \times (b-a)^2 A_2 \qquad (15c)$$

Solving the system (15a), we find the values of the parameters β_i, and then substituting into (3), we get the approximate solution of the BVP (2). If we replace x by $\frac{x-a}{x+a}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (1).

For nonlinear tenth order BVP, we first compute the initial values on neglecting the nonlinear terms and using the systems (14) and (15). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

IV. Numerical Examples and Results

To test the applicability of the proposed method, we consider both linear and nonlinear problems which are available in the literature. For all the examples, we give the results for linear problems in brief depending on corresponding boundary conditions, but the nonlinear problem is formulated in details. All the computations are performed by ***MATLAB***.

Example 1: Consider the tenth order l**inear** differential equation [12, 15]

$$\frac{d^{10}u}{dx^{10}} - \frac{d^2u}{dx^2} = -8e^x, \ 0 \le x \le 1 \qquad (16a)$$

subject to the boundary conditions of type 1 in equation (2b):

$$u(0) = 1, \ u(1) = 0, \ u'(0) = 0, \ u'(1) = -e, \ u''(0) = -1, \ u''(1) = -2e, \ u'''(0) = -2,$$
$$u'''(1) = -3e, \ u^{(iv)}(0) = -3, \ u^{(iv)}(1) = -4e. \qquad (16b)$$

The analytic solution of the above problem is, $u(x) = (1-x)e^x$.

Solution

Employing the method illustrated in section III, we approximate $u(x)$ in a form

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} \beta_i B_i(x), n \ge 2 \qquad (17)$$

Here $\theta_0(x) = (1-x)$ is specified by the essential boundary conditions of equation (16b).

Now the parameters $\beta_i, (i = 1,2, \dots, n-1)$ satisfy the linear system

$$\sum_{i=1}^{n-1} E_{i.j} \beta_i = G_j, \ j = 1,2, \dots, n-1 \qquad (18a)$$

where

$$E_{i.j} = \int_0^1 \left[-\frac{d^9}{dx^9}\left[B_j(x)\right] + \frac{d}{dx}\left[B_j(x)\right]\right] \frac{d}{dx}\left[B_i(x)\right] dx - \left[\frac{d}{dx}\left[B_j(x)\right] \frac{d^8}{dx^8}\left[B_i(x)\right]\right]_{x=1}$$
$$+ \left[\frac{d}{dx}\left[B_j(x)\right] \frac{d^8}{dx^8}\left[B_i(x)\right]\right]_{x=0} + \left[\frac{d^2}{dx^2}\left[B_j(x)\right] \frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=1} - \left[\frac{d^2}{dx^2}\left[B_j(x)\right] \frac{d^7}{dx^7}\left[B_i(x)\right]\right]_{x=0}$$
$$- \left[\frac{d^3}{dx^3}\left[B_j(x)\right] \frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=1} + \left[\frac{d^3}{dx^3}\left[B_j(x)\right] \frac{d^6}{dx^6}\left[B_i(x)\right]\right]_{x=0} + \left[\frac{d^4}{dx^4}\left[B_j(x)\right] \frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=1}$$
$$- \left[\frac{d^4}{dx^4}\left[B_j(x)\right] \frac{d^5}{dx^5}\left[B_i(x)\right]\right]_{x=0} \qquad (18b)$$

$$G_j = \int_0^1 \left\{-8e^x B_j(x) + \left[\frac{d^9}{dx^9}[B_j(x)] - \frac{d}{dx}[B_j(x)]\right]\frac{d\theta_0}{dx}\right\}dx + \left[\frac{d^5}{dx^5}[B_j(x)]\right]_{x=1} \times (-4e)$$
$$-\left[\frac{d^5}{dx^5}[B_j(x)]\right]_{x=0} \times (-3) - \left[\frac{d^6}{dx^6}[B_j(x)]\right]_{x=1} \times (-3e) + \left[\frac{d^6}{dx^6}[B_j(x)]\right]_{x=0} \times (-2)$$
$$+\left[\frac{d^7}{dx^7}[B_j(x)]\right]_{x=1} \times (-2e) - \left[\frac{d^7}{dx^7}[B_j(x)]\right]_{x=0} \times (-1) - \left[\frac{d^8}{dx^8}[B_j(x)]\right]_{x=1} \times (-e), \quad (18c)$$

Solving the system (18a) we obtain the values of the parameters and then substituting these parameters into equation (17), we get the approximate solution of the BVP (16) for different values of n.
The maximum absolute errors obtained by our method for this problem are given in Table 1 to compare with the existing results. On the other hand the accuracy is found nearly the order 10^{-6} in [12] by Mohyud-Din and Yildirim and in [15] by Kasi and Raju respectively.

