

Existence of periodic solutions for second order delay differential equations with a singularity of repulsive type

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Abstract: In this paper, the problem of existence of periodic solution is studied for the second order delay differential equation with a singularity of repulsive type

$$x''(t) + f(x(t))x'(t) + \varphi(t)x(t - \tau_1) - g(x(t - \tau_2)) = h(t),$$

where τ_1 and τ_2 are constants, $g(x)$ is singular at $x = 0$, φ and h are T -periodic functions. By using a continuation theorem of coincidence degree theory, a new result on the existence of positive periodic solutions is obtained. The interesting is that the sign of function $\varphi(t)$ is allowed to change for $t \in [0, T]$.

Keywords: Liénard equation; Continuation theorem; Singularity; Periodic solution.

1. Introduction

The aim of this paper is to search for positive T -periodic solutions for second order delay differential equation with a singularity in the following form

$$x''(t) + f(x(t))x'(t) + \varphi(t)x(t - \tau_1) - g(x(t - \tau_2)) = h(t), \quad (1.1)$$

where τ_1 and τ_2 are constants, $f: [0, \infty) \rightarrow R$ is an arbitrary continuous function, $g \in C((0, +\infty), (0, +\infty))$ and $g(x)$ is singular of repulsive type at $x = 0$, i.e., $g(x) \rightarrow +\infty$, as $x \rightarrow 0^+$, $\varphi, h: R \rightarrow R$ are T -periodic with function $h \in L^1([0, T], R)$, $\varphi \in C([0, T], R)$, while the sign of function φ being changeable for $t \in [0, T]$.

In recent years, the problem of periodic solutions to the second order singular equation

$$x''(t) + f(x(t))x'(t) + \varphi(t)x(t - \tau_1) - \frac{b(t)}{x^{\lambda(t)}} = h(t), \quad (1.2)$$

where $f: [0, +\infty) \rightarrow R$ is an arbitrary continuous function, $\varphi, b, h \in L^1[0, T]$ and $\lambda > 0$, has been studied widely. This is due to the fact that the singular term possesses a significant role in many practical situations [1-11]. For example, the singular term in the equations models the restoring force caused by a compressed perfect gas (see [3-6] and the references therein). Lazer and Solimini in the pioneering paper[12] first used the method of topological degree theory, together with the technique of upper and lower solutions, to study the existence of periodic solution to Eq.(1.2) where $f(x) \equiv 0$, $\varphi(t) \equiv 0$, $b(t) \equiv 1$. They obtained that if $\lambda \geq 1$, a necessary and sufficient condition for existence of a positive periodic solution to Eq.(1.2) is that $\bar{h} := \frac{1}{T} \int_0^T h(s)ds < 0$. After that, the problem of periodic solutions for singular differential equations like Eq.(1.2) has attracted the attention of many researchers[13-19]. We notice that the condition of $\varphi(t) \geq 0$ for a.e. $t \in [0, T]$ is required in [16-19], since it is crucial for obtain the priori estimates over all the possible periodic solutions to the equations

$$x''(t) + \lambda f(x(t))x'(t) + \lambda \varphi(t)x(t - \tau_1) - \frac{\lambda b(t)}{x^{\lambda(t)}} = \lambda h(t), \lambda \in (0, 1). \quad (1.3)$$

We only find [20,21] where the sign of $\varphi(t)$ is allowed to change. In [20,21], a priori bounds of all the possible periodic solutions to Eq.(1.3) are estimated by using the inequality

$$\int_0^T \frac{u''(t)}{u^{\delta(t)}} dt \geq 0, \quad (1.4)$$

where $\delta > 0$ is an arbitrary constant, $u(t)$ is a positive T -periodic function with $u \in C^2([0, T], R)$.

