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Periodic solutions for singular Liénard equations with indefinite weight

Chentong Zhao, Shiping Lu

[1]*School of math and statistics, Nanjing University of Information Science & Technology, Nanjing 210044, China*

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Abstract In this paper, the problem of periodic solutions is studied for singular Liénard equations

$$\ddot{x}(t) + f\big(x(t)\big)\dot{x}(t) + \varphi(t)x^{\mu}(t) = h(t),$$

where $f: (0, +\infty) \to R$ is continuous and has a singularity at the origin, μ is a positive constant. By using a continuation theorem of coincidence degree theory, a new result on the existence of positive periodic solutions is obtained. The interesting thing is that the sign of weight $\varphi(t)$ is allowed to change for $t \in [0, T]$.

Keywords: Liénard equation, Continuation theorem, Periodic solution

1. Introduction

In this paper, we are concerned with the existence of positive T −periodic solutions for the equations

$$\ddot{x}(t) + f\big(x(t)\big)\dot{x}(t) + \varphi(t)x^{\mu}(t) = h(t)\,, \tag{1.1}$$

where $f \in C((0, +\infty), R)$, φ is T −Periodic function with φ in $L([0, T], R)$, μ is a positive constant. In this equation, the function $f(x)$ has a singularity at $x = 0$, i.e., $\lim_{x\to 0^+} f(x) = +\infty$. Besides this, the sign of $\varphi(t)$ is allowed to change. The equations of this type arise in modelling of important problems appearing in many physical contexts (see [1]-[5] and the references therein).

In the past years, under the conditions of $\varphi(t) \geq 0$ and $\alpha(t) \geq 0$ for a.e. $t \in [0, T]$, the problem of existence of periodic solutions to the equation without friction term

$$\ddot{x}(t) + \varphi(t)x(t) - \frac{\alpha(t)}{x^{\mu}} = h(t)$$

has been extensively studied by [6]-[10]. Beginning with the paper of Habets-Sanchez [11], many researchers in [12]-[15] have considered the classical Liénard equation with a singularity of repulsive type

$$\ddot{x}(t) + f\big(x(t)\big)\dot{x}(t) + \varphi(t)x(t) - \frac{\alpha(t)}{x^{\mu}} = h(t).$$

In these papers, apart from the function $\varphi(t)$ satisfies $\varphi(t) \geq 0$ for a.e. $t \in [0, T]$, $f(x)$ being continuous on $[0, +\infty)$ is needed. For the recent development of this area, we refer readers to the literature [16]-[19]. But up to our knowledge, few papers have considered the case where $f(x)$ has a singularity at $x = 0$, and the sign of $\varphi(t)$ is indefinite. The reason for this is that, in such situation, the equation may have no a priori estimates.

Throughout this paper, let $C_T = \{x \in C(R, R): x(t + T) = x(t) \text{ for all } t \in R\}$ with the norm defined by $|x|_\infty = \max_{x\in[0,T]} |x(t)|$, and $C_T^1 = \{x \in C^1(R, R): x(t + T) = x(t) \text{for all } t \in R\}$ with the norm defined by $\| x \|_{C_T} = \max\{ |x|_\infty, |\dot{x}|_\infty\}$. For any $T-$periodic solution $y(t)$ with $y \in L([0, T], R)$, $y_+(t)$ and $y_-(t)$ is denoted by $\max\{y(t), 0\}$ and $\min\{y(t), 0\}$, respectively, and $\bar{y} = \frac{1}{T}\int_0^T y(s)ds$. Clearly, $y(t) = y_+(t) - y_-(t)$ for all $t \in R$, and $\bar{y} = \overline{y_+} - \overline{y_-}$.

2. Preliminary lemmas

Lemma 2.1. [20] Assume that there exist positive constants m_0, m_1 and M^* with $0 < m_0 < m_1$, such that the following conditions hold.

1. For any $\lambda \in (0,1]$, each possible positive T −periodic solution μ to the equation

$$(2.1)\ddot{u}(t) + \lambda f\big(u(t)\big)\dot{u}(t) + \lambda\varphi(t)u^{\mu}(t) = \lambda h(t)$$

satisfies the inequalities $m_0 < u(t) < m_1$ and $|\dot{u}(t)| < M^*$, for all $t \in [0, T]$.

2. The inequality

$$(\bar{h}-\bar{\varphi}m_0^{\mu})(\bar{h}-\bar{\varphi}m_1^{\mu})<0$$

holds.

Then, equation (1.1) has at least one T −periodic solution u such that $m_0 < u(t) < m_1$ for all $t \in [0,T]$.