Table 1: Computed absolute error of example 1

x	Exact Solutions	16, Bezier polynomials	
		Approx. Solutions	**Absolute Error**
0.0	1.0000000000	1.0000000000	0.000000000
0.1	0.9946538263	0.9946538263	4.44×10^{-16}
0.2	0.9771222065	0.9771222065	7.33×10^{-16}
0.3	0.9449011653	0.9449011653	1.22×10^{-15}
0.4	0.8950948186	0.8950948186	1.51×10^{-15}
0.5	0.8243606354	0.8243606354	7.33×10^{-16}
0.6	0.7288475202	0.7288475202	4.55×10^{-15}
0.7	0.6041258122	0.6041258122	1.17×10^{-15}
0.8	0.4451081857	0.4451081857	1.02×10^{-16}
0.9	0.2459603111	0.2459603111	9.99×10^{-16}
1.0	1.0000000000	1.0000000000	0.000000000

In Figure 1, comparison of the approximate solution with the exact solution of example 1 for $n = 16$ is presented.

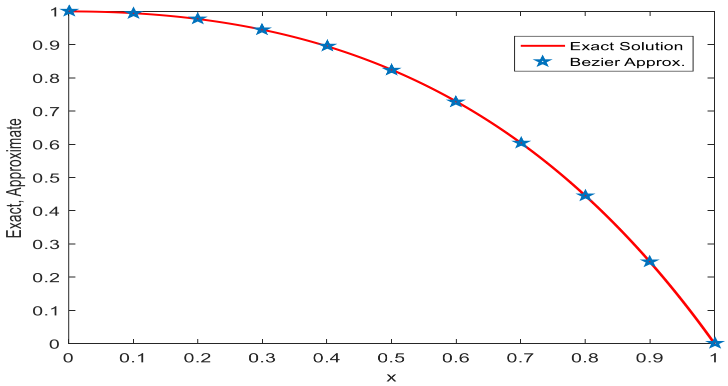

Figure 1: Exact solutions and Numerical solutions for Example 1

Example 2: Consider the tenth order l**inear** differential equation [1]

$$\frac{d^{10}u}{dx^{10}} - xu = -(89 + 21x + x^2 - x^3)\, e^x, \; -1 \le x \le 1 \quad (19a)$$

subject to the boundary conditions of type 2 in equation (2c):

$$u(-1) = u(1) = 0,\; u''(-1) = \frac{2}{e},\; u''(1) = -6e,\; u^{(iv)}(-1) = -\frac{4}{e},\; u^{(iv)}(1) = -20e,\; u^{(vi)}(-1) = -\frac{18}{e},$$
$$u^{(vi)}(1) = -42e,\; u^{(viii)}(-1) = -\frac{40}{e},\; u^{(viii)}(1) = -72e. \quad (19b)$$

The analytic solution of the above problem is, $u(x) = (1 - x^2)e^x$.
The equivalent BVP over [0, 1] to the BVP (19) is

$$\frac{1}{2^{10}}\frac{d^{10}u}{dx^{10}} = (2x-1)u - (89 + 21(2x-1) + (2x-1)^2 - (2x-1)^3)\, e^{(2x-1)}, 0 \le x \le 1 \quad (20a)$$

$u(0) = u(1) = 0,\ \frac{1}{4}u''(0) = \frac{2}{e},\ \frac{1}{4}u''(1) = -6e,\ \frac{1}{16}u^{(iv)}(0) = -\frac{4}{e},\ \frac{1}{16}u^{(iv)}(1) = -20e,\ \frac{1}{64}u^{(vi)}(0) = -\frac{18}{e},$
$\frac{1}{64}u^{(vi)}(1) = -42e,\ \frac{1}{256}u^{(viii)}(0) = -\frac{40}{e},\ \frac{1}{256}u^{(viii)}(1) = -72e.$ (20b)

The maximum absolute errors obtained by our method for this problem are given in Table 2. On the other hand the accuracy is found nearly the order 10^{-3} in [1] by Siddiqi and Twizell.

Table 2: Computed absolute error of example 2

x	Exact Solutions	15, Bezier polynomials	
		Approx. Solutions	Absolute Error
-1	0.0000000000	0.0000000000	0.000000000
-0.8	0.1617584271	0.1617584271	1.44×10^{-14}
-0.6	0.3512394471	0.3512394471	3.17×10^{-13}
-0.4	0.5630688387	0.5630688387	3.21×10^{-13}
-0.2	0.7859815230	0.7859815230	5.66×10^{-13}
0.0	1.0000000000	1.0000000000	4.55×10^{-12}
0.2	1.1725466478	1.1725466478	3.10×10^{-12}
0.4	1.2531327460	1.2531327460	4.88×10^{-13}
0.6	1.1661560323	1.1661560323	4.44×10^{-13}
0.8	0.8011947343	0.8011947343	5.96×10^{-14}
1.0	0.0000000000	0.0000000000	0.000000000

In Figure 2, comparison of the approximate solution with the exact solution of example 2 for $n = 15$ is presented.

