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# A comparative study of the Daubechies wavelet based new Galerkin and Haar wavelet collocation methods for the numerical solution of differential equations

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**Abstract.** In this paper, we have made an attempt to comparative study of the existing Haar wavelet collocation method with proposed new wavelet-Galerkin method for the solution of differential equations. The solutions obtained using proposed scheme are comparably good and give higher accuracy than existing ones with exact solution by increasing the level of resolution.To yield the solutions of differential equations using usal wavelet-Galerkin require use of connection coefficients, which takes large computational complexities and time consumption. The proposed scheme is rather simple and avoids the above complexities, which is demonstrated through the some of the test problems.

**Keywords:** Haar wavelet collocation method; wavelet-Galerkin method; differential equations; resolution level.

## 1. Introduction

The subject of wavelets has been received a much attention because of the comprehensive mathematical power and the good application potential of wavelets in science and engineering problems. Special interest has been devoted to the construction of compactly supported smooth wavelet bases. As we have noted earlier that, spectral bases are infinitely differentiable but have global support. On the other side, basis functions used in finite-element methods have small compact support but poor continuity properties. Already we know that, spectral methods have good spectral localization but poor spatial localization, while finite element methods have good spatial localization, but poor spectral localization. Wavelet bases perform to combine the advantages of both spectral and finite element bases. We can expect numerical methods based on wavelet bases to be able to attain good spatial and spectral resolutions. Daubechies [1] illustrated that these bases are differentiable to a certain finite order. These scaling and corresponding wavelet function bases gain considerable interest in the numerical solutions of differential equations since from many years (see for example [2–4]).

An approach to study differential equations is the use of scaling function bases in place of other conventional piecewise polynomial trial functions in finite element type methods. Because of its implementation simplicity, the Galerkin method is considered the most widely used in applied mathematics (see for example [5–9]). The method combines the advantages of both Galerkin scheme and the refinable shift-invariant spaces of the multiresolution analysis. Since the different scales are introduced automatically through the translates and dilates of a single function, an essentially dimension independent concept, there is no need for developing explicit geometrical refinement strategies which become complicated for higher spatial dimensions. Integrals involving the scaling function and/or its derivatives occur when applying the wavelet-Galerkin procedure to differential and integral equations. These are called the connection coefficients. Since the scaling function and wavelets do not have an explicit analytical expression but are implicitly determined by the wavelet filter coefficients, it is necessary to develop algorithms to compute these coefficients. It takes the large storage requirements and at the same time, increasing the computational complexity drastically. To overcome this, the primary objective of this paper is to develop a new wavelet-Galerkin method (NWGM) for the numerical solution of differential equations involving the variable coefficients and nonhomogenous problems. The superiority of the solutions obtained by the proposed scheme is evident here in comparison with exact ones than the existing ones i. e Haar wavelet collocation method (HWCM) [10-15]. The validity of the proposed method is verified with numerical examples. This paper is summerized as follows. Some basics of Daubechies wavelets are given in section 2. Method of solution is described in section 3. Some of the test problems are illustrated in section 4. Finally, conclusions of the present work are discussed in section 5.

## 2. Daubechies wavelets

Daubechies wavelets are compactly supported functions, introduced by Daubechies [1]. This means that they have non zero values within a finite interval and have a zero value everywhere else. That's why it is useful for representing the solution of differential equation. They are an orthonormal bases for functions in $L^2(R)$. The construction of wavelet functions starts from building the scaling or dilation function, $\phi(x)$ and a set of coefficients $h_k$, $k \in Z$, which satisfies the two-scale relation or refinement equation,

$$\phi(x) = \sum_{k=0}^{L-1} h_k \phi(2x-k) \tag{2.1}$$

where $L$ denotes the order of the Daubechies wavelet. The associated wavelet function is given by

$$\psi(x) = \sum_{k=0}^{L-1} g_k \phi(2x-k) \tag{2.2}$$

where $g_k = (-1)^k h_{L-1-k}$ and $\int \phi(x)dx = 1$.

The translation and dilations of the scaling function $\phi(2^J x - k)$ or the wavelet function $\psi(2^J x - k)$ form a complete and orthogonal basis.

The wavelet basis induces a multiresolution analysis [16] on $L^2(R)$, i.e. the decomposition of the Hilbert space $L^2(R)$ into a chain of closed subspaces

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \tag{2.3}$$

such that

$$\bigcup V_J = L^2(R) \tag{2.4}$$

and

$$\bigcap V_J = \{0\} \tag{2.5}$$

By defining $W_J$ as an orthogonal complement of $V_J$ in $V_{J+1}$,

$$V_{J+1} = V_J \oplus W_J \tag{2.6}$$

The space $L^2(R)$ is represented as a direct sum of $W_j$'s as

$$L^2(R) = \oplus W_J \tag{2.7}$$

On each fixed scale $J(\geq 0)$, the wavelets $\left\{\psi_{J,k}(x) = 2^{J/2}\psi(2^J x - k), k \in Z\right\}$ form an orthonormal basis of $W_J$ and the functions $\left\{\phi_{J,k}(x) = 2^{J/2}\phi(2^J x - k), k \in Z\right\}$ form an orthonormal basis of $V_J$. The set of spaces $V_J$ is called as multiresolution analysis of $L^2(R)$, these spaces will be used to approximate the solutions of differential equations using the new wavelet-Galerkin method.

