ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 13, No. 3, 2018, *pp*.195-200

Primal-dual Simplex Algorithm for Linear Programming

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(Received March 03 2018, accepted August 19 2018)

Abstract. When the linear programming (LP) problem satisfies neither the primal feasibility nor the dual feasibility, the primal-dual simplex method can be used to solve the problem in order to avoid the defect of the increase of decision variables caused by the addition of artificial variables in the two-stage algorithm. In this paper, we use PPSM (primitive - primitive simplex method) and DDSM(dual - dual simplex method) to solve these problems. Compared with the simplex method of two-stages, a better result is obtained.

Keywords: Primal feasibility; Dual feasibility; PPSM; DDSM.

1. Introduction

Linear programming (LP) problem is an important branch of operational research. It studies Mathematical theories and methods of extremum problems under linear objective function and constraint [1].LP problem has been widely used in military operations, economic analysis, and operation management and engineering technology. Reasonable utilization of limited human, material and financial resources will provide a scientific basis for decision makers to make optimal decisions， which assists and guides people to conduct scientific management and planning.

In 1947, American mathematician G.B.Dantzig proposed the general mathematical model and classical algorithm of LP problem, simplex method [2, 3], which laid a foundation of the subject. The algorithm points out that the optimal solution of LP problem must be reached at a vertex (basic feasible solution) of the feasible domain if it exists, since the feasible domain of LP problem is convex set. Two methods can be used to construct the initial basic feasible solution, which are two-stage method and big M method [1]. That is, the initial base matrix can be constructed by adding artificial variables. But, they lead to the increase of decision variables of LP problem, the increase of scale and the increase of calculation quantity.

In 1972, V.Klee and G.J.Minty pointed out that simplex algorithm was not an algorithm of polynomial time in computational complexity. They illustrated by an example that simplex algorithm was exponential time in the worst case. In 1979, Khachian proposed ellipsoid algorithm [4], which proved that there is indeed polynomial time algorithm in LP problem, but the actual effect is not satisfactory. In 1984, Karmarkar proposed a new polynomial time algorithm, Karmarka interior point algorithm [5], which not only has lower order polynomial time complexity than ellipsoid algorithm, but also has encouraging practical performance. Subsequently, the internal point method is hot, and a number of good algorithms are produced. Many scholars believe that solving large-scale problems is better than simplex algorithm. However, it cannot be used to solve the integer LP problem because it cannot be started hot, and the interior point algorithm cannot shake the dominance of simplex algorithm in practice.

In 1993, Konstantinos Paparrizos expounded a new simplex algorithm for solving linear programming problems, the external point simplex algorithm [6-8]. Literature [8] describes the modified form of the external point simplex algorithm, and combines the outer point simplex algorithm with the interior point algorithm to form a hybrid primal-dual simplex algorithm. The algorithm forms two paths: one converges from the inside or the boundary of the feasible region to the optimal solution of the original problem, and the other from the outside of the feasible region to the optimal solution of the original problem. The algorithm extends the original feasible solution to any solution of the original problem.

In 1998, PAN P.Q extends the concept of base, puts forward a new kind of simplex algorithm, a basis-deficiency-allowing variation of the simplex method [9]. It does not require that the basis be square matrix in the iterative process, as long as the definition of generalized basis can be satisfied as the basis in the iteration. Of course, the selection rules of the incoming, outgoing base variables and the construction methods of the initial basic feasible solutions have been changed. In 2003, Pan proposed the two-stage simplex algorithm for the deficient basis [10, 11], which marked the improvement of the deficient basis theory. That is, there is a perfect description from the formation of the initial feasible basis, the selection of the incoming, outgoing basis, the optimal termination criteria and so on.

In this paper, for a LP problem which is neither primal feasible nor dual feasible, if it can construct a basic feasible solution of LP problem or the basic feasible solution of dual problem, the primal simplex method or dual simplex method can be used to solve LP problem.

2. The Generalization of Basis

We consider the following LP problem [12-15]:

$$\min z = c^T x$$
$$s.t.\begin{cases} Ax = b \\ x \geq 0 \end{cases} \tag{1}$$

where $A \in R^{m\times n}$ with $m < n$ and $b \in R^m$, $c, x \in R^n$. It is assumed that the cost vector c, the right-hand side b, and A's columns and rows are all nonzero. In addition, we stress that no assumption is made on the rank of A, except $1 \leq rank(A) \leq m$.

