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On the fractional derivatives of some special functions

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Abstract. Recently, we realize increasing number of researchers that utilizing the modified Riemann-Liouville fractional differential operator, which is called the Jumarie's fractional differential operator. One of the important properties of this operator is the chain rule:

$$D_x^{\alpha} f[g(x)] = D_y^{\alpha} f(y)|_{y=g(x)} \left(g'(x)\right)^{\alpha} = D_y^{\alpha} f(y)|_{y=g(x)} D_x^{\alpha} g(x), \alpha \in (0,1).$$

This rule plays a substantial role to solve nonlinear fractional differential equations such as the fractional sub-equation method. In the present paper, we deal with the above chain rule in the sense of the Jumarie's differential operator. We shall show the chain rule for the modefied Riemann -Liouville that it is not valid for some types of functions. Three cases are illustrated. Also we present new fractional order derivatives of some functions in Caputo derivative sense.

Keywords: Fractional calculus; fractional differential operator; Riemann-Liouville fractional operators; Jumarie's fractional differential operator

1. Introduction

Fractional calculus has been utilized to model physical and engineering processes, which are considered to be best characterized by fractional differential equations. It is worth nothing that the regular mathematical models of integer-order derivatives, consisting nonlinear models, do not work sufficiently in wide cases. Recently, fractional calculus has function a very important role in different fields. The concept of the fractional calculus (that is, calculus of derivatives and integrals operators of any arbitrary real or complex order) was instilled over three centuries ago. For the first time, it was studied by Abel in 1823. He generalized tautochrone problem utilizing fractional calculus techniques. Posterior Liouville employed fractional calculus in potential theory. Nowadays the fractional calculus has exhausted the attention of many investigators in all areas of sciences (see [1-5]). Anywise, there are various of the fractional order derivatives definitions in the literatures such as the Riemann-Liouville [1, 2], Caputo [6], Weyl [1,2], Jumarie [7], Davison and Essex [8], and Coimbra [9]. All these fractional derivatives definitions have their advantages and disadvantages.

Next some preliminaries and notations regarding the fractional calculus are imposed.

Definition 1.1 [1] The fractional (arbitrary) order derivative of the function f of order $0 < \alpha \leq 1$ is defined by

$$D_a^{\alpha} f(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

In the sequel, we use the notation $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$.

Remark 1.1 From Definition 1.1 , we have

$$D^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \mu > -1; 0 < \alpha < 1$$

and

$$I^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \mu > -1; \alpha > 0.$$

The Leibniz rule is

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$$D_a^{\alpha}[f(t)g(t)]=\sum_{k=0}^{\infty}\binom{\alpha}{k}D_a^{\alpha-k}f(t)D_a^{k}g(t)=\sum_{k=0}^{\infty}\binom{\alpha}{k}D_a^{\alpha-k}g(t)D_a^{k}f(t),$$

where

$$\binom{\alpha}{k}=\frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1-k)}.$$

See[1].

Definition 1.2 The Caputo fractional derivative of order $\mu>0$ is defined, for a smooth function $f(t)$ by

$${}^{c}D^{\mu}f(t):=\frac{1}{\Gamma(n-\mu)}\int_0^t\frac{f^{(n)}(\zeta)}{(t-\zeta)^{\mu-n+1}}d\zeta,$$

where $n=[\mu]+1,$ (the notation $[\mu]$ stands for the largest integer not greater than μ).

The Jumarie's definition of fractional differential operator is the modified Riemann-Liouville fractional derivative,for a spot continuous function demands not to be differentiable; the fractional derivative of a constant is equal to zero and more substantially it strips singularity at the origin for all functions for which $f(0)=$ constant for example, the exponential functions and Mittag-Leffler functions. With the Riemann- Liouville fractional derivative, an arbitrary function requires not to be continuous at the origin and it admits not to be differentiable.