Lemma 2.2. Let $u: [0,\omega] \to R$ be an arbitrary absolutely continuous function with $u(0) = u(\omega)$. Then the inequality

$$(\max_{[0,T]} u(t) - \min_{[0,T]} u(t))^2 \leq \frac{\omega}{4}\int_4^{\omega} |\dot{u}(s)|^2 ds$$

holds.

Now, we embed equation (1.1) into the following equations family with a parameter $\lambda \in (0,1]$

$$\ddot{x}(t)+\lambda f(x(t))\dot{x}(t)+\lambda\varphi(t)x^{\mu}(t)=\lambda h(t), \lambda \in (0,1].$$

Let

$$D=\{x \in C_T^1: \ddot{x}(t)+\lambda f(x(t))\dot{x}(t)+\lambda\varphi(t)x^{\mu}(t)=\lambda h(t), \lambda \in (0,1]; x(t)>0, \forall t \in [0,T]\},$$

$$F(x)=\int_1^x f(s)ds, G(x)=F(x)+x^{\mu}T\overline{\varphi_-}, x \in (0,+\infty), \quad (2.2)$$

where $f(x)$ and μ are determined in (1.1).

Lemma 2.3. Assume $\bar{\varphi} > 0$, then for each $u \in D$, there are constants $\xi_1, \xi_2 \in [0,T]$ such that

$$u(\xi_1) \leq \left(\frac{\bar{h}}{\bar{\varphi}}\right)^{\frac{1}{\mu}} := \eta \quad (2.3)$$

and

$$u(\xi_2) \geq \left(\frac{\bar{h}}{\overline{|\varphi|}}\right)^{\frac{1}{\mu}} := \eta_0. \quad (2.4)$$

Proof. Let $u \in D$, then

$$\ddot{u}(t)+\lambda f(u(t))\dot{u}(t)+\lambda\varphi(t)u^{\mu}(t)=\lambda h(t),$$

which together with the fact of $u(t) > 0$ for all $t \in [0,T]$ gives

$$\frac{\ddot{u}(t)}{u^{\mu}(t)}+\frac{\lambda f(u(t))\dot{u}(t)}{u^{\mu}(t)}+\lambda\varphi(t)=\lambda h(t).$$

Integrating the above equality over the interval $[0,T]$, we obtain

$$\int_0^T \frac{\ddot{u}(t)}{u^{\mu}(t)}dt+\lambda\int_0^T \varphi(t)dt=\lambda\int_0^T \frac{h(t)}{u^{\mu}(t)}dt,$$

i. e.,

$$\int_0^T \frac{\ddot{u}(t)}{u^{\mu}(t)}dt+\lambda T\bar{\varphi}=\lambda\int_0^T \frac{h(t)}{u^{\mu}(t)}dt.$$

Since the inequality

$$\int_0^T \frac{\ddot{u}(t)}{u^{\mu}(t)}dt \geq 0$$

is easily obtained by a simple integration by parts, it follows from (2.1) that

$$T\bar{\varphi} \leq \int_0^T \frac{h(t)}{u^{\mu}(t)}dt=\frac{T\bar{h}}{u^{\mu}(\xi_1)}.$$

By using mean value theorem of integrals, we have that there exists a point $\eta \in [0,T]$ such that

$$T\bar{\varphi} \leq \frac{T\bar{h}}{u^{\mu}(\xi_1)},$$

i. e.,

$$u(\xi_1) \le \left(\frac{\bar{h}}{\bar{\varphi}}\right)^{\frac{1}{\mu}} := \eta.$$

So, inequality (2.3) holds.

Multiplying two sides of (1.1) with $u^\mu(t)$ and integrating it over the interval $[0,T]$, we obtain that

$$\int_0^T \varphi(t)u^\mu(t)dt = \int_0^T h(t)dt, \tag{2.5}$$

which together with

$$|\int_0^T \varphi(t)u^\mu(t)dt| = |\int_0^T h(t)dt| = T\bar{h},$$

yields

$$|\int_0^T \varphi(t)u^\mu(t)dt| \le \int_0^T |\varphi(t)|u^\mu(t)dt = u^\mu(\xi_2)T\overline{|\varphi|}.$$

Thus, there is a point $\eta \in [0,T]$ such that

$$u(\xi_2) \ge \left(\frac{\bar{h}}{\overline{|\varphi|}}\right)^{\frac{1}{\mu}} := \eta_0.$$

So, inequality (2.4) holds.

The proof is complete.

Lemma 2.4. Suppose that the following assumptions are satisfied.

