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The Iterative Method for Optimal Control Problems by the Shifted Legendre Polynomials

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Abstract. In this paper an iterative method based on shifted Legendre polynomials is presented to obtain the approximate solutions of optimal control problems subject to integral equations. The operational matrices of integration and product of shifted Legendre polynomials for solving integral equation is employed. The methodology is based on the parametrization of control and state functions. This converted the problem to nonlinear optimization problem in any iteration. In addition, some numerical examples are presented to illustrate the accuracy and efficiency of the proposed method.

Keywords: Optimal control problem, Legendre polynomials, Iterative method.

1. Introduction

The classical theory of optimal control was developed in the last years as a powerful tool to create optimal solutions for real processes in many aspects of science and technology. Complexity of applying analytical methods for obtaining fast and near optimal solutions is the reason for creating numerical approaches. The numerical methods for solving optimal control problems described by ODE or integral equations can be found in [1]. Some interesting iterative schemes with their convergence for optimal control of Volterra integral equations considering some conditions on the kernel of integral equation was introduced in [2-4]. The method of parametrization for solving some classes of optimal control problems were proposed [5-7]. Lee and Chang [8] appeared to be the first to study optimal control problem of nonlinear systems using general orthogonal polynomials. Chebyshev polynomials [9] were used for solving nonlinear optimal control problem. A Fourier based state parameterizations approach for solving linear quadratic optimal control problems proposed in [10]. The hybrid functions consisting of the combination of block-pulse functions with Chebyshev polynomials [11], Legendre polynomials [12] and Taylor series [13, 14] shown to be a mathematical power tool for discretization of selected problems. Also some methods based on approximating the Volterra integral equation can be seen in [15, 16]. In this paper, we considered the numerical solution of a class of optimal control problems subject to integral equations, which is described by the following minimizing problem:

$$J(x,u)=\int_0^1 f\left(t,x(t),u(t)\right)dt, \tag{1}$$

subject to:

$$\mathrm{x(t)}=y(t)+\lambda_1\int_0^t K_1\left(t,s,x(s),u(s)\right)ds++\lambda_2\int_0^1 K_2\left(t,s,u(s)\right)x(s)\,ds, \tag{2}$$

where

$\lambda_1,\lambda_2\in\mathbb{R}$, $f\in C\left([0,1]\times\mathbb{R}\times\mathbb{R}\right)$ and $y(.)\in C^{+\infty}\left([0,1]\right)$ are given function. $x(.),\ u(.)\in C^{+\infty}\left([0,1]\right)$ are the state and control functions, respectively, which to be determined and the given kernel functions, $K_1(t,s,x(s),u(s))$ is smooth in $C^{+\infty}\left([0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\right)$ and $K_2(t,s,u(s))$ is smooth in $C^{+\infty}\left([0,1]\right)\times C^{+\infty}\left([0,1]\right)\times\mathbb{R}$. Here, we assume that the problem (1)- (2) has a unique solution. Due to the absence of an approximate numerical method for solving this kind of optimal control problems the main purpose of this article is to present a direct numerical method for obtaining approximate solutions of the Eqs.

(1) and (2) by using parametrization and shifted Legendre polynomials. The paper is organized as follows: in section 2, the shifted Legendre polynomials and their properties are presented. Section 3 is devoted to the solution method. In section 4, we reported our numerical findings and demonstrated the accuracy of the proposed method.

2. Properties of shifted Legendre polynomials

A set of shifted Legendre polynomials, denote by $\{L_k(t)\}$ for $k=0,1,2,\cdots$ is orthogonal with respect to the weighting function $w(t)=1$, over interval $[0,1]$, can be generated from the recurrence relation [17]

$$(\mathrm{k}+1)\,\mathrm{L}_{k+1}(t)=(2\,\mathrm{k}+1)(2\,\mathrm{t}-1)\,\mathrm{L}_k(t)-kL_{k-1}(t),\quad k=1,2,3,\cdots \tag{3}$$

With

$$L_0(t)=1,\quad L_1(t)=2t-1. \tag{4}$$

The orthogonality of these polynomials is expressed by the relation

$$\int_0^1 L_j(\mathrm{t})L_k(t)dt=\begin{cases}\dfrac{1}{2k+1}, & j=k\\ 0, & j\neq k\end{cases} \tag{5}$$

A function, $f(t)$ square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as

$$f(t)\simeq P_m(t)=\sum_{i=0}^{m}c_iL_i(t),\qquad m\geq 0 \tag{6}$$

Equation (6) can be written as

$$P_m(t)=C^TL(t) \tag{7}$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $L(t)$ are given by

$$C=\left[c_0,c_1,c_2,\cdots,c_m\right]^T, \tag{8}$$

$$L(t)=\left[L_0(t),L_1(t),L_2(t),\cdots,L_m(t)\right]. \tag{9}$$

The use of an orthogonal basis on $[0,1]$ allows us to directly obtain the least-squares coefficients of $P_m(t)$ in that basis, and also ensure permanence of these coefficients with respect to the degree m the approximate, that is, all coefficients of $P_{m+1}(t)$ agree with those of $P_m(t)$, except for that of the newly introduced term. By using Eq. (5) the Legendre coefficients are given by

$$c_j=(2j+1)\int_0^1 L_j(t)f(t)\,\mathrm{dt},\qquad \mathrm{j}=0,1,2,\cdots,\mathrm{m}. \tag{10}$$

Also, the integration of the cross, product of two vector $L(t)$ in Eq. (9) is

$$D=\int_0^1 L(t)L^T(t)\,\mathrm{dt}=diag\left(1,\frac{1}{3},\cdots,\frac{1}{2m+1}\right), \tag{11}$$

where, D is the $(m+1)\times(m+1)$ diagonal matrix.