Recently, Guy Jumarie planned the fractional derivative in the limit form

$$f^{\alpha}(x)=\lim_{h\to 0}\frac{\Delta^{\alpha}(f(x)-f(0))}{h^{\alpha}},\quad \alpha\in(0,1),$$

where

$$\Delta^{\alpha}f(x)=\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(\alpha-k+1)}f(x+(\alpha-k)h).$$

Alternatively,

$$f^{\alpha}(x)=\frac{1}{\Gamma(1+\alpha)}\frac{d}{dx}\int_0^x(x-\zeta)^{-\alpha}(f(\zeta)-f(0))d\zeta,\quad \alpha\in(0,1)$$

and

$$f^{\alpha}(x)=(f^{(n)}(x))^{\alpha-n},\quad \alpha\in n,n+1],n\geq 1.$$

If $0<\alpha<1, a=0$ and $\gamma>0,$ then the definition of Jumarie derivative gives

$$D_x^{\alpha}x^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}x^{\gamma-\alpha}. \tag{1}$$

This modified Riemann-Liouville fractional derivative has the utility for the normal Riemann-Liouville and Caputo fractional derivatives: it is proposed for any continuous (nondifferentiable) functions and the fractional derivative of a constant is equal to zero. Indeed, if the function is not defined at the origin, the fractional derivative will not exist. One of its important property is the chain rule

$$D_x^{\alpha}f[g(x)]=D_y^{\alpha}f(y)|_{y=g(x)}\left(g'(x)\right)^{\alpha}=D_y^{\alpha}f(y)|_{y=g(x)}D_x^{\alpha}g(x), \alpha\in(0,1). \tag{2}$$

In recent work, we shall introduce a counter example to show equation (2) is not true for all functions. A fractional sub-equation method is imposed to solve the nonlinear fractional differential equations. Recently, this method is modified by using the chain rule (2) of Jumarie's derivative see [12-15].

2. Chain rule

Consider two functions $f(y)=y^{\alpha}$ and $g(x)=x^{\beta}$ such that $0<\alpha<1$ and $0<\beta<1$ Then (1) implies that

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$$D_x^{\alpha}(f(g(x))) = D_x^{\alpha}\left(x^{\alpha\beta}\right) = \frac{\Gamma(\alpha\beta+1)}{\Gamma(\alpha\beta-\alpha+1)} x^{\alpha\beta-\alpha}. \tag{3}$$

On the other hand, we have

$$D_y^{\alpha} f(y)|_{y=g(x)} \left(g'(x)\right)^{\alpha} = \Gamma(\alpha+1)(\beta x^{\beta-1})^{\alpha} \tag{4}$$

Since

$$D_y^{\alpha} y^{\alpha} = \Gamma(\alpha+1)$$

Is a constant. But (3) and (4) are different. They contain the same power of x but the constant factor is different. Finally we proceed to evaluate $D_y^{\alpha} f(y)|_{y=g(x)} D_x^{\alpha} g(x)$. A computation implies

$$D_y^{\alpha} f(y)|_{y=g(x)} D_x^{\alpha} g(x) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{5}$$

But (5) is different from (3). Therefore, we conclude that

$$D_x^{\alpha} f[g(x)] \neq D_y^{\alpha} f(y)|_{y=g(x)} \left(g'(x)\right)^{\alpha} \neq D_y^{\alpha} f(y)|_{y=g(x)} D_x^{\alpha} g(x), \alpha \in (0,1).$$

3. New Fractional Order Derivatives of Some Functions in Caputo Derivative Sense

In this section we obtain new fractional order derivative for two functions in the sense of Caputo derivative

Theorem 3.1 The Caputo derivative for the function $f(x) = \sec h^2 x$ is

$$D^{\alpha} f(x) = \sum_{k=0}^{\infty} \frac{(2k+1)!}{\Gamma(2k+3-\alpha)} d_k x^{2k+2-\alpha} \quad \text{for } |x| < \frac{\pi}{2},$$

where

$$d_k = (2k+3)(2k+2)c_{k+2} \text{ and } c_k = \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{(2k)!}.$$

More generally,

$$D_x^{\alpha} \sec h^2 x = \frac{1}{\Gamma(1-\alpha)} \sum_{n=-\infty}^{n=\infty} - \frac{x^{1-\alpha}}{(1-\alpha)\beta^3} F(3,1,2-\alpha, z) \text{for all } x \in \Re,$$

Where

$$\beta = i\left(\frac{1}{2}+n\right)\pi, z = \frac{x}{\beta}.$$

Proof. We know

$$\tanh x = \sum_{k=1}^{\infty} c_k x^{2k-1} \quad \text{for} \quad x < \frac{\pi}{2},$$