Let B be the initial basis matrix, $B \in R^{m\times m}$. Let N is a non-basis matrix, containing the remaining columns. Define the ordered basic and nonbasic (index) sets respectively by

$$J_B = \{j_1, \cdots, j_m\}, J_N = \{k_1, \cdots, k_n\} \tag{2}$$

where $j_i, i = 1, \cdots, m$, is the index of the i^{th} column of B, and $k_j, j = 1, \cdots, n$, is the index of the j^{th} column of N. The subscript of a basic index j_i is called row index, and that of a nonbasic index k_j column index.

Let us develop a tableau version of the LP problem first. Assume the presence of a canonical matrix, which might as well be denoted again by $\begin{pmatrix} B & N & b \end{pmatrix}$, with the associated sets J_B and J_N known, as partitioned:

$$\begin{pmatrix} A & b \\ c^T & 0 \end{pmatrix} \to \begin{pmatrix} B & N & b \\ c_B^T & c_N^T & 0 \end{pmatrix} \to \begin{pmatrix} I & B^{-1}N & B^{-1}b \\ 0 & \bar{z}_N^T & z \end{pmatrix} \tag{3}$$

where I is a unit matrix, z is the objective value. From (3), a solution can be determined immediately:

$$\begin{cases} x_B = B^{-1}b \\ x_N = 0 \end{cases} \tag{4}$$

Assume that the current canonical tableau, say (3).

If $\bar{b} \geq 0$, the basic solution is the basic feasible solution. If, in addition, it holds that $\bar{z}_N^T \geq 0$, the dual feasibility is already obtained, and hence all is done. If not, the LP problem can be solved by the primal simplex method.

When $\bar{b}$ is not all ≥ 0, LP problem does not satisfy the primal feasibility. If the test number of LP problem $\bar{z}_N^T \geq 0$, dual simplex method can be used to solve it. If not, the LP problem does not satisfy the dual feasibility. That is to say neither the primal simplex method nor the dual simplex method can deal with the problem.

In general, two-stage method is used to solve these problems, but it complicates the mathematical model and increases the computer time. To solve LP problem, the following two methods can be adopted [13-20].

Suppose $\bar{a}_j$ is the j^{th} column of $(I_B, B^{-1}N)$, $\bar{a}_{ij}$ represents the elements of i^{th} row and j^{th} column of the coefficient matrix.

Case 1: If the primal feasibility of LP problem can be satisfied, the solution of LP problem can be realized.

In this case, the row index p may be determined such that:

$$p = \arg\min\{\bar{b}_i \mid i = 1, \cdots, m\} \tag{5}$$

So, if $\bar{b}_p < 0$, the basic variable x_{j_p} leave the basis.

We select a basic column to enter the basis, as below:

$$q = \arg\min\{a_{pk_i} \mid i = 1, \cdots n\} \tag{6}$$

$$\alpha = \frac{b_p}{a_{pk_q}} \tag{7}$$

Therefore, as the value of the nonbasic variable x_{k_q} increases from zero up to α .This leads to the following formula for updating the basic solution:

$$\bar{x}_{j_i} = x_{j_i} - \alpha B^{-1}\bar{a}_{k_q}, i = 1, \cdots, m; \bar{x}_{k_q} = \alpha \tag{8}$$

We annihilate the 1^{th} through $(p-1)^{th}$ and $(p+1)^{th}$ through m^{th} components of a_{k_q} by premultiplying $[B \quad N \quad b]$ by an appropriate Householder reflection. Then we bring the p^{th} basic column of the canonical tableau to the end of its nonbasic columns（corresponding to N）, the q^{th} nonbasic column of the canonical tableau to the p^{th} of its basic columns, and adjust J_B and J_N conformably.

The above procedure describes an iterative process. Repeat the above process until the primal feasibility of LP problem is satisfied. If the LP problem satisfies the dual feasibility, the LP problem gets the optimal solution; otherwise, the primal simplex method is adopted to solve it. So, we call this algorithm primal - primal simplex method (PPSM).

We put the preceding steps into the following model.

