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VIM for Determining Unknown Source Parameter in Parabolic Equations

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Abstract. In this paper, an application of the variational iteration method (VIM) is presented. This technique provides a sequence of function which converges to the exact solution of the problem. The main property of the method is in its flexibility and ability to solve nonlinear equation accurately and conveniently. For solving the discussed inverse problem, at first we transform it into a nonlinear direct problem then use the proposed method. Numerical examples are examined to show the efficiency of the technique.

Keywords: VIM; inverse parabolic problem; unknown source parameter; additional condition.

1. Introduction

In this paper, VIM is presented as an alternative method for simultaneously finding the time-dependent source parameter and the temperature distribution in one-dimensional heat equation.

Consider the parabolic equation:

$$u_t = u_{xx} + a(t)u + f(x,t); \qquad 0 < x < \infty, \quad 0 < t < T, \tag{1}$$

with unknown coefficient $a(t)$. Impose the initial and boundary condition:

$$u(x,0) = \varphi(x); \qquad 0 \le x < \infty, \tag{2}$$

$$u(0,\mathrm{t}) = g(\mathrm{t}); \qquad 0 \le t \le T, \tag{3}$$

and the additional condition:

$$-u_x(0,\mathrm{t}) = E(\mathrm{t}); \qquad 0 \le t \le T, \tag{4}$$

where $T > 0$ is final time and φ, f, g and E are known functions. If u is a temperature then (1)-(4) can be regarded as a control problem finding the control $a(t)$ such that the internal constraint is satisfied. If $a(t)$ is known then direct initial-boundary value problem (1)-(4) has a unique smooth solution $u(\mathrm{x},\mathrm{t})$ [4]. For the existence and uniqueness of solutions of these inverse problems and also more applications, the reader can refer to [3, 4, 6, 12, 13, 17].

The VIM is a powerful tool to searching for approximate solutions of nonlinear equation without requirement of linearization or perturbation. This method, which was first proposed by He [7, 8] in 1998, has been proved by many authors to be a powerful mathematical tool for various kinds of nonlinear problems [1, 2, 15, 19]. The interested reader can see [9, 10, 14] for some other applications of the method.

The rest of this paper is organized as follows: In Section 2, the variational iteration method is reviewed. In Section 3, application of the VIM is presented to solve the discussed inverse problem. In Section 4, several numerical examples are presented to confirm the accuracy and efficiency of the new method and finally a conclusion is presented in Section 5.

2. Basic idea of the variational iteration method

To illustrate its basic concepts of VIM, we consider the following general nonlinear differential equation:

$$Lu(t) + Nu(t) = f(t), \tag{5}$$

where L and N are linear and nonlinear operators, respectively and f is source or sink term. According to VIM [1, 2, 7-10, 14, 15, 19], we can write down following correction functional:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,\tau)\{Lu_n(\tau) + N\tilde{u}_n(\tau) - f(\tau)\}d\tau; \quad n \ge 0, \tag{6}$$

where λ is general Lagrange multiplier [11], which can be identified optimally via the variational theory [7, 8]. The subscript n denotes the nth order approximation and is considered as restricted variation [7, 8] which means $\delta\tilde{u}_n = 0$.

It is required first to determine the Lagrange multiplier. Employing the restricted variation in correction functional and using integration by part makes it easy to compute the Lagrange multiplier, see for instance [8].

For linear problems, its exact solution can be obtain by only one iteration step due to the fact that, no nonlinear exist so the Lagrange multiplier can be exactly identified.

Assuming $u_0(t)$ is the solution of $Lu = 0$. Having λ determined, then several approximations $u_{n+1}(t)$; $n \geq 0$, can be determined.