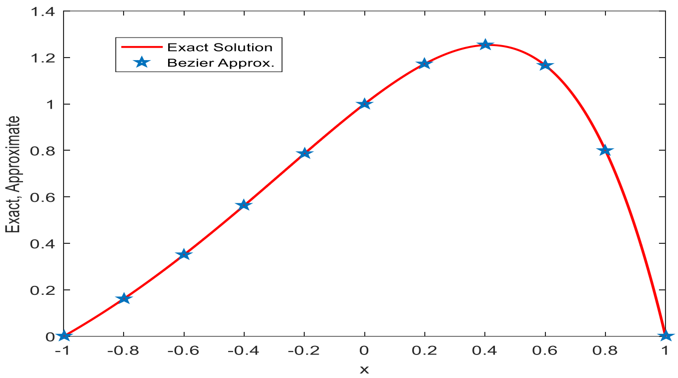

Figure 2: Exact solutions and Numerical solutions for Example 2

Example 3: Consider the twelfth order l**inear** differential equation [8, 11, 13]

$\frac{d^{12}u}{dx^{12}} - u = -12(2x\cos x + 11\sin x),\ -1 \le x \le 1$ (21a)

subject to the boundary conditions of type 1 in equation (2b):

$u(-1) = u(1) = 0,\ u'(-1) = u'(1) = 2\sin 1,\ u''(-1) = -u''(1) = -4\cos 1 - 2\sin 1,$
$u'''(-1) = u'''(1) = 6\cos 1 - 6\sin 1,\ u^{(iv)}(-1) = -u^{(iv)}(1) = 8\cos 1 + 12\sin 1,$
$u^{(v)}(-1) = u^{(v)}(1) = -20\cos 1 + 10\sin 1.$ (21b)

The analytic solution of the above problem is, $u(x) = (x^2 - 1)\sin x.$

The equivalent BVP over [0, 1] to the BVP (21) is

$\frac{1}{2^{12}}\frac{d^{12}u}{dx^{12}} - u = -12(2(2x-1)\cos(2x-1) + 11\sin(2x-1)),\ 0 \le x \le 1$ (22a)

$u(0) = u(1) = 0,\ \frac{1}{2}u'(0) = \frac{1}{2}u'(1) = 2\sin 1,\ \frac{1}{4}u''(0) = -\frac{1}{4}u''(1) = -4\cos 1 - 2\sin 1,\ \frac{1}{8}u'''(0) =$
$\frac{1}{8}u'''(1) = 6\cos 1 - 6\sin 1,\ \frac{1}{16}u^{(iv)}(0) = -\frac{1}{16}u^{(iv)}(1) = 8\cos 1 + 12\sin 1,$
$\frac{1}{32}u^{(v)}(0) = \frac{1}{32}u^{(v)}(1) = -20\cos 1 + 10\sin 1.$ (22b)

Solution

Employing the method illustrated in section III, we approximate $u(x)$ in a form

$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} \beta_i B_i(x),\ n \ge 2$ (23)

Here $\theta_0(x) = 0$ is specified by the essential boundary conditions of equation (22b).

Now the parameters $\beta_i, (i = 1,2, \dots, n-1)$ satisfy the linear system

$$\sum_{i=1}^{n-1} E_{i,j}\beta_i = G_j, \; j = 1,2, \dots, n-1 \tag{24a}$$

where

$$\begin{aligned} E_{i,j} &= \int_0^1 \left[-\frac{d^{11}}{dx^{11}}[B_j(x)]\frac{d}{dx}[B_i(x)] - 2^{12}B_i(x)B_j(x)\right]dx - \left[\frac{d}{dx}[B_j(x)]\frac{d^{10}}{dx^{10}}[B_i(x)]\right]_{x=1} \\ &+ \left[\frac{d}{dx}[B_j(x)]\frac{d^{10}}{dx^{10}}[B_i(x)]\right]_{x=0} + \left[\frac{d^2}{dx^2}[B_j(x)]\frac{d^9}{dx^9}[B_i(x)]\right]_{x=1} - \left[\frac{d^2}{dx^2}[B_j(x)]\frac{d^9}{dx^9}[B_i(x)]\right]_{x=0} \\ &- \left[\frac{d^3}{dx^3}[B_j(x)]\frac{d^8}{dx^8}[B_i(x)]\right]_{x=1} + \left[\frac{d^3}{dx^3}[B_j(x)]\frac{d^8}{dx^8}[B_i(x)]\right]_{x=0} + \left[\frac{d^4}{dx^4}[B_j(x)]\frac{d^7}{dx^7}[B_i(x)]\right]_{x=1} - \\ &\left[\frac{d^4}{dx^4}[B_j(x)]\frac{d^7}{dx^7}[B_i(x)]\right]_{x=0} - \left[\frac{d^5}{dx^5}[B_j(x)]\frac{d^6}{dx^6}[B_i(x)]\right]_{x=1} + \left[\frac{d^5}{dx^5}[B_j(x)]\frac{d^6}{dx^6}[B_i(x)]\right]_{x=0} \end{aligned} \tag{24b}$$