$[H_1]$ $\lim_{x\to 0^+} F(x) = +\infty$,

$$[H_2] \quad \lim_{x\to+\infty}(F(x) + T\overline{\varphi_+}x^\mu) = -\infty,$$

where $F(x)$ is determined in (2.2), $\eta_0 = \left(\frac{\bar{h}}{\overline{|\varphi|}}\right)^{\frac{1}{\mu}}$ is defined by (2.4). Then there exists a constant $\gamma_0 > 0$, such that

$$\min_{t\in[0,T]} u(t) \ge \gamma_0, \text{ uniformly for } u \in D.$$

Proof. Let $u \in D$, then u satisfies

$$\ddot{u}(t) + \lambda f(u(t))\dot{u}(t) + \lambda\varphi(t)u^\mu(t) = \lambda h(t), \lambda \in (0,1],$$

since $u \in D$, it is easy to see that there exist points $t_1, t_2 \in R$ such that $0 < t_2 - t_1 < T$,

$$u(t_1) = \max_{t\in[0,T]} u(t),$$

and

$$u(t_2) = \min_{t\in[0,T]} u(t).$$

Assumptions of $\bar{h} > 0$ and $\varphi(t) \ge 0$ for a. e. $t \in [0,T]$ with $\bar{\varphi} > 0$ holds. This gives

$$\eta_0 \le u(t_1) < +\infty,$$

to which by using $[H_1]$, we have

$$F(u(t_1)) \le \sup_{\eta_0 \le s \le +\infty} F(s) < +\infty.$$

When the condition $\varphi(t) \ge 0$ for $a.e.\ t \in [0,T]$ with $\bar{\varphi} > 0$ that

$$F(u(t_2)) = F(u(t_1)) - \int_{t_1}^{t_2} \varphi(t)u^\mu(t)dt + \int_{t_1}^{t_2} h(t)dt$$

$$\le F(u(t_1)) + \int_0^T \varphi_-(s)u(s)ds + T\bar{h}$$

$$\le F(u(t_1)) + u^\mu(t_1)T\overline{\varphi_-} + T\bar{h},$$

we have

$$G(u) = F(u) + u^\mu T\bar{h}$$

and then

$$F(u(t_2)) \le G(u(t_1)) + T\bar{h} \le \sup_{[\eta_0,+\infty)} G(u).$$

If there exists a constant $\gamma_0 > 0$, combining the above equations, we can get

$$\min_{t\in[0,T]} u(t) = u(t_2) \geq \gamma_0.$$

The proof is complete.

Lemma 2.5. Assume $\overline{\varphi} > 0$ and h(t)≥0 for a.e. $t \in [0,T]$ with $\overline{h} > 0$. Then there exists a constant $\rho > 0$ with $\rho > \gamma_0$, such that

$$\max_{t\in[0,T]} u(t) \leq \rho, \text{ uniformly for } u \in D. \tag{2.6}$$

Proof. Since $u \in D$, it is easy to see that there exist points $t_1, t_2 \in R$ such that $0 < t_2 - t_1 < T$,

$$u(t_1) = \max_{t\in[0,T]} u(t)$$

and

$$u(t_2) = \min_{t\in[0,T]} u(t).$$

Assumptions of $\bar{h} > 0$ and $\varphi(t) \geq 0$ for a.e. $t \in [0,T]$ with $\bar{\varphi} > 0$ holds. When the condition $\varphi(t) \geq 0$ for $a.e.\ t \in [0,T]$ with $\bar{\varphi} > 0$ that

$$F(u(t_2)) - F(u(t_1)) + \int_{t_1}^{t_2} \varphi(t)u^{\mu}(t)dt = \int_{t_1}^{t_2} h(t)dt.$$

So, we get

$$F(u(t_1)) = F(u(t_2)) + \int_{t_1}^{t_2} \varphi(t)u^{\mu}(t)dt - \int_{t_1}^{t_2} h(t)dt$$

$$\geq F(u(t_2)) - \int_{t_1}^{t_2} \varphi_-(t)u^{\mu}(t)dt - \int_0^T h(t)dt$$

Which together with (2.5) yields,

$$F(u(t_1)) = F(u(t_2)) + \int_{t_1}^{t_2} \varphi(t)u^{\mu}(t)dt - \int_{t_1}^{t_2} h(t)dt$$

$$\geq F(u(t_2)) - \int_0^T \varphi_-(t)u^{\mu}(t)dt - \int_0^T \varphi(t)u^{\mu}(t)dt$$

$$= F(u(t_2)) - \int_0^T \varphi_+(t)u^{\mu}(t)dt$$

$$\geq F(u(t_2)) - u^{\mu}(t_1)T\overline{\varphi_+}$$

$$F(u(t_1)) + T\overline{\varphi_+}u^{\mu}(t_1) \geq F(u(t_2)) \geq \min_{t\in[\gamma_0,\eta]} F(x) > -\infty. \tag{2.7}$$

Using $[H_2]$ in Lemma 2.4 we get, exists $\rho > 0$, when $x \in [\rho, +\infty]$,

$$F(x(t)) + T\overline{\varphi_+}x^{\mu}(t) < \min_{t\in[\rho_0,\eta]} F(x(t)).$$

From (2.7) we get $u(t_1) < \rho$

i.e.

$$\max_{t\in[0,T]} u(t) < \rho,$$

for all $u \in D$ are satisfied.