## 3. Method of solution

The Russian engineer V. I. Galerkin had proposed a projection method based on weak form in which a set of test functions are selected such that residual of differential equation becomes orthogonal to test functions [17, 18].

If the base functions in Galerkin method are wavelets, then it is called wavelet-Galerkin method (WGM). Here, we have developed the new wavelet-Galerkin method and has advantages over the usual WGM in terms of time consumption that is we applied the method instead of finding the connection coefficients.

**Theorem:** Let $V_J$, $J \in Z$ be a given MRA with scaling function $\phi$ and $P_J f$ is a projection of $f \in L^2(R)$ onto $V_J$ so that

$$P_J f = \sum_k c_k 2^{J/2} \phi(2^J x - k)$$

Then for sufficiently large $J$, $c_k \cong 2^{-J/2} f(k2^{-J})$ with $\int \overline{\phi(x)}\, dx = 1$ [19].

Here, we develop a NWGM is followed by basics of finite difference scheme.

**Lemma:** For an unknown function $y(x)$ and for large $J \in Z_+$ then

$$y^{(n)}(x) = \frac{1}{h^n} \sum_{i=-1}^{n-1} (-1)^{n+i+1}\, {}^nC_{i+1}\, y(x+ih).$$

**Proof:** For small $h = \frac{1}{2^J}$, finite difference discretizations of $y(x)$ is as follows,

$$y'(x) = \frac{y(x) - y(x-h)}{h},$$

$$y''(x) = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2},$$

and so on upto $n$ th difference, we have

$$y^n(x) = \frac{y(x+(n-1)h) - {}^nC_1 y(x+(n-2)h) + {}^nC_2 y(x+(n-3)h) + \cdots + {}^nC_{n-1} y(x) - y(x-h)}{h^n}$$

$$= \frac{1}{h^n} \sum_{i=-1}^{n-1} (-1)^{n+i+1}\, {}^nC_{i+1}\, y(x+ih)$$

**Method of solution:**

Consider $n$ th order ODE

$$\sum_{p=0}^{n} A_p\; y^p(x) = F(x), \quad a < x < b, \tag{3.1}$$

where $A_p$ is constant/variable coefficient and $F(x)$ is a polynomial of any degree in $x$.

Let the solution $y(x)$ of the problem be approximated by its $J$ th level wavelet series on the interval (*a*, *b*), i.e.

$$y(x) = \sum_k \alpha_k 2^{J/2} \phi(2^J x - k) \tag{3.2}$$

Using above lemma, we have,

$$\begin{aligned} y^p(x) &= \frac{1}{h^p} \sum_{i=-1}^{p-1} {}^pC_{i+1} (-1)^{p+i+1} \sum_k \alpha_k 2^{J/2} \phi\left( 2^J (x + \frac{i}{2^J}) - k \right) \\ &= \frac{1}{h^p} \sum_{i=-1}^{p-1} {}^pC_{i+1} (-1)^{p+i+1} \sum_k \alpha_k 2^{J/2} \phi(2^J x + i - k) \qquad , \\ &= \frac{1}{h^p} \sum_{i=-1}^{p-1} {}^pC_{i+1} (-1)^{p+i+1} \sum_k \alpha_{k+i}\, \phi_k \end{aligned} \tag{3.3}$$

Substituting Eqn. (3.3) in (3.1), we get

$$\sum_{p=0}^{n} A_p \frac{1}{h^p} \sum_{i=-1}^{p-1} {}^pC_{i+1} (-1)^{p+i+1} \sum_k \alpha_{k+i}\, \phi_k = F(x) \tag{3.4}$$

By taking inner product with $\phi_m$, we get,

$$\sum_{p=0}^{n} A_p \frac{1}{h^p} \sum_{i=-1}^{p-1} {}^pC_{i+1} (-1)^{p+i+1} \alpha_{m+i} = G(x), \tag{3.5}$$

where $G(x) = \int_R F(x) \phi_m dx$.

Solve system (3.5), for coefficients $\alpha$. And then finally, substitute these coefficients in Eqn. (3.2), we get the required solution of the given differential equation.

## 4. Numerical examples

**Problem 4.1:** First, consider the second order transcedental equation

$$y''+y=\sin(4\pi x) \tag{4.1}$$

with respect to boundary conditions $y(0)=1,\ y(1)=0$.

The implementation of the Eqn. (4.1) as per the method explained in section 3, is as follows:

Here $A_2=1$, $A_0=1$ and $F(x)=\sin(4\pi x)$.