Motivated by this, in this paper, we study the existence of positive T -periodic solution for the equation (1.1). Since there is a delay τ_1 in (1.1), generally, the inequality like (1.4) for

$$\delta = 1$$

$$\int_0^T \frac{u''(t)}{u(t - \tau_1)} dt \geq 0.$$

may not hold. This means that the work to estimate a priori bounds of all the possible periodic solutions to the equations

$$x''(t) + \lambda f(x(t))x'(t) + \lambda\varphi(t)x(t - \tau_1) - \lambda g(x(t - \tau_2)) = \lambda h(t), \lambda \in (0,1).$$

is more difficult than the corresponding ones associated to (1.3).

2. Preliminary lemmas

Throughout this paper, let $C_T = \{x \in C(R, R) : x(t + T) = x(t) \text{ for all } t \in R\}$ with the norm defined by $|x|_\infty = \max_{t \in [0, T]} |x(t)|$. For any T -periodic solution $y(t)$ with $y \in L^1([0, T], R)$, $y_+(t)$ and $y_-(t)$ is denoted by $\max\{y(t), 0\}$ and $-\min\{y(t), 0\}$ respectively, and $\bar{y} = \frac{1}{T} \int_0^T y(s) ds$. Clearly, $y(t) = y_+(t) - y_-(t)$ for all $t \in R$, and $\bar{y} = \bar{y}_+ - \bar{y}_-$.

The following Lemma is the consequence of Theorem 3.1 in [22].

Lemma 2.1. Assume that there exist positive constants M_0, M_1 and M_2 with $0 < M_0 < M_1$, such that the following conditions hold.

1. For each $\lambda \in (0, 1]$, each possible positive T -periodic solution x to the equation

$$u''(t) + \lambda f(u(t))u'(t) + \lambda\varphi(t)u(t - \tau_1) - \lambda g(u(t - \tau_2)) = \lambda h(t),$$

satisfies the inequalities $M_0 < x(t) < M_1$ and $|x'(t)| < M_2$ for all $t \in [0, T]$.

2. Each possible solution c to the equation

$$g(c) - c\bar{\varphi} + \bar{h} = 0,$$

satisfies the inequality $M_0 < c < M_1$.

3. It holds

$$(g(M_0) - \bar{\varphi}M_0 + \bar{h})(g(M_1) - \bar{\varphi}M_1 + \bar{h}) < 0,$$

Then Eq.(1.1) has at least one T -periodic solution u such that $M_0 < u(t) < M_1$ for all $t \in [0, T]$.

Lemma 2.2.^[19] Let x be a continuous T -periodic continuous differential function. Then, for any $\tau \in (0, T]$,

$$\left(\int_0^T |x(s)|^2 ds\right)^{\frac{1}{2}} \leq \frac{T}{\pi} \left(\int_0^T |x'(s)|^2 ds\right)^{\frac{1}{2}} + \sqrt{T}|x(\tau)|.$$

In order to study the existence of positive periodic solutions to Eq.(1.1), we list the following assumptions.

[H₁] The function $\varphi(t)$ satisfies the following conditions

$$\int_0^T \varphi_+(s) ds > 0, \sigma := \frac{\int_0^T \varphi_-(s) ds}{\int_0^T \varphi_+(s) ds} \in [0, 1) \text{ and } \sigma_1 := \frac{T^{\frac{1}{2}}}{1-\sigma} \left(\int_0^T \varphi_+(t) dt\right)^{\frac{1}{2}} \in (0, 1);$$

[H₂] there are constants $M > 0$ and $A > 0$ such that $g(x) \in (0, A)$ for all $x > M$;

[H₃] $\int_0^1 g(s) ds = +\infty$;

[H₄] $\lim_{x \rightarrow 0^+} g(x) = +\infty$.

Remark 2.1. It is noted that assumption [H₄] can not be deduced from assumption [H₃]. For example, let $g(x) = \frac{1}{x} |\sin \frac{1}{x}|$ for all $x \in (0, +\infty)$, then assumption [H₃] is satisfied. But, assumption [H₄] does not hold.

