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The primal-dual simplex algorithm base on the most obtuse angle principle

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Abstract We present a relaxation algorithm for solving linear programming (LP) problems under the framework of the primal-dual simplex algorithm. Each iteration, based on a heuristic representation of the optimal basis (the principle of the most obtuse Angle), the algorithm constructs and solves a sub-problem, whose objective function is the same as the original problem, but only contains partial constraints.The primal-dual simplex algorithm is used to solve the sub-problem. If the sub-problem has an optimal solution or the sub-problem has no feasible solution, the constraint is added according to the principle of the obtuse Angle and then the final solution of the old sub-problem is taken as the starting point. Our preliminary numerical experiments show that the proposed algorithm can effectively reduce the number of iterations compared with the traditional two-stage simplex algorithm. Iterations for sub-problems take up a significant proportion, which greatly reduces the CPU time required for each iteration. The new algorithm has potential advantages in solving large-scale problems. It's a very promising new algorithm.

Keywords: linear programming, primal-dual simplex algorithm, the most obtuse Angle principle, sub-problem

1. Introduction

Linear programming is an early and far-reaching branch of operational research. After more than 70 years of development, linear programming has been widely used in national economy, science and technology, management and engineering. It has produced enormous economic and social benefits.

In 1947, G.B.Dantzig[1,2]proposed the simplex algorithm for solving LP problems for the first time, which marked the establishment of this discipline. In 1979, Khachian[3], a young Soviet mathematician, proposed the first polynomial time complexity algorithm -- ellipsoid algorithm. It is proved that polynomial time algorithm exists in LP problem, but the actual test shows that its computational effect is far worse than simplex algorithm. Until 1984, Karmarkar[4] proposed another algorithm with polynomial time complexity for solving LP problems, Karmarkar inner point algorithm. Later experiments show that this algorithm not only has lower order polynomial time complexity than ellipsoid algorithm but also has very encouraging performance.

In 1990, professor Pan P.Q proposed the concept of principal element notation[5], and gave a heuristic representation of an optimal basis (the principle of the most obtuse Angle) ,which is used to further improve the computational efficiency of simplex algorithm.

In 1993, Konstantinos Paparrizos expounded a new simplex algorithm for solving LP problems, the external point simplex algorithm[7~9]. In these papers, a hybrid primal-dual simplex algorithm is formed by combining the exterior point simplex algorithm and the interior point algorithm.

In this paper, we will apply these good results to reduce the number of iterations and further improve the efficiency of simplex algorithm. First, we use the most obtuse angle principle to select the partial constraints in the LP problem to form the sub-problem. Then, we use the primal-dual algorithm to solve the problem. If the sub-problem has an optimal solution or no feasible solution, the constraint condition is added according to the obtuse Angle principle to update the sub-problem and then the final solution of the old sub-problem is taken as the starting point.

2. Construction of sub-problem

For LP problem, the fewer the decision variables, the fewer the constraint conditions, the easier the LP problem is to be solved.Therefore, this paper uses the most obtuse angle principle to form sub-problem to reduce the scale of LP problems[5,6].

For LP problems where the constraint condition is inequality:

$$\min z = c^T x \quad s.t.\begin{cases} Ax \ge b \\ x \ge 0 \end{cases} \tag{2.1}$$

where $A \in R^{k\times n}$ with $k<n$ and $b \in R^k$， $c, x \in R^n$。It is assumed that the cost vector c, the right-hand side b, and A's columns and rows are all nonzero. In addition, we stress that no assumption is made on the rank of A, except $1 \le rank(A) \le k$.

Partition matrix A by row: $A = \left(u_1^T, \cdots, u_k^T\right)^T$.

According to the most obtuse Angle principle , the active constraint is some constraints which form obtuse Angle with the objective function.Therefore,the primary dimension of LP problem is calculated by:

$$\alpha_i = u_i^T c (i = 1, \cdots, k) \tag{2.2}$$

$$\text{Set } J = \{i \mid \alpha_i \ge 0, i = 1, \cdots, n\};, \quad |J| = m \tag{2.3}$$

Set $J_s = \{1, \cdots, k\} \setminus J$， then, only the constraints in J are retained and the remaining constraints are omitted to obtain a small scale sub-problem.

$$\min z = c^T x \quad s.t.\begin{cases} A_J x \ge b_J \\ x \ge 0 \end{cases} \tag{2.4}$$

where $A_J \in R^{m\times n}$ and $b_J \in R^m$.