PPSM Algorithm: a tableau version. Given an initial canonical tableau (3) and associated sets J_B and J_N . Given the basic solution x_B

1. If $x_B \geq 0$， go to step 7.
2. else， if $\min(x_B) < 0$， Determine the row index p by rule (5).
3. Determine the column index q by rule (6).If there is x_{k_q} ,annihilate the 1^{th} through $(p-1)^{th}$ and $(p+1)^{th}$ through m^{th} components of a_{k_q} by premultiplying $[B \quad N \quad b]$ by an appropriate Householder reflection.
4. Bring the p^{th} basic column of the canonical tableau to the end of its nonbasic columns（corresponding to N）, the q^{th} nonbasic column of the canonical tableau to the p^{th} of its basic columns ,and adjust J_B and J_N conformably, go to step 1.
5. Annihilate $\bar{z}_{k_q}$ using Gaussian elimination, Compute the vector x_B , defined by (4)
6. If there is x_{k_q} , LP problem has no row solution, stop.
7. Call the primal simplex method to solve LP problem (3).

Theorem: if $\min(x_B) < 0$, for all the depart basic rows, the enter base variables can't be found, then the LP problem has no feasible solution.

Proof: if the $\min(x_B) < 0$, and for all the depart basic rows, the enter base variables cannot be found, that is, there is no feasible solution to LP problem, which can be proved.

Case 2: If the dual feasibility of LP problem can be satisfied, the solution of LP problem can be realized too. So, we select a nonbasic column to enter the basis, as below:

$$q = \arg\min\{\bar{z}_{k_i} \mid i = 1, \cdots, m\} \tag{9}$$

So, $z_{k_q} < 0$, and the nonbasic column x_{k_q} will enter the basis.

We select a basic column to depart the basis, as below:

$$p = \arg\max\{a_{i,k_q} \mid i = 1, \cdots, m\} \tag{10}$$

So, $a_{pk_q} > 0$ we annihilate the 1^{th} through $(p-1)^{th}$ and $(p+1)^{th}$ through m^{th} components of a_{k_q} by premultiplying $[B \quad N \quad b]$ by an appropriate Householder reflection. Then we bring the p^{th} basic column of the canonical tableau to the end of its nonbasic columns（corresponding to N）, the q^{th} nonbasic column of the canonical tableau to the p^{th} of its basic columns, and adjust J_B and J_N conformably.

The above process describes an iterative process, which aims to meet the dual feasibility of LP problems. Such steps are repeated until the dual feasibility of LP problem is satisfied or the departing variable can't be found. If the LP problem also satisfies the original feasibility, the LP problem can obtain the optimal solution; otherwise, the dual simplex method is adopted to solve it. So, we call this algorithm dual - dual simplex method (DDSM)

We put the preceding steps into the following model.

DDSM Algorithm: a tableau version. Given an initial canonical tableau (3) and associated sets J_B and J_N. Given the basic solution x_B

1. If $z_N^T \geq 0$, go to step 7.
2. Else, if $\min(z_N^T) < 0$, Determine the column index q by rule (9).
3. Determine the row index p by rule (10).If there is p, annihilate the 1^{th} through $(p-1)^{th}$ and $(p+1)^{th}$ through m^{th} components of a_{k_q} by premultiplying $[B \quad N \quad b]$ by an appropriate Householder reflection.
4. Bring the p^{th} basic column of the canonical tableau to the end of its nonbasic columns (corresponding to N) ,the q^{th} nonbasic column of the canonical tableau to the p^{th} of its basic columns ,and adjust J_B and J_N conformably, go to step 1.
5. Annihilate $\bar{z}_{k_q}$ using Gaussian elimination,Compute the vector x_B, defined by(4)
6. If there is p ,LP problem has no feasible solution, stop.
7. Call the dual simplex method to solve LP problem (2.3).

Theorem: if $\min(z_N^T) < 0$,for all the enter basic columns, the depart base variables can't be found, then the LP problem has no feasible solution.

Proof: if the $\min(z_N^T) < 0$, for all the enter basic columns, the depart base variables can't be found, that is, there is no feasible solution to dual problem. According to the weak duality theorem, LP problem has no feasible solution.