$$\begin{aligned} G_j &= \int_0^1 2^{12}[-12(2(2x-1)\cos(2x-1) + 11\sin(2x-1))]\,B_j(x)dx \\ &- \left[\frac{d^6}{dx^6}[B_j(x)]\right]_{x=1} \times 32(-20\cos 1 + 10\sin 1) + \left[\frac{d^6}{dx^6}[B_j(x)]\right]_{x=0} \times 32(-20\cos 1 + 10\sin 1) \\ &+ \left[\frac{d^7}{dx^7}[B_j(x)]\right]_{x=1} \times 16(-8\cos 1 - 12\sin 1) - \left[\frac{d^7}{dx^7}[B_j(x)]\right]_{x=0} \times 16(8\cos 1 + 12\sin 1) \\ &+ \left[\frac{d^8}{dx^8}[B_j(x)]\right]_{x=1} \times 8\,(6\cos 1 - 6\sin 1) - \left[\frac{d^8}{dx^8}[B_j(x)]\right]_{x=0} \times 8\,(6\cos 1 - 6\sin 1) \\ &+ \left[\frac{d^9}{dx^9}[B_j(x)]\right]_{x=1} \times 4\,(4\cos 1 + 2\sin 1) - \left[\frac{d^9}{dx^9}[B_j(x)]\right]_{x=0} \times 4\,(-4\cos 1 - 2\sin 1) \\ &- \left[\frac{d^{10}}{dx^{10}}[B_j(x)]\right]_{x=1} \times (4\sin 1) + \left[\frac{d^{10}}{dx^{10}}[B_j(x)]\right]_{x=0} \times (4\sin 1), \quad j = 1,2, \dots, n-1 \end{aligned} \tag{24c}$$

Solving the system (24a) we obtain the values of the parameters and then substituting these parameters into equation (23), we get the approximate solution of the BVP (22) for different values of n. If we replace x by $\frac{x+1}{2}$ in $\tilde{u}(x)$, then we get the desired approximate solution of the BVP (21).
In Table 3, we list the maximum absolute errors for this problem to compare with the existing methods. On the other hand the accuracy is found nearly the order 10^{-9} in [11] by Mirmoradi *et al* and in [13] by Kudri and Mulhem respectively.

Table 3: Computed absolute error of example 3

x	**Exact Solutions**	**16, Bezier polynomials**	
		Approx. Solutions	**Absolute Error**
-1	0.0000000000	0.0000000000	0.00000000
-0.8	0.2582481927	0.2582481927	3.78×10^{-16}
-0.6	0.3613711829	0.3613711829	1.07×10^{-16}
-0.4	0.3271114075	0.3271114075	1.33×10^{-15}
-0.2	0.1907225576	0.1907225576	4.25×10^{-15}
0.0	0.0000000000	0.0000000000	3.91×10^{-15}
0.2	-0.1907225576	-0.1907225576	2.76×10^{-15}
0.4	-0.3271114075	-0.3271114075	1.84×10^{-15}
0.6	-0.3613711830	-0.3613711830	1.14×10^{-15}
0.8	-0.2582481927	-0.2582481927	3.60×10^{-16}
1.0	0.0000000000	0.0000000000	0.00000000

In Figure 3, comparison of the approximate solution with the exact solution of example 3 for $n = 16$ is presented.