Where

$$c_k = \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{(2k)!}.$$

Since $\tanh' x = \sec h^2 x$, we obtain

$$\sec h^2 x = \sum_{k=1}^{\infty} c_k (2k-1) x^{2k-2},$$

And for, $f(x) = \sec h^2 x$,

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$$f'(x) = \sum_{k=2}^{\infty} c_k (2k-1)(2k-2) x^{2k-3} = \sum_{k=0}^{\infty} d_k x^{2k+1}$$

Where

$$d_k = (2k+3)(2k+2)c_{k+2}.$$

Then

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)\,dt,$$

$$= \sum_{k=0}^{\infty} \frac{(2k+1)!}{\Gamma(2k+3-\alpha)} d_k x^{2k+2-\alpha}.$$

This is true only when $|x| < \frac{\pi}{2}$. Now we try to find it for all $x \in \Re$.

Since

$$f(x) = -\sum_{n=-\infty}^{n=\infty} \frac{1}{\left(x - i\frac{\pi}{2} - n\pi i\right)^2}$$

Which is true for all $x \in \Re$.
Then

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \sum_{-\infty}^{\infty} \int_0^x (x-t)^{-\alpha} \frac{1}{\left(x - i\frac{\pi}{2} - n\pi i\right)^3}\,dt$$

After substituting $t = xs$ the integral becomes

$$\int_0^x \frac{(x-t)^{-\alpha}}{(t-\beta)^3}\,dt = x^{1-\alpha} \int_0^1 \frac{(1-s)^{-\alpha}}{(xs-\beta)^3}\,ds = -\frac{x^{1-\alpha}}{\beta^3} \int_0^1 \frac{(1-s)^{-\alpha}}{(1-zs)^3}\,ds,$$

Where

$$\beta = i\left(\frac{1}{2} + n\right)\pi,\ z = \frac{x}{\beta}.$$

Next, use the standard integral representation of the Gauss hypergeometric
Function

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 s^{b-1} \frac{(1-s)^{c-b-1}}{(1-zs)^a}\,ds$$

and use

$$F(a,b,c,z) = \Gamma(c)\mathrm{F}(a,b,c,z).$$

It follows that we should take $a = 3, b = 1, c - b - 1 = -\alpha,$ hence $c = 2 - \alpha$.
This gives

$$\int_0^x \frac{(x-t)^{-\alpha}}{(t-\beta)^3}\,dt = -\frac{x^{1-\alpha}}{\beta^3} \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(3,1,2-\alpha,z)$$

or

$$\int_0^x \frac{(x-t)^{-\alpha}}{(t-\beta)^3}\,dt = -\frac{x^{1-\alpha}}{\beta^3} \frac{\Gamma(1)\Gamma(1-\alpha)}{\Gamma(2-\alpha)} F(3,1,2-\alpha,z) = -\frac{x^{1-\alpha}}{(1-\alpha)\beta^3} F(3,1,2-\alpha,z).$$

So we have

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$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \sum_{n=-\infty}^{n=\infty} - \frac{x^{1-\alpha}}{(1-\alpha)\beta^3} F(3,1,2-\alpha,z).$$

Theorem 2. The Caputo derivative for the function $f(t) = (\sin t)^{\beta}$, $\beta > 0$ is

$$D^{\alpha} f(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\beta + 2k + 1)!}{\Gamma(1 - \alpha + \beta + 2k)} c_k(\beta) x^{-\alpha+\beta+2k} \quad ,$$

Where

$c_0(\beta) = 1, 2kc_k(\beta) = \beta \sum_{m=1}^{k} a_m c_{k-m}(\beta)$, and $t \cos t = \sum_{m=0}^{\infty} a_m t^{2m} = 1 - \frac{1}{3} t^2 - \frac{1}{45} t^4 + \ldots$.

Proof. We write

$$(\sin t)^{\beta} = t^{\beta} \left(\frac{\sin t}{t} \right)^{\beta} = t^{\beta} (1 + f(t))^{\beta}$$

where

$$f(t) = \frac{\sin t}{t} - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}.$$