Remark 2.2. If assumptions [H₁]-[H₂] and [H₄] hold, then there are constants D_1 and D_2 with $0 < D_1 < D_2$ such that

$$g(x) - \bar{\varphi}x + \bar{h} > 0 \text{ for all } x \in (0, D_1)$$

and

$$g(x) - \bar{\varphi}x + \bar{h} < 0 \text{ for all } x \in (D_2, +\infty)$$

Now, we suppose that assumptions [H₁] and [H₂] hold, and embed Eq.(1.1) into the following equations family with a parameter $\lambda \in (0, 1)$

$$x''(t) + \lambda f(x(t))x'(t) + \lambda\varphi(t)x(t - \tau_1) - \lambda g(x(t - \tau_2)) = \lambda h(t), \lambda \in (0, 1]. \tag{2.1}$$

Let

$$\Omega = \{x \in C_T : x''(t) + \lambda f(x(t))x'(t) + \lambda\varphi(t)x(t - \tau_1) - \lambda g(x(t - \tau_2)) = \lambda h(t), \lambda \in (0, 1], x(t) > 0, \forall t \in [0, T]\},$$

and

$$M_0 = \frac{1}{1-\sigma} \left[\frac{T^{\frac{3}{2}}(\bar{h}_-)^{\frac{1}{2}}}{(1-\sigma_1)(1-\sigma)^{\frac{1}{2}}} + \left(\frac{A_0}{1-\sigma_1} \right)^{\frac{1}{2}} \right]^2 + \frac{\max\{A+\bar{h}, 0\}}{(1-\sigma)\bar{\varphi}_+}, \tag{2.2}$$

where

$$A_0 = T^{\frac{1}{2}}(\bar{\varphi}_+)^{\frac{1}{2}} \frac{A + |\bar{h}|}{(1-\sigma)\bar{\varphi}_+} + \left(\frac{T^{\frac{1}{2}}\bar{h}_- \max\{A + \bar{h}, 0\}}{(1-\sigma)\bar{\varphi}_+} \right)^{\frac{1}{2}},$$

A is a constant determined by assumption $[H_2]$. Clearly, M_0 and A_0 are all independent of $(\lambda, x) \in (0, 1] \times \Omega$, and there is a positive integer k_0 such that

$$k_0 M \geq M_0, \tag{2.3}$$

where M is a constant determined by assumption $[H_2]$.

Lemma 2.3. Assume that assumptions $[H_1]$ - $[H_2]$ hold, then there is an integer $k^* > k_0$ such that for each function $u \in \Omega$, there is a point $t_0 \in [0, T]$ satisfying

$$u(t_0) \leq k^* M.$$

Proof: If the conclusion does not hold, then for each $k > k_0$, there is a function $u_k \in \Omega$ satisfying

$$u_k(t) > kM \text{ for all } t \in [0, T]. \tag{2.4}$$

From the definition of Ω , we see

$$u_k'' + \lambda f(u_k)u_k' + \lambda \varphi(t)u_k(t - \tau_1) - \lambda g(u_k(t - \tau_2)) = \lambda h(t), \lambda \in (0, 1], \tag{2.5}$$

and by using assumption $[H_2]$,

$$0 < g(u_k(t)) < A, \text{ for all } t \in [0, T]. \tag{2.6}$$

By integrating (2.5) over the interval $[0, T]$, we have

$$\int_0^T \varphi(t)u_k(t - \tau_1)dt = \int_0^T g(u_k(t - \tau_2))dt + \int_0^T h(t)dt,$$

i.e.,

$$\int_0^T \varphi_+(t)u_k(t - \tau_1)dt = \int_0^T \varphi_-(t)u_k(t - \tau_1)dt + \int_0^T g(u_k(t - \tau_2))dt + \int_0^T h(t)dt,$$