We will rewrite equation (6) in the operator form as follows:

$$u_{n+1}(t) = A[u_n],$$

where the operator A takes the following form:

$$A[u(t)] = u(t) + \int_0^t \lambda(t,\tau)\{Lu(\tau) + Nu(\tau) - f(\tau)\}d\tau.$$

Theorem. Let $(X, \|.\|)$ be a Banach space and $A: X \to X$ is a nonlinear mapping and suppose that:

$$\|A[u] - A[\tilde{u}]\| \leq \gamma \|u - \tilde{u}\|, \quad u, \tilde{u} \in X,$$

for some constant γ. Then, A has a unique fixed point. Furthermore, the sequence (6) using VIM with an arbitrary choice of $u_0 \in X$, converges to the fixed point of A and

$$\|u_n - u_m\| \leq \|u_1 - u_0\| \sum_{j=m-1}^{n-2} \gamma^j.$$

Proof: See [16].

Consequently, the exact solution may be obtained by using the Banach's fixed point Theorem [16]:

$$u = \lim_{n\to\infty} u_n.$$

According to the above theorem, a sufficient condition for the convergence of the VIM is strictly contraction of A. Furthermore, sequence (6) converges to the fixed point of A, which is also the solution of the equation (5). Also, the rate of convergence depends on γ.

3. Application

In this section, the VIM is used for solving the problem (1)-(4). In order to solve this problem by using VIM, we require transforming the problem with only one unknown function. This transformation is proposed by Cannon, Lin and Xu [5]. According to this procedure the term $a(t)$ in (1) is eliminated by introducing some transformation and system (1)-(4) is written in the canonical.

This procedure is as follows: the term $a(t)$ in (1) eliminated by introducing the following transformation:

$$r(t) = \exp\{-\int_0^t a(s)ds\}, \tag{7}$$

$$w(x,t) = u(x,t)r(t). \tag{8}$$

Thus, we have:

$$u(x,t) = \frac{w(x,t)}{r(t)}, \qquad a(t) = \frac{-r'(t)}{r(t)}. \tag{9}$$

We reduce the original inverse problem (1)-(4) to the following auxiliary direct problem:

$$w_t = w_{xx} + r(t)f(x,t); \qquad 0 < x < \infty, \quad 0 < t < T, \tag{10}$$

$$w(x,0) = \varphi(x); \qquad 0 \leq x < \infty, \tag{11}$$

$$w(0,t) = g(t)r(t); \qquad 0 \leq t \leq T, \tag{12}$$

Subject to:

$$r(t) = -\frac{w_x(0,t)}{E(t)}; \qquad 0 \le t \le T. \tag{13}$$

This system can be solved by VIM. Now, we can write following correction functional:

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda(t,\tau)\{w_{n\tau}(x,\tau) - \tilde{w}_{nxx}(x,\tau) + \frac{\tilde{w}_x(0,\tau)}{E(\tau)} f(x,\tau)\} d\tau; \quad n \ge 0.$$

Making the above correction functional stationary, note that $\delta\tilde{w}_n(x,t) = 0$, we have: $\delta w_{n+1}(x,t) = \delta w_n(x,t) + \delta\int_0^t \lambda(t,\tau)\{w_{n\tau}(x,\tau) - \tilde{w}_{nxx}(x,\tau) + \frac{\tilde{w}_x(0,\tau)}{E(\tau)} f(x,\tau)\} d\tau.$

Therefore:

$$\delta w_{n+1}(x,t) = \delta w_n(x,t) + \delta\int_0^t \lambda(t,\tau)\{w_{n\tau}(x,\tau)\} d\tau.$$

Thus, its stationary condition can be obtained as follow:

$$\begin{cases} \lambda'(t,\tau) = 0, \\ 1 + \lambda(t,\tau)|_{\tau=t} = 0. \end{cases}$$

Therefore $\lambda(t,\tau) = -1$.

Note that $\delta w_n(x,0) = 0$. Now, the following iteration formula can be obtained as:

$$w_{n+1}(x,t) = w_n(x,t) - \int_0^t \{w_{n\tau}(x,\tau) - w_{nxx}(x,\tau) + \frac{\tilde{w}_x(0,\tau)}{E(\tau)} f(x,\tau)\} d\tau; \quad n \ge 0. \tag{14}$$

According to Adomian's decomposition method in t -direction which is equivalent to the VIM in t -direction [18], we choose its initial approximate solution as $w_0(x,t) = w(x,0)$. Having $w = \lim_{n\to\infty} w_n$ determined [16], then the unknown (u, a) can be calculated by using the equation (9). If the exact solution of w is not obtainable, it was found that a few number of approximations can be used for numerical purposes.