For sufficiently large $J$, $\alpha_0=\langle y,\phi_0\rangle=2^{-J/2}y(0),\ \alpha_{2^J}=\langle y,\phi_{2^J}\rangle=2^{-J/2}y(1)$ for $k=0, k=2^J$ and

$$\alpha_k=2^{-J/2}y(k/2^J) \tag{4.2}$$

Let us take,

$$y''(x)=\frac{1}{h^2}\sum_{i=-1}^{1}{}^{2}C_{i+1}(-1)^{2+i+1}\alpha_{m+i} \tag{4.3}$$

$$y(x)=\sum_{i=-1}^{-1}{}^{0}C_{i+1}(-1)^{i+1}\alpha_{m+i} \tag{4.4}$$

Substituting Eqns. (4.3) and (4.4) in Eqn. (4.1), we get system of algebraic equations

$$\frac{1}{h}\sum_{i=-1}^{0}{}^{1}C_{i+1}(-1)^{1+i+1}\alpha_{m+i}+\sum_{i=-1}^{-1}{}^{0}C_{i+1}(-1)^{i+1}\alpha_{m+i}=\sin(4\pi x) \tag{4.5}$$

Now, we have the system of $2^J-1$ equations with $2^J-1$ unknown coefficients. We obtain the coefficients $\alpha$ by solving Eqn. (4.5), i. e for J=4, $\alpha$ = {2.3875e-01, 2.2722e-01, 2.1573e-01, 2.0404e-01, 1.9151e-01, 1.7749e-01, 1.6175e-01, 1.4462e-01, 1.2686e-01, 1.0923e-01, 9.2081e-02, 7.5193e-02, 5.7946e-02, 3.9715e-02, 2.0280e-02}. Substitute these coefficients in Eqn. (4.2), we get required numerical solution, the results are presented in figure 1 in comparison with exact equation $y(x)=\cos(x)+\frac{1}{\sin(1)}\left(\frac{\sin(4\pi)}{16\pi^2-1}-\cos(1)\right)\sin(x)-\frac{\sin(4\pi x)}{16\pi^2-1}$.

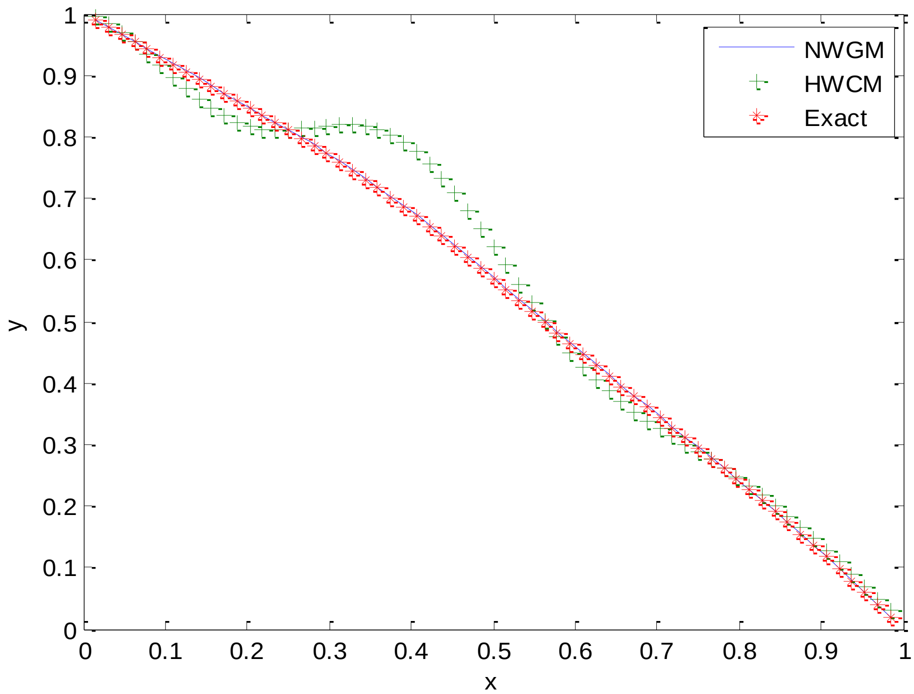



Figure 1. Comparison of numerical solutions with exact solution for J=6 of problem 4.1.

**Problem 4.2:** Now, consider the equation,

$$y''+y=-x \tag{4.6}$$

subjected to boundary conditions $y(0)=0,\ y(1)=0$. As in the previous example, we obtained the numerical solution and are presented in compariison with exact equation $y(x)=\dfrac{\sin(x)}{\sin(1)}-x$ in figure 2.

**Problem 4.3:** Next, consider the transcedental equation,

$$-y''=3xe^{x}(x+3) \tag{4.7}$$

with respect to Dirichlet boundary conditions. The exact solution is known as the Bessel function of zeroth order denoted by $y(x)=3xe^{x}(1-x)$. As in the previous examples, we get the numerical solutions and are presented in figure 3.