Since $\varphi_+(t) \geq 0$ and $\varphi_-(t) \geq 0$ for all $t \in [0, T]$, it follows from the integral mean value theorem that there are two points $\xi, \eta \in R$ such that

$$\begin{aligned} u_k(\xi)T\bar{\varphi}_+ &= T\bar{\varphi}_-u_k(\eta) + \int_0^T g(u_k(t))dt + T\bar{h} \\ &\leq T\bar{\varphi}_-|u_k|_\infty + \int_0^T g(u_k(t))dt + T\bar{h}, \end{aligned}$$

which together with (2.6) yields

$$u_k(\xi)T\bar{\varphi}_+ < T\bar{\varphi}_-|u_k|_\infty + TA + T\bar{h},$$

i.e.,

$$u_k(\xi) < \sigma|u_k|_\infty + \frac{A+\bar{h}}{\bar{\varphi}_+}. \tag{2.7}$$

In view of the inequality

$$|u_k|_\infty \leq u_k(\xi) + T^{\frac{1}{2}} \left(\int_0^T |u_k'(s)|^2 ds \right)^{\frac{1}{2}},$$

it follows from (2.7) and the condition of $\sigma \in [0, 1)$, which is determined in assumption $[H_1]$, that

$$|u_k|_\infty \leq \frac{T^{\frac{1}{2}}}{1-\sigma} \left(\int_0^T |u_k'(s)|^2 ds \right)^{\frac{1}{2}} + \frac{\max\{A+\bar{h}, 0\}}{(1-\sigma)\bar{\varphi}_+}. \tag{2.8}$$

On the other hand, by multiplying (2.5) with $u_k(t)$, and integrating it over the interval $[0, T]$, we obtain

$$\int_0^T |u_k'(t)|^2 dt = -\lambda \int_0^T g(u_k(t - \tau_2))u_k(t)dt + \lambda \int_0^T \varphi(t)u_k(t - \tau_1)u_k(t)dt - \lambda \int_0^T h(t)u_k(t)dt,$$

which together with the fact of $g(x) > 0$ for all $x > 0$ gives

$$\begin{aligned} \int_0^T |u_k'(t)|^2 dt &< \lambda \int_0^T \varphi_+(t)u_k(t - \tau_1)u_k(t)dt + \lambda \int_0^T h_-(t)u_k(t)dt, \\ &\leq T\bar{\varphi}_+|u_k|_\infty^2 + T\bar{h}_-|u_k|_\infty, \end{aligned}$$

i.e.,

$$\left(\int_0^T |u_k'(t)|^2 dt\right)^{\frac{1}{2}} < (T\overline{\varphi_+})^{\frac{1}{2}}|u_k|_{\infty} + (T\overline{h_-})^{\frac{1}{2}}|u_k|_{\infty}^{\frac{1}{2}}. \tag{2.9}$$

Substituting (2.8) into the above formula,

$$\begin{aligned} \left(\int_0^T |u_k'(t)|^2 dt\right)^{\frac{1}{2}} &< (T\overline{\varphi_+})^{\frac{1}{2}} \left[\frac{T^{\frac{1}{2}}}{1-\sigma} \left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{2}} + \frac{\max\{A + \overline{h}, 0\}}{(1-\sigma)\overline{\varphi_+}} \right] + \\ &(T\overline{h_-})^{\frac{1}{2}} \left[\frac{T^{\frac{1}{2}}}{1-\sigma} \left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{2}} + \frac{\max\{A + \overline{h}, 0\}}{(1-\sigma)\overline{\varphi_+}} \right]^{\frac{1}{2}} \\ &= \sigma_1 \left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{2}} + \frac{T^{\frac{3}{4}}(\overline{h_-})^{\frac{1}{2}}}{(1-\sigma)^{\frac{1}{2}}} \left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{4}} + A_0, \end{aligned}$$

where $\sigma_1 = \frac{T^{\frac{1}{2}}}{(1-\sigma)} \left(\int_0^T \varphi_+(t) dt\right)^{\frac{1}{2}} \in (0,1)$, which is determined by assumption $[H_1]$, and

