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Existence of Periodic Solutions of Rayleigh Equations with Singularties

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Abstract: The existence of positive periodic solutions is studied for the Rayleigh equation with a singularity of attractive type

$$x''(t)+f(x'(t))-\varphi(t)x(t)+\frac{\alpha(t)}{x^{\mu}}=h(t),$$

Where $f:R\to R$ is continuous function, which has a singularity at $x=0$, that is $\lim\limits_{x\to 0}\frac{\alpha(t)}{x^{\mu}}=\infty$. A new result on the existence of T-periodic positive solution to the above equation is obtained. The methods are based on Mawhin's continuation theorem of coincidence degree theory.

Keywords: Differential Equation; Singularity; Periodic Solution; Mawhin's Continuation Theorem, Coincidence Degree Theory.

1. Introduction

Second order differential equations are often used to model some dynamic problems in real world, where the problems are related to the first derivative with respect to the state variable, such as the resistance of air to a body in free falling movement, the resistance of moving charges in liquid, etc. Differential equation with a singularity is derived from physics, biology and other fields. For example, the famous Rayleigh-plesset equation[1-2]

$$u''(t)-\frac{4\mu}{u^{\frac{4}{5}}(t)}u'(t)+\varphi(t)u^{\frac{1}{5}}(t)+\frac{A}{u^{\frac{6k-1}{5}}(t)}-\frac{5S}{u^{\frac{1}{5}}(t)}=0$$

describes the oscillation of spherical bubbles affected by periodic sound fields in a liquid. The Rayleigh-plesset equation plays an important role in fluid dynamics. It can be derived by taking spherical coordinates in the Euler equation and assuming some physically acceptable simplifications. Many physical, biological, and medical models related to cavitations and luminescence depend on this equation.

The study of differential equations with singularities began with the publication of Nagumo [3], and the interest in this field increased with the paper [4] of Lazer and Solimini. In that paper, the periodic problem of equations as follows

$$u''(t)-\frac{1}{u^{\alpha}(t)}=h(t)\ (repulsive\ type) \tag{1.1}$$

and

$$u''(t)+\frac{1}{u^{\alpha}(t)}=h(t)\ (attractive\ type) \tag{1.2}$$

was investigated. Inspired by this, later, many scholars devoted themselves to the study of the problems of existence and stability of periodic solutions for singular equations [5-9]. Among these papers, most of them considered the problem of periodic solutions for differential equations with repulsive singularities. The methods are mainly based on the theory of coincidence degree established by Mawhin [10], which has become one of the key methods to study the existence of periodic solutions for differential equations of many types. For example, Lu and Kong studied the periodic solution problem for mean curvature equation[8]

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)'+f(u(t))u'(t)+g(u(t-\sigma))=e(t)$$

where $0\le\sigma\le T, g:(0,+\infty)\to R$ is a continuous function, and $g(x)$ has a singularity at $x=0$.

The behaviors of Rayleigh equation have been widely investigated due to their applications in many fields such as physics, mechanics and the engineering technique fields [11-19]. The authors in [14] studied the existence of periodic positive solutions for a Rayleigh equation with a singularity of attractive type

$$x''(t) - f(x'(t)) + g(t,x) = 0. \tag{1.3}$$

By using a continuation theorem of coincidence degree principle, they obtained the following result.

Theorem 1.1 Suppose $f(0) < 0$ and the following conditions are satisfied:

1. $\lim_{x\to 0^+} \inf_{t\in[0,T]} g(t,x) = +\infty$;

2.There is a constant $M > 0$ for any $(t,u) \in [0,T] \times (M,\infty)$, the relation $g(t,u) < -f(0)$ holds.

Then, Rayleigh equation (1.3) has at least one T-periodic positive solution.

Later, Guo and others made a further research[17], in which they studied the existence of periodic solutions of Rayleigh equation with a singularity of repulsive type

$$x''(t) + f(x'(t)) + \varphi(t)x(t) - \frac{1}{x^{\alpha}(t)} = p(t),$$

where $f: R \to R$ is continuous, $f(0) = 0$, and $\alpha \geq 1$ is a constant, $\varphi, p: R \to R$ are continuous with T −periodic with respect to variable t.