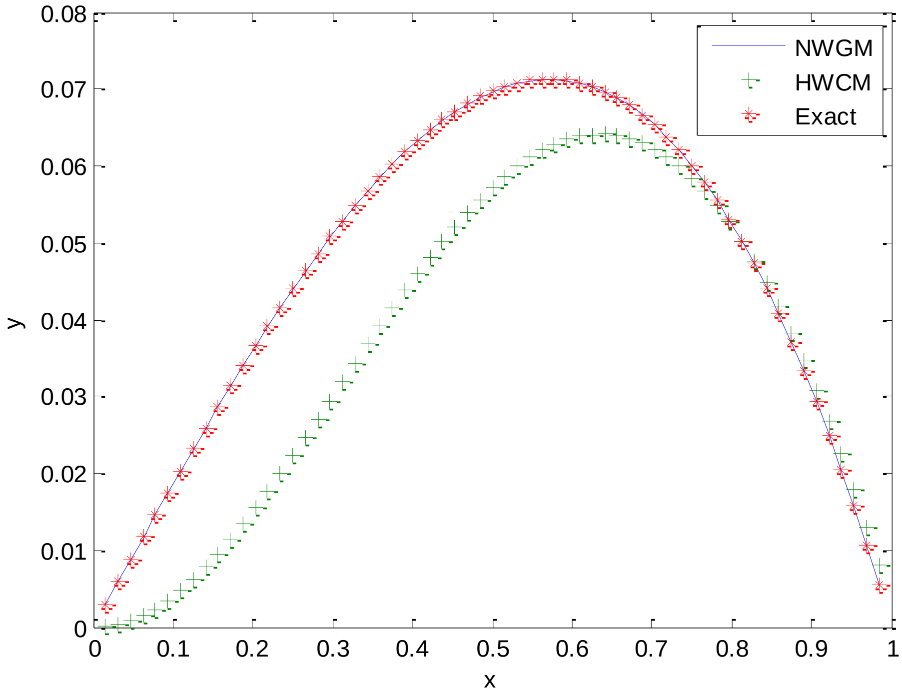



Figure 2. Comparison of numerical solutions with exact solution for J=6 of problem 4.2.

**Problem 4.4:** Next, consider the other transcedental equation,

$$y''+y'+y=-e^{-5x}\left(9979\sin(100x)+900\cos(100x)\right) \tag{4.8}$$

Subjected to boundary conditions $y(0)=0,\ y(1)=\sin(100)e^{-5}$. The exact equation is $y(x)=e^{-5x}\sin(100x)$. The method is applied as in the previous examples; we get numerical results and are presented in figure 4 ((a) & (b)).

**Problem 4.5:** Now, consider the another transcedental equation,

$$y''+\frac{16\pi^2}{9}y=\frac{7\pi^2}{9}\sin(\pi x) \tag{4.9}$$

with respect to boundary conditions $y(0)=0,\ y(1)=0$. As in the previous examples, we obtained the numerical solution and is presented in compariison with exact equation $y(x)=\sin(\pi x)$ in figure 5 ((a) & (b)).

**Problem 4.6:** Finally, consider another transcedental equation,

$$y''+\frac{16\pi^2}{9}y=-256\pi^2\sin\left(\frac{4\pi}{3}x\right)\sin\left(\frac{16\pi}{3}x\right)+128\pi^2\cos\left(\frac{4\pi}{3}x\right)\cos\left(\frac{16\pi}{3}x\right) \tag{4.10}$$

subjected to boundary conditions $y(0)=1,\ y(1)=6.25$. As in the previous examples, we obtained the numerical solution and is presented in compariison with exact equation $y(x)=9\sin\left(\frac{4\pi}{3}x\right)\sin\left(\frac{16\pi}{3}x\right)+\cos\left(\frac{4\pi}{3}x\right)$ in figure 6. The error analysis of all the problems is given in table 1.