$$A_0 = (T\overline{\varphi_+})^{\frac{1}{2}} \frac{\max\{A + \overline{h}, 0\}}{(1-\sigma)\overline{\varphi_+}} + \left(\frac{T\overline{h} \max\{A + \overline{h}, 0\}}{(1-\sigma)\overline{\varphi_+}}\right)^{\frac{1}{2}},$$

and then

$$\left(\int_0^T |u_k'(t)|^2 dt\right)^{\frac{1}{2}} < \frac{T^{\frac{3}{4}}(\overline{h_-})^{\frac{1}{2}}}{(1-\sigma)(1-\sigma_1)^{\frac{1}{2}}} \left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{4}} + \frac{A_0}{1-\sigma_1},$$

which results in

$$\left(\int_0^T |u_k'(t)|^2 dt\right)^{\frac{1}{4}} < \frac{T^{\frac{3}{4}}(\overline{h_-})^{\frac{1}{2}}}{(1-\sigma)(1-\sigma_1)^{\frac{1}{2}}} + \left(\frac{A_0}{1-\sigma_1}\right)^{\frac{1}{2}}. \tag{2.10}$$

Substituting (2.10) into (2.8), we have

$$|u_k|_{\infty} < M_0,$$

where M_0 is determined by (2.2). This is

$$u_k(t) < M_0 \text{ for all } t \in [0, T]. \tag{2.11}$$

By the definition of k_0 , we see from (2.3) that (2.11) contradicts to (2.4). This contradiction implies that the conclusion of Lemma 2.3 is true.

3. Main results

Theorem 3.1. Assume that $[H_1]$ - $[H_4]$ hold. Then Eq.(1.1) has at least one positive T -periodic solution.

Proof. Firstly, we will show that there exist M_1, M_2 with $M_1 > k^*M$ and $M_2 > 0$ such that each positive T -periodic solution $u(t)$ of Eq.(2.1) satisfies the inequalities

$$u(t) < M_1, |u'(t)| < M_2, \text{ for all } t \in [0, T]. \tag{3.1}$$

In fact, if u is an arbitrary positive T -periodic solution of Eq.(2.1), then

$$u''(t) + \lambda f(u(t))u'(t) + \lambda \varphi(t)u(t - \tau_1) - \lambda g(u(t - \tau_2)) = \lambda h(t), \lambda \in (0,1]. \tag{3.2}$$

This implies $u \in \Omega$. So by using Lemma 2.2 that there is a point $t_0 \in [0, T]$ such that

$$u(t_0) \leq k^*M, \tag{3.3}$$

and then

$$|u_k|_{\infty} \leq u_k(\xi) + T^{\frac{1}{2}} \left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{2}} \tag{3.4}$$

Integrating (3.2) over the interval $[0, T]$, we have

$$\int_0^T \varphi(t)u(t - \tau_1) dt - \int_0^T g(u(t - \tau_2)) dt = \int_0^T h(t) dt. \tag{3.5}$$

By assumption $[H_4]$, we see from (3.5) that there is a point $t_1 \in [0, T]$ such that

$$u(t_1) \geq \gamma, \tag{3.6}$$

where $\gamma < k^*M$ is a positive constant, which is independent of $\lambda \in (0,1]$. Similar to the proof of (2.9), we have

$$\left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} < (T\overline{\varphi_+})^{\frac{1}{2}}|u|_{\infty} + (T\overline{h_-})^{\frac{1}{2}}|u|_{\infty}^{\frac{1}{2}}. \tag{3.7}$$