The aim of this paper is to study the periodic problem for the Rayleigh equation with a singularity of attractive type.

$$x''(t) + f(x'(t)) - \varphi(t)x(t) + \frac{\alpha(t)}{x^{\mu}(t)} = h(t), \tag{1.4}$$

where $f: R \to R$ is continuous, $f(0) = 0$, and φ, α, h are T-periodic continuous functions, $\max_{t\in[0,T]} h(t) > 0$, $\bar{\varphi} < 0, \alpha(t) > 0$ for $t \in [0,T], \mu > 0$ is constant. By means of a continuation theorem of coincidence degree theory, a new result on the existence of periodic solutions to equation (1.4) is obtained. The interesting is that the sign of $\varphi(t)$ is allowed to change. In such a situation, the priori upper bound of periodic solutions associated with equation (1.4) is more difficult to estimate than the case of $\varphi(t) \geq 0$ for $t \in [0,T]$. In order to overcome this difficulty, the priori upper bound of periodic solutions to equation (1.4) is obtained by using extreme value principle. This method is essentially different to the corresponding ones used by the known literature.

2.Preliminary lemmas

In this section, let $C_T := \{x \in C(\mathbb{R},\mathbb{R}), x(t+T) = x(t), \forall t \in \mathbb{R}\}$ with the norm $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$, $C_T^1 := \{x \in C^1(\mathbb{R},\mathbb{R}), x(t+T) = x(t), \forall t \in \mathbb{R}\}$ with the norm $\|x\|_{C_T^1} := max\{\|x\|_\infty, \|x'\|_\infty\}$. Clearly, C_T and C_T^1 are both Banach spaces. For any $y \in C_T$, let $y_+(t) := max\{y(t),0\}$, $y_-(t) := -max\{y(t),0\}, \bar{y} = \frac{1}{T}\int_0^T y(s)ds$, $y_M = \max_{t\in[0,T]} y(t)$, $y_m = \min_{t\in[0,T]} y(t)$, $\|y\|_p := (\int_0^T |y(s)|^p ds)^{\frac{1}{p}}, p \in [1,+\infty)$. Clearly, $y(t) = y_+(t) - y_-(t)$, $\bar{\varphi} = \bar{\varphi}_+ - \bar{\varphi}_-$ for all $t \in R$.

Lemma 2.1 Assume that there exist positive constants of N_0, N_1 and N_2 with $N_0 < N_1$ such that the following conditions hold.

[C1] For each $\lambda \in (0,1]$, each possible positive T-periodic solution x to the equation

$$u''(t) + \lambda f(u'(t)) - \lambda\varphi(t)u(t) + \frac{\lambda\alpha(t)}{u^{\mu}(t)} = \lambda h(t)$$

satisfies the inequalities $N_0 < u(t) < N_1$ and $|u'(t)| < N_2$ for all $t \in [0,T]$.

[C2] Each possible solution c to the equation

$$\frac{\bar{\alpha}}{c^{\mu}} - c\bar{\varphi} - \bar{h} = 0$$

satisfies the inequality $N_0 < c < N_1$.

[C3] The inequality

$$\left(\frac{\bar{\alpha}}{{N_0}^{\mu}} - N_0\bar{\varphi} - \bar{h}\right)\left(\frac{\bar{\alpha}}{{N_1}^{\mu}} - N_1\bar{\varphi} - \bar{h}\right) < 0$$

holds.

Then equations (1.4) has at least one positive T-periodic solution u such that $N_0 \leq u(t) \leq N_1$ and $|u'(t)| \leq N_2$ for all $t \in [0,T]$.

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Now, we list the following assumptions, which will be used in the study of T-periodic solution to equation (1.4).