4. Numerical results

In this section we report some results of our numerical calculations using the numerical procedures described in the previous section.

Example 1:

We consider the following inverse problem:

$$u_t = u_{xx} + a(t)u + \exp(x+t); \qquad x > 0, \quad 0 < t < 1,$$
$$u(x,0) = \exp(x); \qquad x \ge 0,$$
$$u(0,t) = \exp(t); \qquad 0 < t < 1,$$
$$-u_x(0,t) = \exp(t); \qquad 0 < t < 1.$$

The true solution is $u(x,t) = \exp(x+t)$ while $a(t) = 1$.

Let $w_0(x,t) = w(x,0) = \exp(x)$.

By using the equation (14), we obtain:

$$w_1(x,t) = \exp(x),$$
$$w_2(x,t) = w_1(x,t).$$

Thus:

$$w = \lim_{n\to\infty} w_n = \exp(x).$$

Therefore, we obtain a series which is convergent to the exact solution of the problem (10)-(13).

Also from (13), we can obtain:

$$r(t) = \exp(-t).$$

Thus, using (9) we obtain:

$$u(x,t) = \exp(x+t),$$

$$a(\mathrm{t}) = 1,$$

which is equal to the exact solution of this example. From this example, it can be seen that the exact solution is obtain by using one iteration step only.

Example 2:

We consider the following inverse problem:

$$u_t = u_{xx} + a(t)u; \qquad x > 0, \quad 0 < t < 1,$$

$$\mathrm{u(x,0) = cos(x) - sin(x)}; \qquad x \geq 0,$$

$$\mathrm{u(0,t) = exp}(\frac{t^2}{2} - \mathrm{t}); \qquad 0 < t < 1,$$

$$-\mathrm{u}_x(0,\mathrm{t}) = \mathrm{exp}(\frac{t^2}{2} - \mathrm{t}); \qquad 0 < t < 1.$$

The true solution is $\mathrm{u(x,t) = exp}(\frac{t^2}{2} - \mathrm{t})(\mathrm{cos(x) - sin(x)})$ while $a(\mathrm{t}) = -\mathrm{t}$.

Let $\mathrm{w}_0(\mathrm{x,t}) = \mathrm{w(x,0) = cos(x) - sin(x)}.$

By using the equation (14), we obtain:

$$\mathrm{w}_1(\mathrm{x,t}) = (\mathrm{cos(x) - sin(x)}) - \mathrm{t}(\mathrm{cos(x) - sin(x)}),$$

$$\mathrm{w}_2(\mathrm{x,t}) = (\mathrm{cos(x) - sin(x)}) - \mathrm{t}(\mathrm{cos(x) - sin(x)}) + \frac{1}{2}\mathrm{t}^2(\mathrm{cos(x) - sin(x)}),$$

$$\mathrm{w}_3(\mathrm{x,t}) = (\mathrm{cos(x) - sin(x)}) - \mathrm{t}(\mathrm{cos(x) - sin(x)}) + \frac{1}{2}\mathrm{t}^2(\mathrm{cos(x) - sin(x)}) - \frac{1}{6}\mathrm{t}^3(\mathrm{cos(x) - sin(x)}),$$

$$\vdots$$

$$\mathrm{w}_n(\mathrm{x,t}) = (\mathrm{cos(x) - sin(x)}) - \mathrm{t}(\mathrm{cos(x) - sin(x)}) + \frac{1}{2}\mathrm{t}^2(\mathrm{cos(x) - sin(x)}) + \cdots + (-1)^n \frac{1}{n!}\mathrm{t}^n(\mathrm{cos(x) - sin(x)}).$$

We know that $\mathrm{w}_n(\mathrm{x,t}) = (1 - \mathrm{t} + \frac{1}{2}\mathrm{t}^2 + \cdots + (-1)^n \frac{1}{n!}\mathrm{t}^n)$ is the n order Taylor series of $\mathrm{exp}(-\mathrm{t})$. Now using the fact that:

$$w = \lim_{n \to \infty} w_n,$$

that leads to the exact solution:

$$\mathrm{w(x,t) = exp(-t)(cos(x) - sin(x))}.$$

Therefore, we obtain a series which is convergent to the exact solution of the problem (10)-(13).