Substituting (3.4) into (3.7), we have

$$\begin{aligned} \left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} &\leq (T\overline{\varphi_+})^{\frac{1}{2}} \left[k^*M + T^{\frac{1}{2}} \left(\int_0^T |u'(s)|^2 ds\right)^{\frac{1}{2}} \right] + \\ &\quad (T\overline{h_-})^{\frac{1}{2}} \left[k^*M + T^{\frac{1}{2}} \left(\int_0^T |u'(s)|^2 ds\right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \\ &\leq (T\overline{\varphi_+})^{\frac{1}{2}} \left(\int_0^T |u'(s)|^2 ds\right)^{\frac{1}{2}} + T^{\frac{3}{4}}(\overline{h_-})^{\frac{1}{2}} \left(\int_0^T |u'(s)|^2 ds\right)^{\frac{1}{4}} + \\ &\quad k^*M(T\overline{\varphi_+})^{\frac{1}{2}} + (Tk^*M\overline{h_-})^{\frac{1}{2}}, \end{aligned}$$

which results in

$$\begin{aligned} &\left[1 - T(\overline{\varphi_+})^{\frac{1}{2}} \right] \left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} \\ &\leq T^{\frac{3}{4}}(\overline{h_-})^{\frac{1}{2}} \left(\int_0^T |u'(s)|^2 ds\right)^{\frac{1}{4}} + k^*M(T\overline{\varphi_+})^{\frac{1}{2}} + (Tk^*M\overline{h_-})^{\frac{1}{2}}. \end{aligned} \tag{3.8}$$

Since

$$(T\overline{\varphi_+})^{\frac{1}{2}} = T^{\frac{1}{2}} \left(\int_0^T \varphi_+(s) ds\right)^{\frac{1}{2}} < \frac{T^{\frac{1}{2}}}{1-\sigma} \left(\int_0^T \varphi_+(s) ds\right)^{\frac{1}{2}},$$

it follows from assumption $[H_2]$ that

$$1 - T(\overline{\varphi_+})^{\frac{1}{2}} > 0,$$

which together (3.8) yields that there is a constant $\rho > 0$, which is independent of $\lambda \in (0,1]$, such that

$$\left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} < \rho,$$

and then by (3.4), we have

$$u(t) < k^*M + T^{\frac{1}{2}}\rho =: M_1, \text{ for all } t \in [0, T]. \tag{3.9}$$

Now, if u attains its maximum over $[0, T]$ at $t_2 \in [0, T]$, then $u'(t_2) = 0$ and we deduce from (3.2) that

$$u'(t) = \lambda \int_{t_2}^t [-f(u)u' - \varphi(t)u(t - \tau_1) + g(u(t - \tau_2)) + h(t)] dt$$

For all $t \in [t_2, t_2 + T]$. Thus, if $F' = f$, then

$$\begin{aligned} |u'(t)| &\leq \lambda |F(u(t)) - F(u(t_2))| + \lambda \int_{t_2}^{t_2+T} |\varphi(s)|u(s - \tau_1) ds + \lambda \int_{t_2}^{t_2+T} g(u(t - \tau_2)) dt + \\ &\quad \lambda \int_{t_2}^{t_2+T} |h(s)| ds \\ &\leq 2\lambda \max_{0 \leq u \leq M_1} |F(u)| + \lambda \int_0^T g(u(s)) ds + \lambda T|\overline{\varphi}| |u|_{\infty} + \lambda |\overline{h}|. \end{aligned} \tag{3.10}$$

From (3.5), we see that

$$\begin{aligned} \int_0^T g(u(s)) ds &= \int_0^T \varphi(t)u(t - \tau_1) dt - T\overline{h} \\ &\leq T\overline{\varphi_+}|u|_{\infty} + T\overline{h_-}. \end{aligned}$$

It follows from (3.10) that

$$\begin{aligned} |u'(t)| &\leq 2\lambda \left(\max_{0 \leq u \leq M_1} |F(u)| + \lambda T|\overline{\varphi}| |u|_{\infty} + |\overline{h}| \right) \\ &< 2\lambda \left(\max_{0 \leq u \leq M_1} |F(u)| + M_1 T|\overline{\varphi}| + T|\overline{h}| \right) \\ &:= \lambda M_2, \text{ for all } t \in [0, T], \end{aligned} \tag{3.11}$$

and then

$$|u'(t)| < M_2, \text{ for all } t \in [0, T]. \tag{3.12}$$

(3.10) and (3.12) imply that (3.1) holds.