[H1] There exist constants of $L > 0, \sigma > 0$, and $n \geq 1$ such that

$$\int_0^T f(x'(t))\,dt \leq L\int_0^T |x'(t)|dt, \qquad \forall x \in C_T^1.$$

and

$$yf(y) \geq \sigma y^{n+1}, \qquad \forall y \in R.$$

[H2] The function φ satisfies

$$\sigma\bar{\varphi} > \|\varphi\|_\infty(L + \bar{\varphi}_+ T), \text{ if } n > 1,$$

where n is determined by [H1].

Remark 1.1 If constants D_1, D_2 with $0 < D_1 < D_2$ such that

$$\frac{\bar{\alpha}}{x^\mu} - x\bar{\varphi} - \bar{h} > 0, \qquad x \in (0, D_1]$$

$$\frac{\bar{\alpha}}{x^\mu} - x\bar{\varphi} - \bar{h} < 0, \qquad x \in (D_2, \infty]$$

Now, we embed equation (1.4) into the following equations family with parameter $\lambda \in (0,1]$:

$$x''(t) + \lambda f(x'(t)) - \lambda\varphi(t)x(t) + \frac{\lambda\alpha(t)}{x^\mu(t)} = \lambda h(t), \qquad \lambda \in (0,1]. \tag{2.1}$$

Let

$\Omega = \{x \in C_T: x''(t) + \lambda f(x'(t)) - \lambda\varphi(t)x(t) + \frac{\lambda\alpha(t)}{x^\mu(t)} = \lambda h(t), \lambda \in (0,1]; x(t) > 0, \forall t \in [0,T]\}$, and

$$M_0 = max\{\frac{LT^{-\frac{1}{n+1}} + \bar{\varphi}_+ T^{\frac{n}{n+1}}}{\bar{\varphi}}B + \frac{\bar{h}}{\bar{\varphi}}\},$$

here B will be determined by (3.9).

3. Main results

Lemma 3.1 There exists a positive constant $\gamma > 0$, such that for each $x \in \Omega$, we have the following inequality

$$\min_{t\in[0,T]} x(t) \geq \gamma. \tag{3.1}$$

Proof. For each $x \in \Omega$, and let $t_0 \in [0,T]$, such that $\min_{t\in[0,T]} x(t) = x(t_0)$, then $x'(t_0) = 0, x''(t_0) \geq 0$. From (2.1), we have

$$f(x'(t_0)) - \varphi(t_0)x(t_0) + \frac{\alpha(t_0)}{x^\mu(t_0)} \leq h(t_0).$$

Assumption [H1] implies $f(0) = 0$, which together with the above inequality yields

$$-\varphi(t_0)x(t_0) + \frac{\alpha(t_0)}{x^\mu(t_0)} \leq h(t_0).$$

So the following inequality is true

$$\frac{\alpha_m}{x^\mu(t_0)} \leq \frac{\alpha(t_0)}{x^\mu(t_0)} \leq h(t_0) + \varphi(t_0)x(t_0),$$

that is,

$$\frac{\alpha_m}{x^\mu(t_0)} \leq h_M + \varphi_M x(t_0).$$

If $x(t_0) < 1$, we have $\frac{\alpha_m}{x^\mu(t_0)} \leq h_M + \varphi_M$, which gives that

$$x(t_0) > (\frac{\alpha_m}{h_M + \varphi_M})^{\frac{1}{\mu}}.$$

In conclusion, we know

$$x(t_0) \geq min\{1, (\frac{\alpha_m}{h_M + \varphi_M})^{\frac{1}{\mu}}\} := \gamma.$$

The proof is complete.