Also from (13), we can obtain:

$$r(\mathrm{t}) = \mathrm{exp}(-\frac{t^2}{2} + 2\,\mathrm{t}).$$

Thus, using (9) we obtain:

$$\mathrm{u(x,t) = exp}(\frac{t^2}{2} - \mathrm{t})(\mathrm{cos(x) - sin(x)}),$$

$$a(\mathrm{t}) = -\mathrm{t}.$$

This is equal to the exact solution of this example.

Example 3:

Solve the following inverse problem:

$$u_t = u_{xx} + a(t)u + (\pi^2 + 2\,\mathrm{t})\,\mathrm{exp(t)}\cos(\pi\,\mathrm{x}); \qquad x > 0, \quad 0 < t < 1,$$

$$\mathrm{u(x,0)} = x + \cos(\pi\,\mathrm{x}); \qquad x \geq 0,$$

$$\mathrm{u(0,t) = exp(t)}; \qquad 0 \leq t \leq 1,$$

$$-\mathrm{u}_x(0,\mathrm{t}) = -\mathrm{exp(t)}; \qquad 0 \leq t \leq 1.$$

The true solution is $\mathrm{u(x,t) = exp(t)(x + \cos(\pi\,x))}$ while $a(\mathrm{t}) = 1 - 2\,\mathrm{t}$.

We can select $w_0(x,t) = w(x,0) = x + \cos(\pi x)$; by using the given initial value. According to (14), one can obtain the successive approximations $w_n(x,t)$ of $w(x,t)$ as follow:

$$w_1(x,t) = (x + \cos(\pi x)) + t^2(x + \cos(\pi x)),$$

$$w_2(x,t) = (x + \cos(\pi x)) + t^2(x + \cos(\pi x)) + \frac{t^4}{2}(x + \cos(\pi x)),$$

$$w_3(x,t) = (x + \cos(\pi x)) + t^2(x + \cos(\pi x)) + \frac{t^4}{2}(x + \cos(\pi x)) + \frac{t^6}{6}(x + \cos(\pi x)),$$

$$\vdots$$

$$w_n(x,t) = (x + \cos(\pi x)) + t^2(x + \cos(\pi x)) + \frac{t^4}{2}(x + \cos(\pi x)) + \cdots + \frac{t^{2n}}{n!}(x + \cos(\pi x)).$$

We know that $(1 + t^2 + \frac{t^4}{2} + \cdots + \frac{t^{2n}}{n!})$ is the n th order Taylor series of $\exp(t^2)$.

Now using the fact that:

$$w = \lim_{n\to\infty} w_n,$$

that leads to the exact solution:

$$w(x,t) = \exp(t^2)(x + \cos(\pi x)).$$

Therefore, we obtain a series which is convergent to the exact solution of the problem (10)-(13).

Also from (13), we can obtain:

$$r(t) = \exp(t^2 - t).$$

Thus, using (9) we obtain:

$$u(x,t) = \exp(t)(x + \cos(\pi x)),$$
$$a(t) = 1 - 2t$$

which is equal to the exact solution of this example.

Example 4:

We solve the problem (1)-(4) as:

$$u_t = u_{xx} + a(t)u + 1 - \sin(t)(x+t); \qquad x > 0, \quad 0 < t < 1,$$
$$u(x,0) = x; \qquad x \geq 0,$$
$$u(0,t) = t; \qquad 0 \leq t \leq 1,$$
$$-u_x(0,t) = -1; \qquad 0 \leq t \leq 1.$$

for which the exact solution is $u(x,t) = x + t$ and $a(t) = \sin(t)$.