Below, we will show that then there exists a constant $\gamma_0 \in (0, \gamma)$, such that each positive T -periodic solution of Eq.(2.1) satisfies

$$u(t) > \gamma_0, \text{ for all } t \in [0, T]. \tag{3.13}$$

Suppose that $u(t)$ is an arbitrary positive T -periodic solution of Eq.(2.1), then

$$u''(t) + \lambda f(u)u' + \lambda \varphi(t)u(t - \tau_1) - \lambda g(u(t - \tau_2)) = \lambda h(t), \lambda \in (0, 1]. \tag{3.14}$$

Let t_1 be determined in (3.6). Multiply (3.14) by $u'(t)$ and integrating it over the interval $[t_1 + \tau_1, t + \tau_1]$ (or $[t + \tau_1, t_1 + \tau_1]$), we get

$$\begin{aligned} & \frac{|u'(t + \tau_1)|^2}{2} - \frac{|u'(t_1 + \tau_1)|^2}{2} + \lambda \int_{t_1 + \tau_1}^{t + \tau_1} f(u(s))(u'(s))^2 ds \\ &= \lambda \int_{t_1 + \tau_1}^{t + \tau_1} g(u(s - \tau_2))(u'(s)) ds - \lambda \int_{t_1 + \tau_1}^{t + \tau_1} \varphi(s)u(s - \tau_1)(u'(s)) ds + \lambda \int_{t_1 + \tau_1}^{t + \tau_1} h(t)u'(s) ds \\ &= \lambda \int_{t_1 + \tau_1}^{t + \tau_1} g(u(s - \tau_2))(u'(s - \tau_2)) ds + \lambda \int_{t_1 + \tau_1}^{t + \tau_1} g(u(s - \tau_2))[u'(s) - u'(s - \tau_2)] ds - \\ & \quad \lambda \int_{t_1 + \tau_1}^{t + \tau_1} \varphi(s)u(s - \tau_1)(u'(s)) ds + \lambda \int_{t_1 + \tau_1}^{t + \tau_1} h(t)u'(s) ds \\ &= \lambda \int_{t_1 + \tau_1}^{t + \tau_1} g(u(s))u'(s) ds + \lambda \int_{t_1 + \tau_1}^{t + \tau_1} g(u(s - \tau_2))[u'(s) - u'(s - \tau_2)] ds - \\ & \quad \lambda \int_{t_1 + \tau_1}^{t + \tau_1} \varphi(s)u(s - \tau_1)(u'(s)) ds + \lambda \int_{t_1 + \tau_1}^{t + \tau_1} h(t)u'(s) ds, \end{aligned}$$

which yields the estimate

$$\begin{aligned} & \lambda \left| \int_{u(t)}^{u(t_1)} g(s) ds \right| \leq \frac{|u'(t + \tau_1)|^2}{2} + \frac{|u'(t_1 + \tau_1)|^2}{2} + \lambda \int_0^T |f(u)|(u')^2 dt + \\ & \lambda \int_0^T |\varphi(t)uu'| dt + \lambda \int_0^T |h(t)u'| dt. \end{aligned}$$

From (3.10) and (3.11), we get

$$\lambda \left| \int_{u(t)}^{u(t_1)} g(s) ds \right| \leq \lambda M_2^2 + \lambda \max_{0 \leq u \leq M_1} |f(u)|TM_2^2 + \lambda M_1M_2T|\overline{\varphi}| + \lambda M_2T|\overline{h}|$$

which gives

$$\left| \int_{u(t)}^{u(t_1)} g(s) ds \right| \leq M_3, \text{ for all } t \in [t_1, t_1 + T] \tag{3.15}$$

with

$$M_3 = M_2^2 + \max_{0 \leq u \leq M_1} |f(u)|TM_2^2 + M_1M_2T|\overline{\varphi}| + M_2T|\overline{h}|.$$