Lemma 3.2 If assumptions of [H1] and [H2] hold, then there is a constant $M_0 > 0$, such that for each function $x \in \Omega$, the following inequality holds.

$$\max_{t\in[0,T]} x(t) < M_0 \tag{3.2}$$

Proof. For each $x \in \Omega$, from (2.1) we get

$$x''(t) + \lambda f(x'(t)) - \lambda\varphi(t)x(t) + \frac{\lambda\alpha(t)}{x^{\mu}(t)} = \lambda h(t), \tag{3.3}$$

Integrating (3.3) over the interval $[0,T]$, we get

$$\int_0^T f(x'(t))\,dt - \int_0^T \varphi(t)x(t)\,dt + \int_0^T \frac{\alpha(t)}{x^{\mu}(t)}\,dt = \int_0^T h(t)dt,$$

that is,

$$\int_0^T \varphi_+(t)x(t)\,dt = \int_0^T f(x'(t))\,dt + \int_0^T \varphi_-(t)x(t)\,dt + \int_0^T \frac{\alpha(t)}{x^{\mu}(t)}\,dt - \int_0^T h(t)dt.$$

Since $\varphi_+(t) \geq 0$ and $\varphi_-(t) \geq 0$ for all $t \in [0,T]$, it follows from the integral mean value theorem and condition [H1] that there are three points $\xi,\zeta,\eta \in [0,T]$ such that

$$x(\xi)T\bar{\varphi}_+ \leq L\int_0^T |x'(t)|\,dt + x(\zeta)T\bar{\varphi}_- + \frac{T\bar{\alpha}}{x^{\mu}(\eta)} - \int_0^T h(t)dt$$
$$\leq L\int_0^T |x'(t)|\,dt + \|x\|_\infty T\bar{\varphi}_- + \frac{T\bar{\alpha}}{\gamma^{\mu}} + T\bar{h}_-,$$

i.e.,

$$x(\xi) \leq \frac{L}{T\bar{\varphi}_+}\int_0^T |x'(t)|\,dt + \frac{\bar{\varphi}_-}{\bar{\varphi}_+}\|x\|_\infty + \frac{\bar{\alpha}}{\bar{\varphi}_+\gamma^{\mu}} + \frac{\bar{h}_-}{\bar{\varphi}_+}. \tag{3.4}$$

Besides this, it is clear to see

$$\|x\|_\infty \leq x(\xi) + \int_0^T |x'(t)|\,dt \leq x(\xi) + T^{\frac{n}{n+1}}(\int_0^T |x'(t)|^{n+1}\,dt)^{\frac{1}{n+1}}. \tag{3.5}$$

Substituting (3.4) into (3.5), we have

$$\|x\|_\infty \leq \frac{L}{T\bar{\varphi}_+}\int_0^T |x'(t)|\,dt + \frac{\bar{\varphi}_-}{\bar{\varphi}_+}\|x\|_\infty + \frac{\bar{\alpha}}{\bar{\varphi}_+\gamma^{\mu}} + \frac{\bar{h}_-}{\bar{\varphi}_+} + T^{\frac{n}{n+1}}(\int_0^T |x'(t)|^{n+1}\,dt)^{\frac{1}{n+1}}$$
$$\leq \frac{LT^{-\frac{1}{n+1}}}{\bar{\varphi}_+}(\int_0^T |x'(t)|^{n+1}\,dt)^{\frac{1}{n+1}} + \frac{\bar{\varphi}_-}{\bar{\varphi}_+}\|x\|_\infty + \frac{\bar{\varphi}}{\bar{\varphi}_+\gamma^{\mu}} + \frac{\bar{h}_-}{\bar{\varphi}_+} + T^{\frac{n}{n+1}}(\int_0^T |x'(t)|^{n+1}\,dt)^{\frac{1}{n+1}}$$
$$\leq (\frac{LT^{-\frac{1}{n+1}}}{\bar{\varphi}_+} + T^{\frac{n}{n+1}})(\int_0^T |x'(t)|^{n+1}\,dt)^{\frac{1}{n+1}} + \frac{\bar{\varphi}_-}{\bar{\varphi}_+}\|x\|_\infty + \frac{\bar{\varphi}}{\bar{\varphi}_+\gamma^{\mu}} + \frac{\bar{h}_-}{\bar{\varphi}_+},$$

that is

$$\|x\|_\infty \leq (\frac{LT^{-\frac{1}{n+1}} + \bar{\varphi}_+T^{\frac{n}{n+1}}}{\bar{\varphi}})(\int_0^T |x'(t)|^{n+1}\,dt)^{\frac{1}{n+1}} + \frac{\bar{h}_-}{\bar{\varphi}} + \frac{1}{\gamma^{\mu}}. \tag{3.6}$$