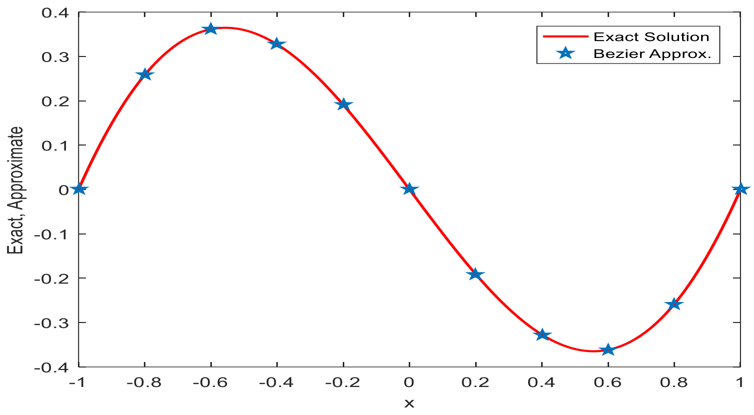

Figure 3: Exact solutions and Numerical solutions for Example 3

Example 4: Consider the tenth order **nonlinear** differential equation [15]

$$\frac{d^{10}u}{dx^{10}} - \frac{d^3u}{dx^3} = 2e^x u^2,\ 0 \le x \le 1 \tag{25a}$$

subject to the boundary conditions of type 1 in equation (2b):

$$u(0) = 1,\ u(1) = e^{-1},\ u'(0) = -1,\ u'(1) = -e^{-1},\ u''(0) = 1,\ u''(1) = e^{-1},\ u'''(0) = -1, u'''(1) = -e^{-1},$$
$$u^{(iv)}(0) = 1,\ u^{(iv)}(1) = e^{-1}. \tag{25b}$$

The analytic solution of the above problem is, $u(x) = e^{-x}$.

Solution

Applying the method illustrated in section III, we approximate $u(x)$ in a form

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} \beta_i B_i(x)\ , n \ge 2 \tag{26}$$

Here $\theta_0(x) = 1 - x\left(1 - \frac{1}{e}\right)$ is specified by the essential boundary conditions of equation (25b).

Now the parameters $\beta_i, (i = 1,2, \dots, n-1)$ satisfy the system

$$(E + F)D = G \tag{27a}$$

where the elements of D, E, F, G are $\beta_i, e_{i,j}, f_{i,j}$ and g_j respectively, given by

$$e_{i.j} = \int_0^1 \left[\left\{-\frac{d^9}{dx^9}[B_j(x)] - \frac{d^2}{dx^2}[B_j(x)]\right\}\frac{d}{dx}[B_i(x)] - 4\theta_0 e^x B_i(x)B_j(x)\right]dx - \left[\frac{d}{dx}[B_j(x)]\frac{d^8}{dx^8}[B_i(x)]\right]_{x=1} + \left[\frac{d}{dx}[B_j(x)]\frac{d^8}{dx^8}[B_i(x)]\right]_{x=0} + \left[\frac{d^2}{dx^2}[B_j(x)]\frac{d^7}{dx^7}[B_i(x)]\right]_{x=1} - \left[\frac{d^2}{dx^2}[B_j(x)]\frac{d^7}{dx^7}[B_i(x)]\right]_{x=0} - \left[\frac{d^3}{dx^3}[B_j(x)]\frac{d^6}{dx^6}[B_i(x)]\right]_{x=1} + \left[\frac{d^3}{dx^3}[B_j(x)]\frac{d^6}{dx^6}[B_i(x)]\right]_{x=0} + \left[\frac{d^4}{dx^4}[B_j(x)]\frac{d^5}{dx^5}[B_i(x)]\right]_{x=1} - \left[\frac{d^4}{dx^4}[B_j(x)]\frac{d^5}{dx^5}[B_i(x)]\right]_{x=0} \tag{27b}$$

$$f_{i.j} = -2\sum_{k=1}^{n-1}\beta_k \int_0^1 B_i(x)B_j(x)B_k(x)e^x dx \tag{27c}$$

$$g_j = \int_0^1 \left[\left\{\frac{d^9}{dx^9}[B_j(x)] + \frac{d^2}{dx^2}[B_j(x)]\right\}\frac{d\theta_0}{dx} + 2{\theta_0}^2 e^x B_j(x)\right]dx + \left[\frac{d}{dx}[B_j(x)]\frac{d^8\theta_0}{dx^8}\right]_{x=1} - \left[\frac{d}{dx}[B_j(x)]\frac{d^8\theta_0}{dx^8}\right]_{x=0} - \left[\frac{d^2}{dx^2}[B_j(x)]\frac{d^7\theta_0}{dx^7}\right]_{x=1} + \left[\frac{d^2}{dx^2}[B_j(x)]\frac{d^7\theta_0}{dx^7}\right]_{x=0} + \left[\frac{d^3}{dx^3}[B_j(x)]\frac{d^6\theta_0}{dx^6}\right]_{x=1} - \left[\frac{d^3}{dx^3}[B_j(x)]\frac{d^6\theta_0}{dx^6}\right]_{x=0} - \left[\frac{d^4}{dx^4}[B_j(x)]\frac{d^5\theta_0}{dx^5}\right]_{x=1} + \left[\frac{d^4}{dx^4}[B_j(x)]\frac{d^5\theta_0}{dx^5}\right]_{x=0} + \left[\frac{d^5}{dx^5}[B_j(x)]\right]_{x=1} \times (e^{-1}) - \left[\frac{d^5}{dx^5}[B_j(x)]\right]_{x=0} - \left[\frac{d^6}{dx^6}[B_j(x)]\right]_{x=1} \times (-e^{-1}) + \left[\frac{d^6}{dx^6}[B_j(x)]\right]_{x=0} \times (-1) + \left[\frac{d^7}{dx^7}[B_j(x)]\right]_{x=1} \times (e^{-1}) - \left[\frac{d^7}{dx^7}[B_j(x)]\right]_{x=0} - \left[\frac{d^8}{dx^8}[B_j(x)]\right]_{x=1} \times (-e^{-1}) + \left[\frac{d^8}{dx^8}[B_j(x)]\right]_{x=0} \times (-1) - \left[\frac{d}{dx}[B_j(x)]\right]_{x=1} \times (-e^{-1}) + \left[\frac{d}{dx}[B_j(x)]\right]_{x=0} \times (-1),\ \ j = 1,2, \dots, n-1 \tag{27d}$$