Since the simplex method of LP problem must be solved under the condition of equality, the relaxation variable is added in the constraint condition of LP problem (2.4), and the inequality is changed into the equation.

$$\min z = c^T x \quad s.t.\begin{cases} A_J x - x_s = b_J \\ x, x_s \ge 0 \end{cases} \tag{2.5}$$

where x_s is the relaxation variable.

In this process, due to the introduction of the most obtuse Angle algorithm, the constraint of the sub-problem is reduced (compared with the original LP problem), which reduces the scale of the problem.

3. Primal-dual simplex method

When the LP sub-problem (2.5) is neither primal feasible nor dually feasible, the sub-problem is usually standardized and the initial basis is constructed by adding artificial variables. Then, the two-stage method is adopted to solve the problem. However, this method leads to the expansion of LP problem (2.5).In this paper, the primal-dual simplex method is used to solve the sub-problem[10~14].

Let us develop a tableau version of the LP problem first. Assume the presence of a canonical matrix, which might as well be denoted again by $\begin{pmatrix} B & N & b \end{pmatrix}$, with the associated sets J_B and J_N known, as partitioned:

$$\begin{pmatrix} A & b \\ c^T & 0 \end{pmatrix} \to \begin{pmatrix} B & N & b \\ c_B^T & c_N^T & 0 \end{pmatrix} \to \begin{pmatrix} I & B^{-1}N & B^{-1}b \\ 0 & \bar{z}_N^T & z \end{pmatrix} \tag{3.1}$$

where I is a unit matrix, z is the objective value. From (2.6), a solution can be determined immediately:

$$\begin{cases} x_B = B^{-1}b \\ x_N = 0 \end{cases} \tag{3.2}$$

If $x_B \geq 0$,the basic solution is the basic feasible solution of the sub-problem. If, in addition, it holds that $\bar{z}_N^T \geq 0$, the dual feasibility is already obtained. If not ,the sub-problem can be solved by the primal simplex method.

When x_B is not all ≥ 0, LP problem does not satisfy the primal feasibility. If the test number of LP problem $\bar{z}_N^T \geq 0$, dual simplex method can be used to solve it.If not, the LP problem does not satisfy the dual feasibility.That is to say neither the primal simplex method nor the dual simplex method can deal with the problem.We can use primal-dual simplex method to deal with it.

The primal-dual simplex method first satisfies the dual feasibility or primal feasibility of the LP problem.

Take an iteration.We select a nonbasic column to enter the basis :

$$q = \arg\min\{z_{k_i} \mid i = 1, \cdots, m\} \tag{3.3}$$

If $z_{k_q} < 0$,the nonbasic column a_{k_q} will enter the basis.Then we will select a basic column leave the basis:

$$p = \arg\max\{a_{i,k_q} \mid i = 1, \cdots, m\} \tag{3.4}$$

If $a_{pk_q} > 0$, the basic variable x_{j_p} will leave the basis. Replace the $j_p{}^{th}$ column with the $k_q{}^{th}$ column to form a new basis matrix B, and get a new basis solution (3.2), then adjust J_B, J_N.

The above process describes an iteration.Repeat the process until the dual feasibility of the LP problem is satisfied.If LP problem also satisfies primal feasibility, LP problem gets optimal solution. Otherwise, the dual simplex method is used.

4. The primal-dual simplex algorithm base on the most obtuse angle principle

ALGORITHM:a tableau version. Given an initial canonical tableau(2.4) and associated sets J_B and J_N. Given the basic solution x_B:

1. Calculate the principal element of the LP problem $\alpha_i = u_i^T c (i = 1, \cdots, k)$, construct sub-problem (2.6).
2. If $x_B \geq 0$ and $\min(z_N^T) < 0$,the primal simplex method is used to solve (2.6),go to 5.
3. If $\min(z_N^T) < 0$ and $\min(x_B) < 0$, the row index p is determined by (3.4) and the base variable x_{k_q} is determined by (3.3). a_{pk_q} is the principle .Principal unitization and all other elements in the column of the pivot are eliminated to 0.Replace the $j_p{}^{th}$ column with the $k_q{}^{th}$ column to form a new basis matrix B, and get a new basis solution (3.2),then adjust J_B, J_N,go to 3.
4. If $z_N^T \geq 0$ and $\min(x_B) < 0$,the dual simplex method is used to solve the problem (2.6).
5. If $|J| = m = k$, $z_N^T \geq 0$ and $x_B \geq 0$, the LP problem has an optimal solution, stop; If $x_B \geq 0$, $\min(z_N^T) < 0$ and $|J| = m = k$,LP problem has unbounded solutions, stop. If $|J| = m = k$, $z_N^T \geq 0$ and $\min(x_B) < 0$, there is no solution to the LP problem, stop.
6. Otherwise, the optimal solution of the sub-problem is substituted into the constraint with subscript J_s. If all the constraints are satisfied, the optimal solution of the sub-problem is the optimal solution of the original problem, stop. If not, the most unsatisfied constraint condition is added to the sub-problem to form a new sub-problem. Adjust J and J_s, then go to step 4.