On the other hand, multiplying both sides of (2.1) by $x'(t)$ and integrating it over the interval [0,T], we get

$$\int_0^T x'(t)\,f(x'(t))dt - \int_0^T x'(t)\varphi(t)x(t)\,dt + \int_0^T x'(t)\frac{\alpha(t)}{x^{\mu}(t)}\,dt = \int_0^T x'(t)h(t)\,dt$$

From assumption[H1], we have

$$\sigma\int_0^T |x'(t)|^{n+1}\,dt \leq \int_0^T x'(t)f(x'(t))\,dt$$
$$= \int_0^T x'(t)h(t)\,dt - \int_0^T x'(t)\frac{\alpha(t)}{x^{\mu}(t)}dt + \int_0^T x'(t)\varphi(t)x(t)\,dt \leq \|x\|_\infty\int_0^T |x'(t)||\varphi(t)|\,dt +$$
$$\int_0^T |x'(t)|\frac{\alpha(t)}{x^{\mu}(t)}dt + \int_0^T |x'(t)||h(t)|\,dt$$
$$\leq (\|x\|_\infty\|\varphi\|_\infty + \frac{\|\alpha\|_\infty}{\gamma^{\mu}} + \|h\|_\infty)\int_0^T |x'(t)|\,dt,$$

that is,

$$\int_0^T |x'(t)|^{n+1}\,dt \leq \sigma^{-1}(\|x\|_\infty\|\varphi\|_\infty + \frac{\|\alpha\|_\infty}{\gamma^{\mu}} + \|h\|_\infty)\int_0^T |x'(t)|\,dt.$$

We infer from the above formula that

$$(\int_0^T |x'(t)|^{n+1}\,dt)^{\frac{n}{n+1}} \leq \sigma^{-1}T^{\frac{1}{n+1}}(\|x\|_\infty\|\varphi\|_\infty + \frac{\|\alpha\|_\infty}{\gamma^{\mu}} + \|h\|_\infty). \tag{3.7}$$

Substituting (3.6) into (3.7), we have

$$\left(\int_0^T |x'(t)|^{n+1} dt\right)^{\frac{n}{n+1}} \le \|\varphi\|_\infty \left(\frac{L+\bar{\varphi}_+T}{\sigma\bar{\varphi}}\right)\left(\int_0^T |x'(t)|^{n+1} dt\right)^{\frac{1}{n+1}} + T^{\frac{1}{n+1}}\frac{\|\varphi\|_\infty}{\sigma\gamma^\mu} + \frac{T^{\frac{1}{n+1}}\bar{h}_-\|\varphi\|_\infty}{\sigma\bar{\varphi}} + \frac{T^{\frac{1}{n+1}}\|h\|_\infty}{\sigma} + T^{\frac{1}{n+1}}\frac{\|\alpha\|_\infty}{\sigma\gamma^\mu}. \tag{3.8}$$

According to this formula, we will prove (3.2) in two cases.

1. If $n > 1$, then we see from (3.8) that there exists $B_0 > 0$, such that

$$\left(\int_0^T |x'(t)|^{n+1} dt\right)^{\frac{1}{n+1}} \le B_0$$

2. If $n = 1$, we have

$$\left(\int_0^T |x'(t)|^2 dt\right)^{\frac{1}{2}} \le \|\varphi\|_\infty \frac{L+\bar{\varphi}_+T}{\sigma\bar{\varphi}}\left(\int_0^T |x'(t)|^2 dt\right)^{\frac{1}{2}} + T^{\frac{1}{2}}\frac{\|\varphi\|_\infty}{\sigma\gamma^\mu} + \frac{T^{\frac{1}{2}}\bar{h}_-\|\varphi\|_\infty}{\sigma\bar{\varphi}} + \frac{T^{\frac{1}{2}}\|h\|_\infty}{\sigma} + T^{\frac{1}{2}}\frac{\|\alpha\|_\infty}{\sigma\gamma^\mu}$$