The proof is complete.

3. Main results

Theorem 3.1. Ammuse $\bar{\varphi} > 0$, and $h(t) \geq 0$ for $a.e.\ t \in [0,T]$ with $\bar{h} > 0$, there exist a constant $M^* = 2\left(\max_{\gamma_0\leq\mu\leq M_1} |F(u)| + T\overline{h} + \| u \|_\infty T\overline{\varphi_-}\right)$, such that

$$|\dot{u}|_\infty \leq M^*. \tag{3.1}$$

Proof. If u attains its maximum over $[0,T]$ at $t_1 \in [0,T]$, then $\dot{u}(t_1) = 0$ and we deduce from (2.1) that

$$\dot{u}(t) = \lambda \int_{t_1}^{t} [-f(u(s))\dot{u}(s) - \varphi(s)u^{\mu}(s) + h(s)]ds,$$

for all $t \in [t_1, t_1 + T]$. Thus, if $\dot{F} = f$, then

$$|\dot{u}(t)| \leq \lambda\left|F\big(u(t)\big) - F\big(u(t_1)\big)\right| + \lambda \int_{t_1}^{t_1+T} h(t)dt - \lambda\varphi(t)u^{\mu}(t)$$

$$\leq 2\lambda(\max_{\gamma_{0\leq\mu\leq M_1}} |F(u)| + T\bar{h} + \| u \|_\infty \int_0^T \varphi_-(t)dt)$$

$$\leq 2\lambda(\max_{\gamma_{0\leq\mu\leq M_1}} |F(u)| + T\bar{h} + \| u \|_\infty T\overline{\varphi_-})$$

$$:= \lambda M^*,$$

and then

$$\max_{t\in[0,T]} |\dot{u}(t)| < M^*, \text{ uniformly for } t \in [0,T]. \quad (3.2)$$

Equation (3.2) implies that (3.1) holds.

Let $m_0 = \gamma_0$ and $m_1 = \rho$ be two constants, then we see each possible positive T −periodic solution u to equation satisfies

$$m_0 < u(t) < m_1, |\dot{u}(t)| < M^* \text{ for all } t \in [0,T].$$

This implies that condition 1 of Lemma (2.1) is satisfied. Also, we can deduce that

$$\bar{h} - \bar{\varphi}x^{\mu} > 0, \text{ for } x \in (0, m_0]$$

and

$$\bar{h} - \bar{\varphi}x^{\mu} < 0, \text{ for } x \in [m_1, +\infty).$$

Furthermore, we have

$$\left(\bar{h} - \bar{\varphi}m_0^{\mu}\right)\left(\bar{h} - \bar{\varphi}m_1^{\mu}\right) < 0.$$

Which gives that condition 2 of Lemma 2.1 holds. By using Lemma 2.1, we see that equation (1.1) has at least one T −periodic solution.

Example 3.1: Consider the following equation

$$\ddot{x}(t) - \frac{1}{x^2}\dot{x}(t) + a(1 + 2\sin 2t)\, x^{\mu}(t) = \cos 2t, \quad (3.3)$$

where $a, \mu \in (0, +\infty)$ are constants.

Corresponding to (1.1), we have $f(x) = -\frac{1}{x^2}, \varphi(t) = a(1 + 2\sin 2t), h(t) = \cos 2t$, and $T = \pi$. Clearly, $\bar{h} = 0$, and $h(t) \geq 0$ for all $t \in [0,T]$ with $\bar{\varphi} = a > 0$. Since $\eta = \left(\frac{\bar{h}}{\bar{\varphi}}\right)^{\frac{1}{\mu}} = 0$ and

$$F(x) = \int_1^x f(s)ds = \frac{1}{x} - 1, \quad (3.4)$$

we have

$$C_0 = \sup_{s\in[A_1,+\infty)} F(s) = F(1) = 0 < +\infty. \quad (3.5)$$

Obviously, (3.4) and (3.5) imply that assumptions of $[H_1]$ and $[H_2]$ hold. Thus, by using Theorem 3.1, we have that for each $\mu \in [0, +\infty)$, equation (3.3) has at least one positive π −periodic solution.

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