The initial values of these coefficients β_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (25a). That is, to find initial coefficients we solve the system

$$\mathrm{E}D = G \tag{28a}$$

whose matrices are constructed from

$$e_{i,j} = \int_0^1 \left[\left\{-\frac{d^9}{dx^9}[B_j(x)] - \frac{d^2}{dx^2}[B_j(x)]\right\}\frac{d}{dx}[B_i(x)]\right]dx - \left[\frac{d}{dx}[B_j(x)]\frac{d^8}{dx^8}[B_i(x)]\right]_{x=1} + \left[\frac{d}{dx}[B_j(x)]\frac{d^8}{dx^8}[B_i(x)]\right]_{x=0} + \left[\frac{d^2}{dx^2}[B_j(x)]\frac{d^7}{dx^7}[B_i(x)]\right]_{x=1} - \left[\frac{d^2}{dx^2}[B_j(x)]\frac{d^7}{dx^7}[B_i(x)]\right]_{x=0} - \left[\frac{d^3}{dx^3}[B_j(x)]\frac{d^6}{dx^6}[B_i(x)]\right]_{x=1} + \left[\frac{d^3}{dx^3}[B_j(x)]\frac{d^6}{dx^6}[B_i(x)]\right]_{x=0} + \left[\frac{d^4}{dx^4}[B_j(x)]\frac{d^5}{dx^5}[B_i(x)]\right]_{x=1} - \left[\frac{d^4}{dx^4}[B_j(x)]\frac{d^5}{dx^5}[B_i(x)]\right]_{x=0} \quad (28b)$$

$$g_j = \int_0^1 \left[\frac{d^9}{dx^9}[B_j(x)] + \frac{d^2}{dx^2}[B_j(x)]\right]\frac{d\theta_0}{dx}dx + \left[\frac{d}{dx}[B_j(x)]\frac{d^8\theta_0}{dx^8}\right]_{x=1} - \left[\frac{d}{dx}[B_j(x)]\frac{d^8\theta_0}{dx^8}\right]_{x=0} - \left[\frac{d^2}{dx^2}[B_j(x)]\frac{d^7\theta_0}{dx^7}\right]_{x=1} + \left[\frac{d^2}{dx^2}[B_j(x)]\frac{d^7\theta_0}{dx^7}\right]_{x=0} + \left[\frac{d^3}{dx^3}[B_j(x)]\frac{d^6\theta_0}{dx^6}\right]_{x=1} - \left[\frac{d^3}{dx^3}[B_j(x)]\frac{d^6\theta_0}{dx^6}\right]_{x=0} - \left[\frac{d^4}{dx^4}[B_j(x)]\frac{d^5\theta_0}{dx^5}\right]_{x=1} + \left[\frac{d^4}{dx^4}[B_j(x)]\frac{d^5\theta_0}{dx^5}\right]_{x=0} + \left[\frac{d^5}{dx^5}[B_j(x)]\right]_{x=1} \times (e^{-1}) - \left[\frac{d^5}{dx^5}[B_j(x)]\right]_{x=0} - \left[\frac{d^6}{dx^6}[B_j(x)]\right]_{x=1} \times (-e^{-1}) + \left[\frac{d^6}{dx^6}[B_j(x)]\right]_{x=0} \times (-1) + \left[\frac{d^7}{dx^7}[B_j(x)]\right]_{x=1} \times (e^{-1}) - \left[\frac{d^7}{dx^7}[B_j(x)]\right]_{x=0} - \left[\frac{d^8}{dx^8}[B_j(x)]\right]_{x=1} \times (-e^{-1}) + \left[\frac{d^8}{dx^8}[B_j(x)]\right]_{x=0} \times (-1) - \left[\frac{d}{dx}[B_j(x)]\right]_{x=1} \times (-e^{-1}) + \left[\frac{d}{dx}[B_j(x)]\right]_{x=0} \times (-1), \quad j = 1,2,\ldots,n-1 \quad (28c)$$

Once the initial values of the coefficients β_i are obtained from equation (28a), they are substituted into equation (27a) to obtain new estimates for the values of β_i .