that is,

$$\left(1 - \|\varphi\|_\infty \frac{L+\bar{\varphi}_+T}{\sigma\bar{\varphi}}\right)\left(\int_0^T |x'(t)|^2 dt\right)^{\frac{1}{2}} \le T^{\frac{1}{2}}\frac{\bar{h}\|\varphi\|_\infty}{\sigma\gamma^\mu} + \frac{T^{\frac{1}{2}}\|\varphi\|_\infty}{\sigma\bar{\varphi}} + \frac{T^{\frac{1}{2}}\|h\|_\infty}{\sigma} + T^{\frac{1}{2}}\frac{\|\alpha\|_\infty}{\sigma\gamma^\mu}$$

$$\left(\int_0^T |x'(t)|^2 dt\right)^{\frac{1}{2}} \le \frac{T^{\frac{1}{2}}\frac{\bar{h}\|\varphi\|_\infty}{\sigma\gamma^\mu} + \frac{T^{\frac{1}{2}}\|\varphi\|_\infty}{\sigma\bar{\varphi}} + \frac{T^{\frac{1}{2}}\|h\|_\infty}{\sigma} + T^{\frac{1}{2}}\frac{\|\alpha\|_\infty}{\sigma\gamma^\mu}}{1 - \|\varphi\|_\infty \frac{L+\bar{\varphi}_+T}{\sigma\bar{\varphi}}} := B_1.$$

Let $B = max\{B_0, B_1\}$, then in either case of $n > 1$or the case of $n = 1$, we always have

$$\left(\int_0^T |x'(t)|^{n+1} dt\right)^{\frac{1}{n+1}} \le B, \tag{3.9}$$

and so

$$\|x\|_\infty \le \frac{LT^{-\frac{1}{n+1}} + \bar{\varphi}_+T^{\frac{n}{n+1}}}{\bar{\varphi}}B + \frac{\bar{h}}{\bar{\varphi}} + \frac{1}{\gamma^\mu} := M_0. \tag{3.10}$$

Theorem 3.3 Assume that [H1], [H2] hold, then equation (1.4) has at least one positive T-periodic solution.
Proof. Firstly, we will show that $\max_{t\in[0,T]}|x'(t)|$ has a priori estimate, for each function $x \in \Omega$. In order to do it, multiplying both sides of equation (2.1) with $x''(t)$ and integrating it over the interval $[0, T]$, we get

$$\begin{aligned}\int_0^T |x''(t)|^2 dt &= \int_0^T |x''(t)||\varphi(t)|x(t)\,dt + \int_0^T |x''(t)|\frac{\alpha(t)}{x^\mu(t)}dt + \int_0^T |x''(t)|h(t)\,dt \\ &\le \|x\|_\infty \int_0^T |\varphi(t)||x''(t)|dt + \frac{1}{\gamma^\mu}\|\alpha\|_2\|x''\|_2 + \|x''\|_2\|h\|_2 \\ &\le \|x\|_\infty\|\varphi\|_2\|x''\|_2 + \|x''\|_2\left(\frac{1}{\gamma^\mu}\|\alpha\|_2 + \|h\|_2\right),\end{aligned}$$

which results in

$$\|x''\|_2 \le M_0\|\varphi\|_2 + \frac{1}{\gamma^\mu}\|\alpha\|_2 + \|h\|_2.$$

Since $x \in \Omega$, it is easy to see that there exists a $t_2 \in [0, T]$, such that $x'(t_2) = 0$, and so

$$\begin{aligned}|x'(t)| &= \left|x'(t_2) + \int_{t_2}^t x''(s)ds\right| \\ &\le \int_{t_2}^t x''(s)ds \le \int_0^T |x''(s)|ds \\ &\le T^{\frac{1}{2}}\|x''\|_2 < 1 + T^{\frac{1}{2}}\left[M_0\|\varphi\|_2 + \frac{1}{\gamma^\mu}\|\alpha\|_2 + \|h\|_2\right] := N_2.\end{aligned} \tag{3.11}$$