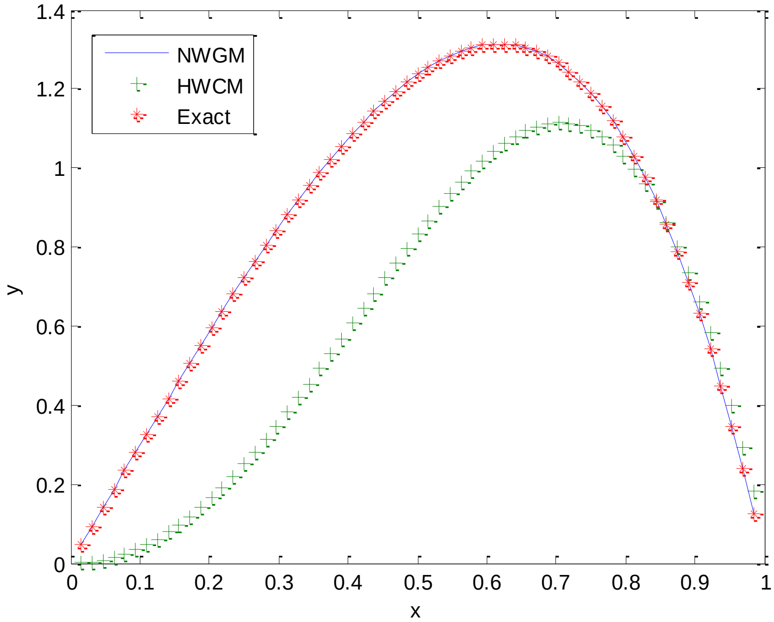
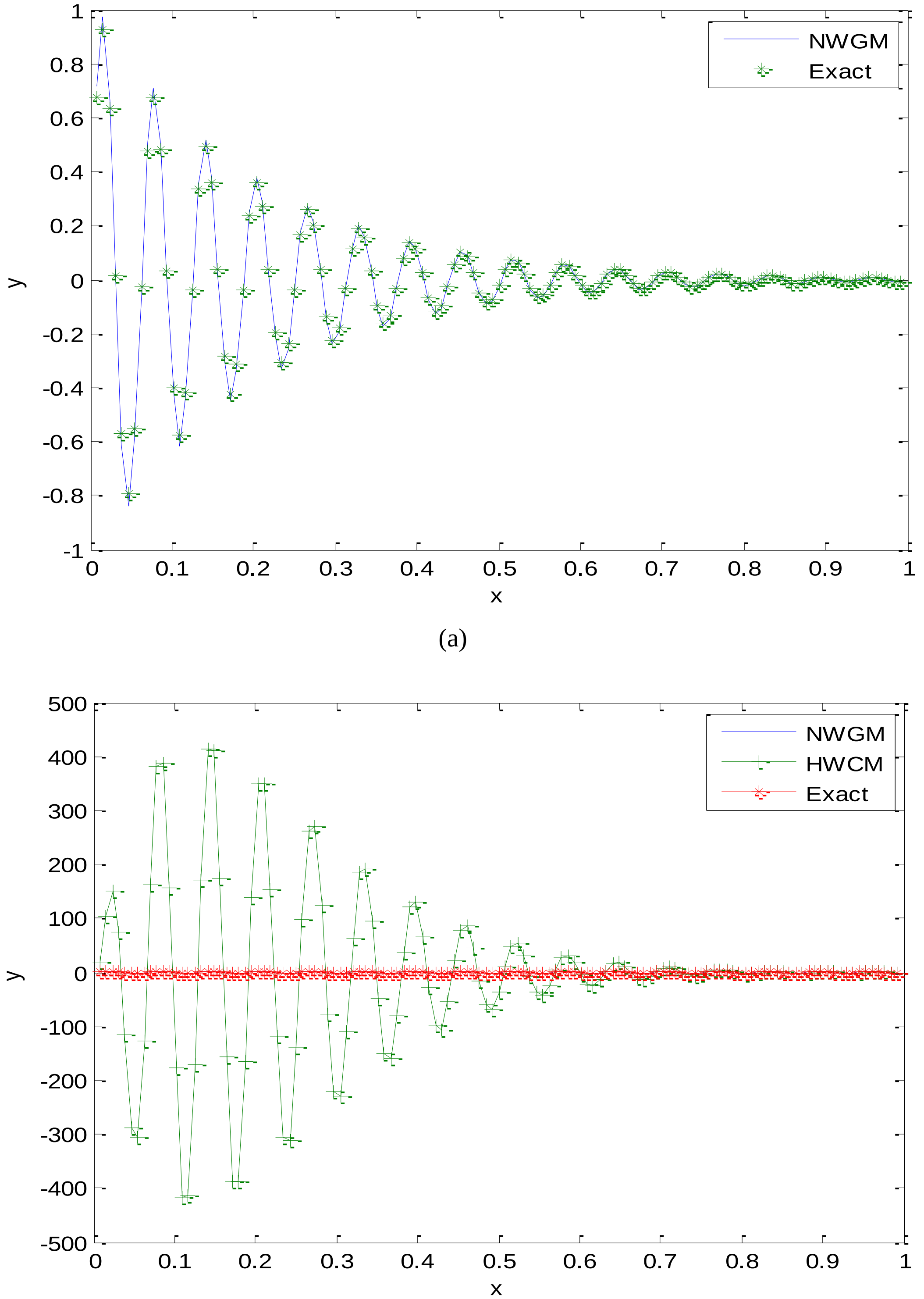
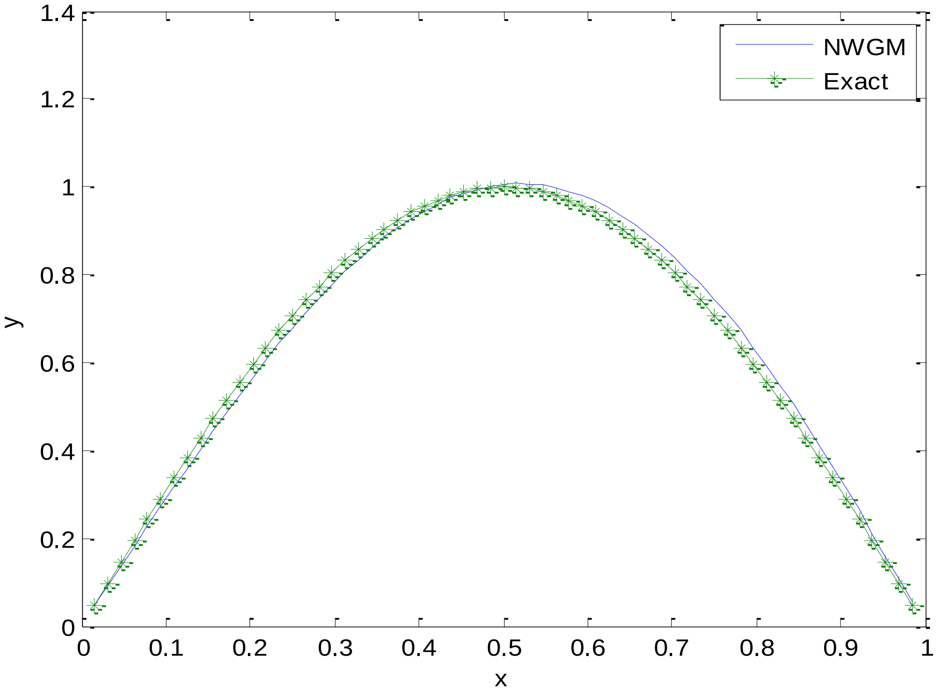