5. Data experiment

In order to understand the actual performance of the algorithm, we conducted a preliminary numerical experiment. In this section, we reported the results of the data experiment and made corresponding analysis on the results.

The algorithm proposed in this paper is programmed with MATLAB

Mpdsm primal-dual simplex algorithm base on the most obtuse angle principle.

Compared with the traditional simplex algorithm FORTRAN program in mathematics department of southeast university:

CLS two-stage simplex algorithm.

CLS compiles and runs on the WINDOWS XP operating system using the VISUAL FORTRAN9.0 environment. Mpdsm compile and run on the WINDOWS 7 flagship system using the MATLABR2014a environment.The computer processor used is Intel(R) Celeron(R) CPU G1610@2.60Ghz, with memory of 8.00Gb and machine precision of 64 bits.All procedures are original and dual feasible tolerances of $\delta = 10^{-6}$. The Harris pivot rule [9] is adopted and the absolute value of the pivot is not lower than $\delta = 10^{-6}$.

A total of 60 LP problems were randomly collected for the test, each with no more than 22 decision variables or inequality constraints. Table 1 shows the total numerical results of all 60 LP problems.

Table 1 numerical results of all 60 LP problems

Problem	m	n	m+n	Inter(CLS)	Inter(Mpdsm)	Problem	m	n	m+n	Inter(CLS)	Inter(Mpdsm)
1	10	3	13	20	1	31	2	5	7	6	2
2	4	4	8	10	4	32	2	5	7	5	2
3	3	4	7	7	4	33	2	5	7	6	2
4	3	4	7	9	5	34	6	4	10	13	4
5	3	4	7	8	3	35	6	4	10	15	6
6	7	2	9	14	7	36	6	4	10	13	2
7	12	4	16	24	5	37	5	4	9	12	7
8	15	22	37	59	9	38	6	6	12	14	4
9	8	16	24	24	5	39	3	5	8	9	3
10	8	16	24	32	8	40	4	6	10	9	2
11	8	6	14	22	9	41	3	8	11	9	3
12	6	8	14	19	4	42	4	4	8	10	2
13	11	17	28	42	11	43	3	3	6	8	2
14	11	14	25	32	5	44	8	3	11	16	3
15	3	2	5	7	3	45	6	9	15	19	4
16	5	2	7	13	4	46	6	6	12	14	4
17	5	5	10	15	6	47	8	7	15	19	4
18	5	4	9	13	7	48	5	3	8	10	5
19	3	2	5	6	3	49	5	4	9	10	8
20	3	2	5	7	3	50	4	7	11	9	1
21	6	6	12	14	4	51	4	4	8	12	5
22	7	8	15	14	7	52	6	9	15	19	4
23	3	5	8	8	5	53	8	7	15	22	7
24	4	5	9	9	5	54	7	6	13	3	5
25	7	4	11	16	5	55	6	6	12	15	6
26	4	4	8	8	1	56	4	5	9	8	0
27	4	4	8	11	4	57	6	4	10	12	4
28	2	6	8	5	2	58	6	6	12	14	4
29	4	4	8	9	5	59	7	8	15	20	5
30	3	3	6	8	2	60	9	6	15	18	5

It can be seen from the table that it takes 872 iterations to solve with CLS, while Mpdsm only needs 261 iterations. And the iteration ratio is 872/261 =3.3409.The advantage of Mpdsm algorithm is that it firstly constructs sub-problems through the principal dimension theory, thus reducing the size of the LP problem and the calculation amount of the problem. Then the sub-problem is solved by the primal-dual simplex method. The main idea of this method is that it can start from an infeasible solution of the sub-problem to construct the dual feasible solution of the sub-problem and even the optimal solution of the sub-problem. The advantage of this method is that the solution of sub-problems can be realized without adding artificial variables, which greatly reduces the scale of problem solving. Finally, the nature of the algorithm is simplex method, so the algorithm has convergence. These results are very encouraging, showing that the new algorithm has great potential advantages, and it is worth further investigating its computational efficiency in solving large-scale sparse problems.

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