This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into equation (26), we obtain an approximate solution of the BVP (25).

The maximum absolute errors for this problem are shown in Table 4 to compare with the existing methods. On the other hand, maximum absolute error has been found by Kasi and Raju [15] is 5.72×10^{-6}.

Table 4: Computed absolute error of example 4 using 6 iterations

x	Exact Solutions	16, Bezier polynomials	
		Approx. Solutions	Absolute Error
0.0	1.0000000000	1.0000000000	0.000000000
0.1	0.9048374181	0.9048374181	6.27×10^{-15}
0.2	0.8187307531	0.8187307531	3.62×10^{-15}
0.3	0.7408182207	0.7408182207	3.76×10^{-16}
0.4	0.6703200461	0.6703200461	9.69×10^{-16}
0.5	0.6065306597	0.6065306597	2.94×10^{-15}
0.6	0.5488116361	0.5488116361	1.28×10^{-16}
0.7	0.4965853038	0.4965853038	4.39×10^{-16}
0.8	0.4493289642	0.4493289642	4.43×10^{-15}
0.9	0.4065696597	0.4065696597	1.19×10^{-15}
1.0	0.3678794412	0.3678794412	0.000000000

In Figure 4, comparison of the approximate solution with the exact solution of example 4 for $n = 16$ is presented.

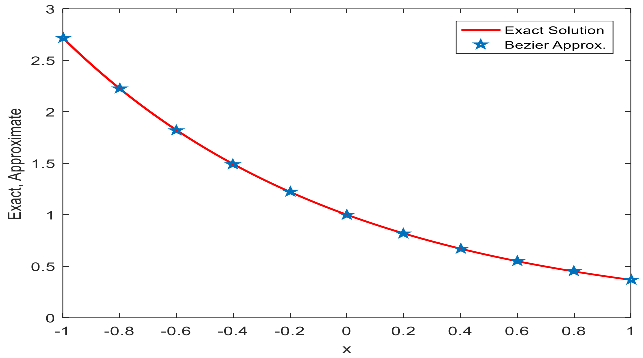

Figure 4: Exact solutions and Numerical solutions for Example 4

Example 5: Consider the twelfth order **nonlinear** differential equation [10, 14, 16]

$$\frac{d^{12}u}{dx^{12}} = \frac{1}{2}e^{-x}u^2, \ 0 \le x \le 1 \quad (29a)$$

subject to the boundary conditions of type 2 in equation (2c):

$$u(0) = 2, \ u(1) = 2e, \ u''(0) = 2, \ u''(1) = 2e, \ u^{(iv)}(0) = 2, \ u^{(iv)}(1) = 2e, \ u^{(vi)}(0) = 2, \ u^{(vi)}(1) = 2e, \ u^{(viii)}(0) = 2, \ u^{(viii)}(1) = 2e, \ u^{(x)}(0) = 2, \ u^{(x)}(1) = 2e \quad (29b)$$

The analytic solution of the above problem is, $u(x) = 2e^x$.

The maximum absolute errors for this problem are shown in Table 5 to compare with the existing methods. On the other hand, maximum absolute error has been found by Kasi and Showri Raju [14] and Noor and Mohy-ud-Din [16] are 2.62×10^{-5} and 6.61×10^{-4} respectively.

Table 5: Computed absolute error of example 5 using 5 iterations

x	Exact Solutions	15, Bezier polynomials	
		Approx. Solutions	**Absolute Error**
0.0	2.0000000000	2.0000000000	0.000000000
0.1	2.2103418361	2.2103418361	2.59×10^{-14}
0.2	2.4428055163	2.4428055163	7.34×10^{-14}
0.3	2.6997176152	2.6997176152	2.64×10^{-13}
0.4	2.9836493953	2.9836493953	4.99×10^{-13}
0.5	3.2974425414	3.2974425414	6.20×10^{-13}
0.6	3.6442376008	3.6442376008	5.08×10^{-13}
0.7	4.0275054149	4.0275054149	2.57×10^{-13}
0.8	4.4510818569	4.4510818569	1.74×10^{-14}
0.9	4.9192062223	4.9192062223	3.61×10^{-15}
1.0	5.4365636569	5.4365636569	0.000000000

In Figure 5, comparison of the approximate solution with the exact solution of example 5 for $n = 15$ is presented.