From $[H_3]$ there exists $\gamma_0 \in (0, \gamma)$ such that

$$\int_{\eta}^{\gamma} g_2(u) du > M_3, \text{ for all } \eta \in (0, \gamma_0] \tag{3.16}$$

Therefore, if there is a $t^* \in [t_1, t_1 + T]$ such that $u(t^*) \leq \gamma_0$, then from (3.16) we get

$$\int_{u(t^*)}^{\gamma} g(s) ds > M_3,$$

and then

$$\int_{u(t^*)}^{u(t_1)} g(s) ds > \int_{u(t^*)}^{\gamma} g(s) ds > M_3,$$

which contradicts (3.15). This contradiction gives that $u(t) > \gamma_0$ for all $t \in [0, T]$. So (3.13) holds.

Let $m_0 \in \min\{D_1, \gamma_0\}$ and $m_1 \in (M_1 + D_2, +\infty)$ be two constants, then from (3.1) and (3.12), we see that each possible positive T -periodic solution u satisfies

$$m_0 < u(t) < m_1, |u'(t)| < M_2$$

This implies that condition 1 and condition 2 of Lemma 2.1 are satisfied. Also, we can deduce from Remark 2.2 that

$$g(c) - \overline{\varphi}c + \overline{h} > 0, \text{ for } c \in (0, m_0]$$

and

$$g(c) - \overline{\varphi}c + \overline{h} < 0, \text{ for } c \in [m_1, +\infty)$$

which results in

$$(g(m_0) - \overline{\varphi}m_0 + \overline{h})(g(M_1) - \overline{\varphi}M_1 + \overline{h}) < 0.$$

So condition 3 of Lemma 2.1 holds. By using Lemma 2.1, we see that Eq.(1.1) has at least one positive T -periodic solution. The proof is complete.

Example 3.1: Considering the following equation

$$x''(t) + f(x(t))x'(t) + a(1 + 2 \sin t)x(t - \tau_1) - \frac{1}{x^2(t-\tau_2)} = \cos t, \quad (3.17)$$

where f is an arbitrary continuous function, $\tau_1, \tau_2 \in [0, +\infty)$ and $a \in (0, +\infty)$ are constants. Corresponding to Eq.(1.1), we have $g(u) = \frac{1}{u^2}$, $\varphi(t) = a(1 + 2 \sin t)$ and $h(t) = \cos t$. By simple calculating, we can verify that assumptions $[H_2]$ - $[H_4]$ are all satisfied. Furthermore,

$$\int_0^T \varphi_+(t) dt = \left(\frac{4\pi}{3} + 2\sqrt{3}\right)a, \quad \int_0^T \varphi_-(t) dt = (2\sqrt{3} - \frac{2\pi}{3})a,$$

and then

$$\sigma = \frac{\int_0^T \varphi_-(s) ds}{\int_0^T \varphi_+(s) ds} = \frac{2\sqrt{3} - \frac{2\pi}{3}}{\frac{4\pi}{3} + 2\sqrt{3}} \in (0, 1)$$

and

$$\sigma_1 = \frac{T^{\frac{1}{2}}}{1 - \sigma} \left(\int_0^T \varphi_+(t) dt \right)^{\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \left(\frac{4\pi}{3} + 2\sqrt{3} \right)^{\frac{3}{2}}.$$

If

$$a < \frac{2\pi}{\left(\frac{4\pi}{3} + 2\sqrt{3}\right)^3},$$

then $\sigma_1 \in (0, 1)$, this implies that assumption $[H_1]$ holds. Thus, by using Theorem 3.1, we have that Eq.(3.17) has at least one positive 2π -periodic solution.

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5. References

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