Figure 3. Comparison of numerical solutions with exact solution for J=6 of problem 4.3.

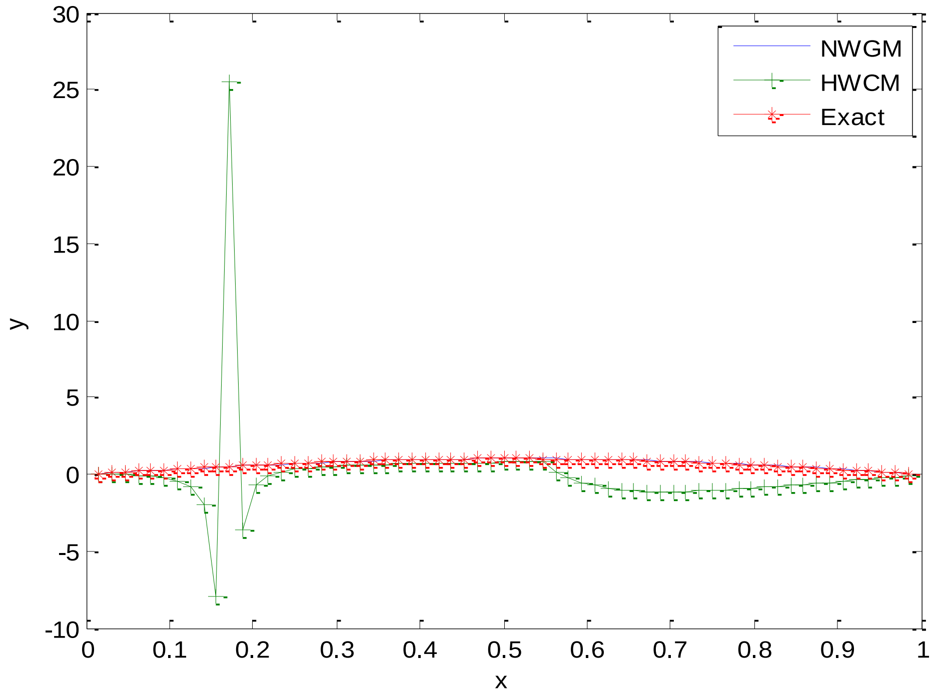



(b)

Figure 4. Comparison of numerical solutions with exact solution for J=7 of problem 4.4