The binomial series is

$$(1+z)^{\beta} = \sum_{k=0}^{\infty} \binom{\beta}{k} z^k .$$

By substituting $z = f(t)$ we get

$$(\sin t)^{\beta} = t^{\beta} \sum_{k=0}^{\infty} \binom{\beta}{k} f(t)^k .$$

We collect powers of t and get

$$(\sin t)^{\beta} = t^{\beta} \sum_{m=0}^{\infty} c_m(\beta) t^{2m}$$

and this is valid for $0 < t < \pi$. To compute c_m we note that

$$f(t) = -\frac{t^2}{6} + \frac{t^4}{120} - \frac{t^6}{5040} - \ldots$$

$$f(t)^2 = \frac{t^4}{36} - \frac{t^6}{360} + \ldots$$

$$f(t)^3 = -\frac{t^6}{216} - \ldots \ .$$

Then

$$c_0(\beta) = 1$$

$$c_1(\beta) = -\binom{\beta}{1} \frac{1}{6} = -\frac{1}{6} \beta$$

$$c_2(\beta) = \binom{\beta}{1} \frac{1}{120} + \binom{\beta}{2} \frac{1}{36} = \beta \frac{1}{120} + \frac{\beta(\beta-1)}{2} \frac{1}{36}$$

$$= -\frac{1}{180} \beta + \frac{1}{32} \beta^2 .$$

One can derive a recursion formula for $c_k(\beta)$. We have

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$$(\sin t)^{\beta} = \sum_{k=0}^{\infty} c_k(\beta) t^{\beta+2k} .$$

By differentiating,

$$\beta \cos t (\sin t)^{\beta-1} = \sum_{k=0}^{\infty} c_k(\beta)(\beta+2k) t^{\beta+2k-1}$$

Therefore,

$$\beta\, t \cos t \sum_{k=0}^{\infty} c_k(\beta) t^{\beta+2k} = \sum_{k=0}^{\infty} c_k(\beta)(\beta+2k) t^{\beta+2k-1} .$$

We know the expansion

$$t \cos t = \sum_{m=0}^{\infty} a_m t^{2m} = 1 - \frac{1}{3} t^2 - \frac{1}{45} t^4 + \ldots .$$

We obtain

$$(\beta+2k) c_k(\beta) = \beta \sum_{m=0}^{k} a_m c_{k-m}(\beta).$$

One can simplify to

$$c_k(\beta) = \beta \sum_{m=1}^{k} a_m c_{k-m}(\beta).$$

Now

$$f'(t) = \sum_{k=0}^{\infty} c_k(\beta)(\beta+2k) t^{\beta+2k-1}$$

and the caputo derivative of f is

$$\begin{aligned}
D^{\alpha} f(x) &= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)\,dt \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} \sum_{k=0}^{\infty} c_k(\beta)(\beta+2k) t^{\beta+2k-1}\,dt \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{\infty} c_k(\beta)(\beta+2k) \int_0^x (x-t)^{-\alpha} t^{\beta+2k-1}\,dt \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(\beta+2k+1)}{\Gamma(1-\alpha+\beta+2k)} c_k(\beta) x^{-\alpha+\beta+2k} .
\end{aligned}$$

4. Conclusions and future works

We constructed a counter example to show that the chain rule, in the sense of the modified Riemann-Liouville fractional differential operator, which is called the Jumarie's fractional differential operator, is not valid. Also we presented new fractional order derivatives of some functions in Caputo derivative sense

5. References

[1] I. Podlubny, Fractional Differential Equations, vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, 1999.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, vol. 204 of North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[3] J. Sabatier, O. P. Agrawal, and J. A. Machado, Advance in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht, The Netherlands, 2007.

[4] V. Lakshmikantham, S. Leela, J. Vasundhara, Theory of Fractional Dynamic Systems. Cambridge Academic Publishers, Cambridge 2009.

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[5] D. Baleanu, B. Guvenc and J. A. Tenreiro, New Trends in Nanotechnology and Fractional Calculus Applications, Springer, New York, NY, USA, 2010.
[6] M. Caputo, Linearmodels of dissipation whoseQis almost frequency independent, part II, Geophysical Journal International, vol. 13, no. 5, pp. 529--539, 1967.
[7] G. Jumarie, On the solution of the stochastic differential equation of exponential growth driven by fractional Brownian motion, AppliedMathematics Letters, vol. 18, no. 7, pp. 817--826, 2005.
[8] M. Davison and C. Essex, " Fractional differential equations and initial value problems,' The Mathematical Scientist, vol. 23, no. 2, pp. 108--116, 1998.
[9] C. F. M. Coimbra, Mechanics with variable-order differential operators, Annalen der Physik, vol. 12, no. 11-12, pp. 692--703, 2003.

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