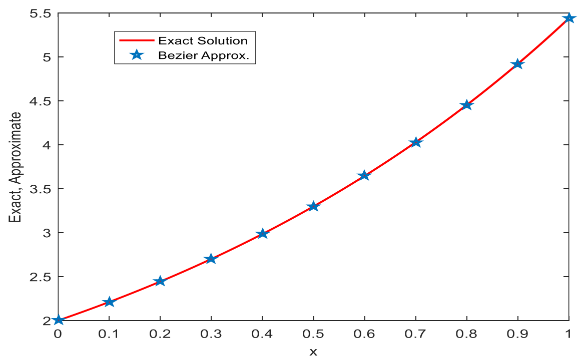

Figure 5: Exact solutions and Numerical solutions for Example 5

V. Conclusions

In this paper, we have solved numerically tenth and twelfth order linear and nonlinear boundary value problems using Galerkin method with Bezier polynomials as trial functions for two different types of boundary conditions. The nonlinear BVPs take long time in testing and calculating to get more accurate results. These methods enable us to approximate the solutions at every points of the domain of integration. Computational procedure and results of numerical examples considered shows that the method is simple, effective and straightforward, and hence make the method suitable for this class of problems. The algorithm can be coded easily and may be used for solving any higher order BVP. The method for the solution of similar problems that arises in engineering and physical sciences are under consideration.

References

[1] Siddiqi S.S.,Twizell E.H., Spline solutions of linear tenth-order boundary value problems, *Int. J. Comput. Math.,* **68**, 1998, 345-362.

[2] Siddiqi S. S. and Ghazala Akram, Solutions of tenth order boundary value problems using eleventh degree spline, *Appl. Math. Comput*, **185**, 2007, 115-127.

[3] Noor M.A., Eisa Al-said and Mohyud-Din S.T., A reliable algorithm for solving tenth order boundary value problems, *Applied Mathematics & Information Sciences,* **6(1)**, 2012, 103-107.

[4] Fazhan Geng, Xiuying Li, Variational iteration method for solving tenth order boundary value problems, *Mathematical Sciences*, **3(2),** 2009, 161-172.

[5] Inayat Ullah, Hamid Khan and Rahim M.T., Numerical solutions of higher order nonlinear boundary value problems by new iterative method, *Applied Mathematical Sciences*, **7(49)**, 2013, 2429-2439.

[6] Siddiqi S. S., Twizell E.H., Spline solutions of linear twelfth order boundary value problems, *J. Comp. Appl. Maths.,* **78**, 1997, 371-390.

[7] Siddiqi S. S., Twizell E.H., Spline solutions of linear tenth order boundary value problems, *Intern. J. Computer Math*., **68(3)**, 1998, 345-362.

[8] Siddiqi S. S. and Ghazala Akram, Solutions of twelfth order boundary value problems using thirteen degree spline, *Appl. Math. Compu.,* **182**, 2006, 1443-1453.

[9] Siddiqi S. S. and Ghazala Akram, Solutions of tenth order boundary value problems using eleventh degree spline, *Appl. Math. Comput*, **185**, 2007, 115-127.

[10] Mohyud-Din S.T., Noor M.A., and Noor K.I., Apprroximate solutions of twelfth order boundary value problems, *Journal of Applied Mathematics, Statistics and Informatics,* **4(2)**, 2008, 139-152.

[11] Mirmoradi H., Mazaheripour H., Ghanbarpour S., Barari A., Homotopy perturbation method for solving twelfth order boundary value problems, *Int. J. of Research and Reviews in Applied Sciences*, **1(2)**, 2009, 163-173.

[12] Mohyud-Din S.T. and Yildirim A., Solutions of tenth and ninth order boundary value problems by modified variational iteration method, *Applications and Applied Mathematics,* **5(1)**, 2010, 11-25.

[13] Ahmad Al-Kudri and Saleh Mulhem, Solution of twelfth order boundary value problems using adomain decomposition method, *Journal of Applied Sciences Research,* **7(6)**, 2011, 922-934.

[14] Kasi Viswanadham K.N.S., Showri Raju Y., Quintic B-spline collocation method for eighth order boundary value problems, *Advances in Computational Mathematics and its Applications,* **1(1)**, 2012, 47-52.

[15] Kasi Viswanadham K.N.S., Showri Raju Y., Quintic B-spline collocation method for tenth order boundary value problems, *International Journal of Computer Applications*, **51(15)**, 2012, 7-13.

[16] Noor M.A., Mohyud-Din S.T., Variational iteration method for solving twelve order boundary value problems using He's polynomials, *Computational Mathematics and Modeling,* **21(2)**, 2010, 239-251.