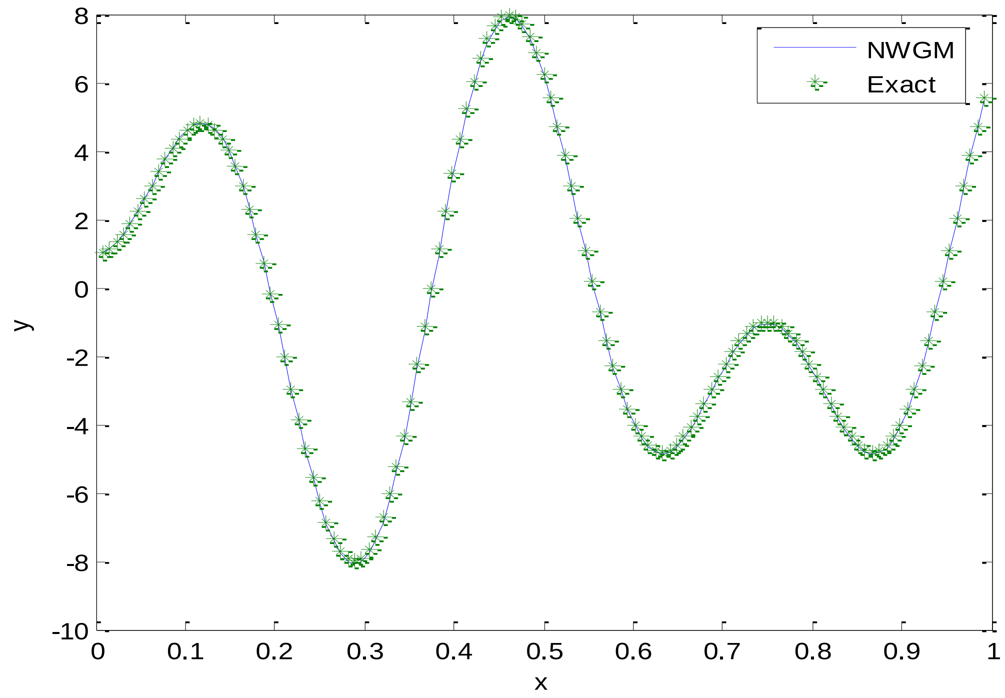



(a)

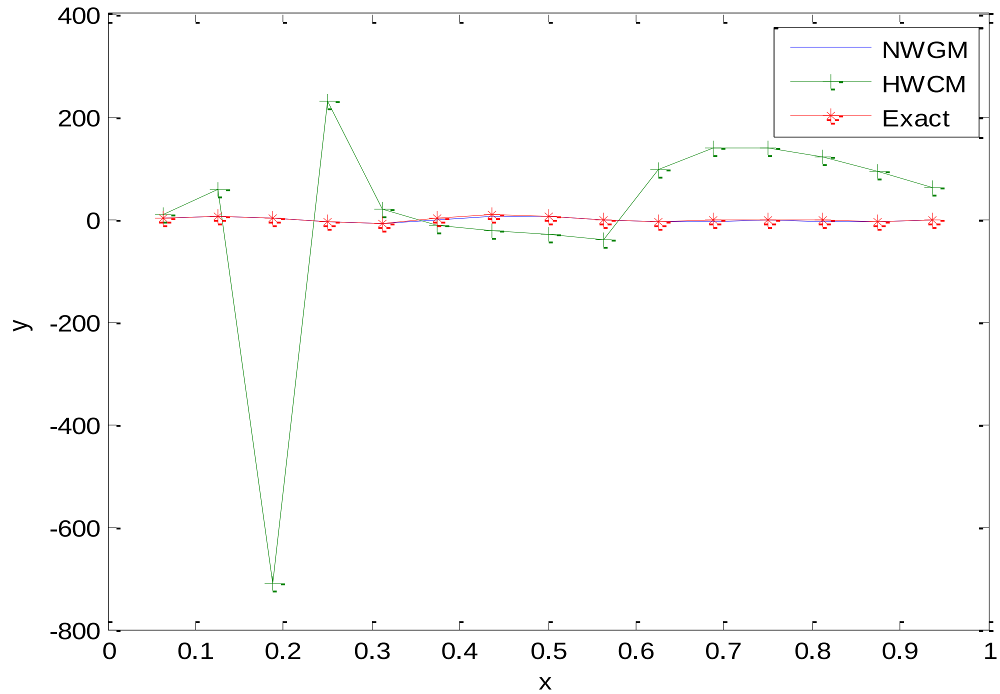



(b)

Figure 5. Comparison of numerical solutions with exact solution for J=6 of problem 4.5.




(a) for J=7




(b) for J=4

Figure 6. Comparison of numerical solutions with exact solution of problem 4.6.

Table 1. Error analysis of the problems.