In addition, let $n_0 = \frac{1}{2}min\{D_1, \gamma\}$and $n_1 \in (M_0 + D_2, +\infty)$ be two constants, Then from (3.1), (3.10) and (3.11) , we see that each possible positive T-periodic solution x to (2.1) satisfies

$$n_0 < x(t) < n_1, \qquad |x'(t)| < N_2.$$

This implies that conditions of [C1] and [C2] in Lemma 2.1 hold, we can infer that

$$\frac{\bar{\alpha}}{c^{\mu}}-c\bar{\varphi}-\bar{h}>0, \quad c\in(0,n_0].$$

and

$$\frac{\bar{\alpha}}{c^{\mu}}-c\bar{\varphi}-\bar{h}<0, \quad c\in(n_1,+\infty].$$

which results in

$$(\frac{\bar{\alpha}}{c^{\mu}}-c\bar{\varphi}-\bar{h})(\frac{\bar{\alpha}}{c^{\mu}}-c\bar{\varphi}-\bar{h})<0.$$

Therefore, condition [C3] of Lemma 2.1 holds, too. Thus, by Lemma 2.1 we see that equation (1.4) has at least one positive T-periodic solution. The proof is complete.

Example 3.1 Consider the following Rayleigh equation

$$x''(t)+10x'(t)-\frac{(x'(t))^9}{1+(x'(t))^8}-a(1+2\sin t)x(t)+\frac{\sin t+2}{x^3(t)}=\cos t, \qquad (3.12)$$

where $a>0$ is a constant. Corresponding to (1.4), we have $f(x'(t))=10x'(t)-\frac{(x'(t))^9}{1+(x'(t))^8}, \varphi(t)=a(1+2\sin x), \alpha(t)=\sin t+2, \mu=3, h(t)=\cos t, T=2\pi$. Clearly, from (3.12) we see that $f(0)=0$, and $\bar{\varphi}=a>0$

$$\bar{\varphi}_+=\frac{1}{T}\int_0^T\varphi_+dt=\frac{\frac{2}{3}\pi+\sqrt{3}}{\pi}a, \bar{\varphi}_-=\frac{1}{T}\int_0^T\varphi_-dt=\frac{-\frac{1}{3}\pi+\sqrt{3}}{\pi}a$$

Secondly, integrating $f(x'(t))$ over the interval [0,T], we get

$$\begin{aligned}\left|\int_0^T f(x'(t))\,dt\right| &= \left|\int_0^T[10x'(t)-\frac{(x'(t))^9}{1+(x'(t))^8}]\,dt\right| \\ &= \left|-\int_0^T\frac{(x'(t))^9}{1+(x'(t))^8}dt\right| \\ &= \left|\int_0^T\frac{(x'(t))^9}{1+(x'(t))^8}dt\right| \\ &\le \int_0^T|x'(t)|\,dt\end{aligned}$$

which implies that we can chose $L=1$ such that the first condition of assumption [H1] holds, Beside, from

$$yf(y)=10y^2-\frac{y^{10}}{1+y^8}\ge 9y^2.$$

we see that the constant σ can be chosen as $\sigma=9$ such that the second condition of assumption [H1] is also satisfied. Last, since $L=1, \sigma=9, n=1$, we get

$$1-\frac{(L+T\bar{\varphi}_+)\|\varphi\|_\infty}{\sigma\bar{\varphi}}=\frac{2}{3}-\left(\frac{4}{9}\pi+\frac{2}{3}\sqrt{3}\right)a>0,$$

if

$$a<\frac{3}{2\pi+3\sqrt{3}},$$

This implies that [H2] holds. Thus, by Theorem 3.3, we have that equation (3.12) has at least one positive 2π −periodic solution.

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