We can select $w_0(x,t) = x$; by using the given initial value. According to (14), one can obtain the successive approximations $w_n(x,t)$ of $w(x,t)$ as follow:

$$w_1(x,t) = t + \cos(t)(x+t) - \sin(t),$$

$$w_2(x,t) = -\frac{1}{4}t + \frac{1}{2}x + \frac{1}{2}\cos^2(t)(x+t) - \frac{1}{4}\sin(t)\cos(t).$$

And the rest of the components of iteration formula (14) are obtained using the Maple 13Package. Now form (13), we can obtain the successive approximations $r_n(t)$ of $r(t)$ as:

$$r_n(t) = -\frac{w_{nxx}(0,t)}{E(t)}.$$

Finally, using (9), we can obtain the successive approximations $u_n(x,t)$ of $u(x,t)$ and $a_n(t)$ of $a(t)$ as following:

$$u_n(x,t) = \frac{w_n(x,t)}{r_n(t)}, \qquad a_n(t) = \frac{-r_n'(t)}{r_n(t)}.$$

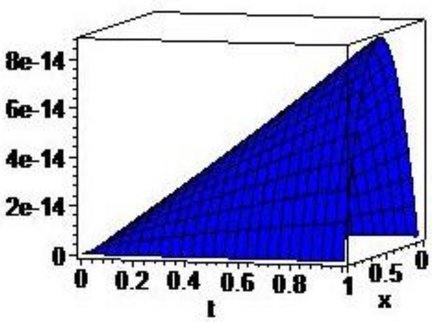
The obtained numerical results are summarized in Tables 1 and 2. In addition, the graphs of the error functions $|u-u_8|$ and $|a-a_8|$ are plotted in Figure 1.

Table1. Absolute errors of u_n at $t=0.5$ for Example 4.

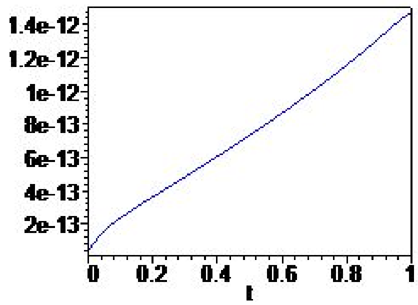
x	$\lvert u-u_2\rvert$	$\lvert u-u_4\rvert$	$\lvert u-u_6\rvert$	$\lvert u-u_8\rvert$	$\lvert u-u_{10}\rvert$
0.0	8.856E-4	5.966E-7	2.066E-10	4.234E-14	5.600E-18
1.5	8.856E-4	5.966E-7	2.066E-10	4.234E-14	5.600E-18
3.0	8.856E-4	5.966E-7	2.066E-10	4.234E-14	5.600E-18
4.5	8.856E-4	5.966E-7	2.066E-10	4.234E-14	5.600E-18
6.0	8.856E-4	5.966E-7	2.066E-10	4.234E-14	5.600E-18
7.5	8.856E-4	5.966E-7	2.066E-10	4.234E-14	5.600E-18
9.0	8.856E-4	5.966E-7	2.066E-10	4.234E-14	5.600E-18

Table 2: Absolute errors of a_n for Example 4.

t	$\lvert a-a_2\rvert$	$\lvert a-a_4\rvert$	$\lvert a-a_6\rvert$	$\lvert a-a_8\rvert$	$\lvert a-a_{10}\rvert$
0.2	4.026E-4	1.334E-7	1.765E-10	1.300E-13	1.001E-18
0.4	1.312E-3	6.817E-7	1.416E-10	1.575E-13	1.081E-18
0.6	1.024E-2	2.607E-5	2.651E-8	1.444E-11	1.896E-15
0.8	4.442E-2	3.425E-4	1.050E-6	1.725E-9	1.763E-12



a) $|u(x,t)-u_8(x,t)|$ on the $x\in[1,10]$ and $t\in[0,1]$.



b) $|a(t)-a_8(t)|$ on the $t\in[0,1]$.

Figure 1. Graph of absolute error by using VIM by for Example 4.

5. Conclusion

In the present work, we have demonstrated the applicability of the VIM for solving a class of parabolic inverse problem. The illustrative examples show the efficiency of the method. This method provides the solutions of the problems in closed form, Moreover, by using only one iteration step; we may get the exact solution. It can be concluded that the VIM is a very powerful and relatively easy tool for solving inverse heat problem.

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