| J | $N=2^J$ | Problem 4.1 | | | |
|---|---|---|---|---|---|
| | | $E_{max}$ | | $E_{RMS}$ | |
| | | NWGM | HWCM | NWGM | HWCM |
| 3 | 8 | 1.7110e-02 | 7.9336e-02 | 6.4670e-03 | 2.8049e-02 |
| 4 | 16 | 8.7450e-03 | 9.4520e-02 | 2.2580e-03 | 2.3630e-02 |
| 5 | 32 | 4.4132e-03 | 9.6665e-02 | 7.9263e-04 | 1.7088e-02 |
| 6 | 64 | 2.2158e-03 | 9.6405e-02 | 2.7917e-04 | 1.2051e-02 |
| 7 | 128 | 1.1110e-03 | 9.6679e-02 | 9.8581e-05 | 8.5453e-03 |
| 8 | 256 | 5.5623e-04 | 9.6741e-02 | 3.4833e-05 | 6.0463e-03 |
| 9 | 512 | 2.7831e-04 | - | 1.2312e-05 | - |
| 10 | 1024 | 1.3920e-04 | - | 6.4670e-03 | - |
| **Problem 4.2** | | | | | |
| 3 | 8 | 1.1015e-03 | 1.9539e-02 | 4.1632e-04 | 6.9079e-03 |
| 4 | 16 | 4.7582e-04 | 2.0339e-02 | 1.2286e-04 | 5.0847e-03 |
| 5 | 32 | 2.1633e-04 | 2.0533e-02 | 3.8854e-05 | 3.6298e-03 |
| 6 | 64 | 1.0258e-04 | 2.0578e-02 | 1.2924e-05 | 2.5722e-03 |
| 7 | 128 | 4.9917e-05 | 2.0587e-02 | 4.4294e-06 | 1.8196e-03 |
| 8 | 256 | 2.4611e-05 | 2.0588e-02 | 1.5412e-06 | 1.2868e-03 |
| 9 | 512 | 1.2085e-05 | - | 5.3463e-07 | - |
| 10 | 1024 | 6.0878e-06 | - | 1.9034e-07 | - |
| **Problem 4.3** | | | | | |
| 3 | 8 | 9.9923e-03 | 4.7806e-01 | 3.7767e-03 | 1.6902e-01 |
| 4 | 16 | 2.5370e-03 | 4.8068e-01 | 6.5506e-04 | 1.2017e-01 |
| 5 | 32 | 6.3449e-04 | 4.8228e-01 | 1.1396e-04 | 8.5256e-02 |
| 6 | 64 | 1.5864e-04 | 4.8226e-01 | 1.9986e-05 | 6.0282e-02 |
| 7 | 128 | 3.9664e-05 | 4.8246e-01 | 3.5196e-06 | 4.2644e-02 |
| 8 | 256 | 9.9161e-06 | 4.8250e-01 | 6.2097e-07 | 3.0156e-02 |
| 9 | 512 | 2.4790e-06 | - | 1.0966e-07 | - |
| 10 | 1024 | 6.1951e-07 | - | 1.9369e-08 | - |
| **Problem 4.4** | | | | | |
| 3 | 8 | 1.5465e+01 | 2.2873e+01 | 5.8453e+00 | 8.0869e+00 |
| 4 | 16 | 1.4569e+01 | 2.0770e+01 | 3.7618e+00 | 5.1924e+00 |
| 5 | 32 | 3.7738e-01 | 4.4760e+02 | 6.7779e-02 | 7.9126e+01 |
| 6 | 64 | 2.1222e-01 | 3.2010e+02 | 2.6737e-02 | 4.0013e+01 |
| 7 | 128 | 4.9279e-02 | 4.1526e+02 | 4.3728e-03 | 3.6705e+01 |
| 8 | 256 | 1.2772e-02 | 4.4046e+02 | 7.9983e-04 | 2.7529e+01 |
| 9 | 512 | 3.5824e-03 | - | 1.5848e-04 | - |
| 10 | 1024 | 1.1259e-03 | - | 3.5201e-05 | - |
| **Problem 4.5** | | | | | |
| 3 | 8 | 6.3802e-01 | 8.5402e-01 | 2.4115e-01 | 3.0194e-01 |
| 4 | 16 | 2.0757e-01 | 5.7697e+00 | 5.3593e-02 | 1.4424e+00 |
| 5 | 32 | 8.3806e-02 | 7.2582e+00 | 1.5052e-02 | 1.2831e+00 |
| 6 | 64 | 3.7365e-02 | 2.5051e+01 | 4.7075e-03 | 3.1313e+00 |
| 7 | 128 | 1.7525e-02 | - | 1.5551e-03 | - |
| 8 | 256 | 8.4248e-03 | 3.7514e+02 | 5.2758e-04 | 2.3446e+01 |

| | | | | | |
|---|---|---|---|---|---|
| 9 | 512 | 6.5532e-03 | - | 2.8990e-04 | - |
| 10 | 1024 | 5.8501e-03 | - | 1.8291e-04 | - |
| **Problem 4.6** | | | | | |
| 3 | 8 | 9.6535e+00 | 3.1801e+02 | 3.6487e+00 | 1.1243e+02 |
| 4 | 16 | 1.6321e+00 | 7.1442e+02 | 4.2140e-01 | 1.7861e+02 |
| 5 | 32 | 6.2106e-01 | - | 1.1155e-01 | - |
| 6 | 64 | 3.0719e-01 | - | 3.8702e-02 | - |
| 7 | 128 | 1.5469e-01 | - | 1.3726e-02 | - |
| 8 | 256 | 7.7811e-02 | - | 4.8727e-03 | - |
| 9 | 512 | 3.5620e-02 | - | 1.5757e-03 | - |
| 10 | 1024 | 1.4840e-02 | - | 4.6397e-04 | - |

## 5. Conclusions

This paper reveals that, a compaertive study of the Haar wavelet collocation and wavelet-Galerkin methods. It can be observed from the figures, the proposed scheme NWGM is good agreement with exact ones than HWCM. From the table we conlude that, NWGM gives higher accuracy than the HWCM, significantly HWCM does not converge or takes large time to converge for some *N*, so for that *N*, we shown ‘dash’. Hence, the proposed technique is a powerful scheme for the fast and accurate solution of certain class of differential equations.

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