3. Data experiment

In order to understand the actual performance of the algorithm, we conducted a preliminary numerical experiment. In this section, we reported the results of the data experiment and made corresponding analysis on the results.

The algorithm proposed in this paper is programmed with MATLAB

PPSM primal - primal simplex algorithm

DDSM dual - dual simplex algorithm

Compared with the traditional simplex algorithm FORTRAN program in mathematics department of southeast university:

CLS two-stage simplex algorithm.

CLS compiles and runs on the WINDOWS XP operating system using the VISUAL FORTRAN9.0 environment. PPSM and DDSM compile and run on the WINDOWS 7 flagship system using the MATLABR2014a environment. The computer processor used is Intel(R) Celeron(R) CPU G1610@2.60Ghz, with memory of 8.00 GB and machine precision of 64 bits. All procedures are original and dual feasible tolerances of $\delta = 10^{-6}$. The Harris pivot rule is adopted and the absolute value of the pivot is not lower than $\delta = 10^{-6}$.

A total of 50 LP problems were randomly collected for the test, each with no more than 22 decision variables or inequality constraints. The table below shows the total numerical results of all 50 LP problems.

Problem	m	n	m+n	Inter(CLS)	Inter(PPSM)	Inter(DDSM)	Problem	m	n	m+n	Inter(CLS)	Inter(PPSM)	Inter(DDSM)
1	10	3	13	20	7	1	26	4	4	8	8	7	4
2	4	4	8	10	5	5	27	4	4	8	11	5	5
3	3	4	7	7	5	4	28	2	6	8	5	2	2
4	3	4	7	9	4	4	29	4	4	8	9	5	5
5	3	4	7	8	6	3	30	3	3	6	8	4	4

Problem	m	n	m+n	Inter(CLS)	Inter(PPSM)	Inter(DDSM)	Problem	m	n	m+n	Inter(CLS)	Inter(PPSM)	Inter(DDSM)
6	7	2	9	14	7	9	31	2	5	7	6	3	2
7	12	4	16	24	13	12	32	2	5	7	5	3	2
8	15	22	37	59	54	15	33	2	5	7	6	2	2
9	8	16	24	24	20	5	34	6	4	10	13	7	7
10	8	16	24	32	14	14	35	6	4	10	15	9	6
11	8	6	14	22	10	9	36	6	4	10	13	9	8
12	6	8	14	19	18	12	37	5	4	9	12	6	7
13	11	17	28	42	27	11	38	6	6	12	14	11	6
14	11	14	25	32	23	5	39	3	5	8	9	5	4
15	3	2	5	7	4	4	40	4	6	10	9	7	4
16	5	2	7	13	4	3	41	3	8	11	9	6	3
17	5	5	10	15	7	10	42	4	4	8	10	6	5
18	5	4	9	13	9	10	43	3	3	6	8	5	5
19	3	2	5	6	3	3	44	8	3	11	16	8	6
20	3	2	5	7	4	3	45	6	9	15	19	10	5
21	6	6	12	14	4	4	46	6	6	12	14	5	9
22	7	8	15	14	11	8	47	8	7	15	19	15	8
23	3	5	8	8	4	8	48	5	3	8	10	4	7
24	4	5	9	9	4	2	49	5	4	9	10	6	7
25	7	4	11	16	9	10	50	4	7	11	9	1	1

It can be seen from the table that it takes 872 iterations to solve with CLS, while PPSM only needs 515 iterations and DDSM only needs 380 iterations .And the iteration ratio is 872/515 =1.69 and 872/380=2.29. The advantage of PPSM and DDSM algorithm lies in using the relaxation variable of LP problem to construct the initial basis of LP problem and obtain a basic solution. By satisfying the primal feasibility or dual feasibility, the basic feasible solution of LP problem or dual problem can be obtained. Then the optimal solution of LP problem is obtained by the primal simplex method or dual simplex method. The experimental results are very encouraging, showing that the new algorithm has great potential advantages, and it is worthy to further investigate its computational efficiency in solving large-scale sparse problems. The algorithm also has some shortcomings, especially for the degradation problem, the algorithm cannot solve the degradation problem.

Acknowledgement

This work was supported by National Nature Science Foundation of China (41805049); Graduate Research and Innovation Project of Jiangsu Province (KYLX_0850).

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