The integrating of the vector $L(t)$ defined in Eq. (9) is given by

$$\int_0^1 L(x)\,\mathrm{d}\,x\simeq PL(t), \tag{12}$$

where P is the $(m+1)\times(m+1)$ operational matrix of integration of the shifted Legendre polynomials is given by [18], and it is a tridiagonal matrix.

$$P=\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{2m+1} & 0 & \frac{1}{2m+1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2m+1} & 0 \end{pmatrix} \tag{13}$$

The following property of the product of two Legendre function vectors is given by [12]

$$L(t)L^T(t)C \simeq \tilde{C}L(t), \tag{14}$$

where C is defined as Eq. (8) and $\tilde{C}$ is a $(m+1)\times(m+1)$ product operational matrix. To illustrate the calculation, we choose $m=2$, we have

$$\tilde{C}=\frac{1}{2}\begin{pmatrix} c_0 & c_1 & c_2 \\ \frac{1}{3}c_1 & c_0+\frac{2}{5}c_2 & \frac{2}{3}c_2 \\ \frac{1}{5}c_1 & \frac{2}{5}c_2 & c_0+\frac{2}{7}c_2 \end{pmatrix}. \tag{15}$$

3. The iterative method and numerical solution of optimal control problem

Let Q be the subset of the product space $C^\infty([0,1])\times C^\infty([0,1])$ contains all pairs $(x_m(.),u_n(.))$, which satisfy the Eq. (2). Also, let $Q_{m,n}$ be the subset of Q consisting of all pairs $(x_m(.),u_n(.))$, where $u_n(.)$ and $x_m(.)$ is a parameterized control and state functions as following polynomials:

$$u_n(t)=\sum_{i=0}^{n} a_i L_i(t)=A^T L(t), \tag{16}$$

$$x_m(t)=\sum_{j=0}^{m} b_j L_j(t)=B^T L(t), \tag{17}$$

where:

$$A^T=[a_0,a_1,\cdots,a_n] \;\; if \;\; n<m, \;\; a_{n+1}=0, \;\; i=1,2,\cdots,m-n,$$

$$B^T=[b_0,b_1,\cdots,b_m] \;\; if \;\; m<n, \;\; b_{m+1}=0, \;\; i=1,2,\cdots,n-m.$$

Suppose $y(t)$ can be expressed as

$$y(t)\simeq Y^T L(t), \tag{18}$$

the coefficients vector Y is known and can be calculated from Eq. (10). Replacing Eqs. (16), (17) and (18), into Eq. (2) we have:

$$B^T L(t)=Y^T L(t)+\lambda_1\int_0^t k_1(t,s,B^T L(s),A^T L(s))ds+\lambda_2\int_0^1 k_2(t,s,A^T L(s))B^T L(s)ds, \tag{19}$$

Eq. (19) reducing to a set of algebraic equations, find vector B corresponding vector A, where satisfying Eq. (17) therefore, we get:

$$x_m(t)=\sum_{j=0}^{m} b_j\left(a_0,a_1,\cdots,a_n\right)L_j(t), \tag{20}$$

Suppose, $\left(x_m^*(.), u_m^*(.)\right)$ be the solution minimizing J on $Q_{m,n}$, $m = 1, 2, \cdots$, $n = 1, 2, \cdots$ .Here, $b_j : \mathbb{R}^{n+1} \to \mathbb{R}$, $j = 0, 1, \cdots, m$ are continuous functions. Substituting Eqs. (16) and (20) into Eq. (1), $J(x_m, u_m)$ may be considered as a function $J\left(a_0, a_1, \cdots, a_n\right)$. The necessary conditions for the minimum are given by

$$\frac{\partial J}{\partial a_i} = 0, \quad i = 0, 1, 2, \cdots, n. \tag{21}$$

Equation (21) corresponds to a set of $n+1$ algebraic equations with the unknown $a_0, a_1, \cdots, a_n$ coefficients. Thus, the vector A can be obtained by solving Eq. (21) and then the unknown vector B will be obtained from Eq. (19). Also the optimal control and optimal state will be calculated from Eqs. (16) and (17), respectively. Finally, we can find cost function.

An algorithm on the basis of the above discussions is proposed here:

Algorithm

Choose $\varepsilon_1 > 0, \varepsilon_2 > 0$ for accuracy of the solution.

Step 1: Let $m = n = k = 1, u_1(t) = a_0 L_0(t) + a_1 L_1(t), x_1(t) = b_0 L_0(t) + b_1 L_1(t)$ and $\alpha_1 = J(x_1(.), u_1(.))$, where $b_0 = b_0(a_0, a_1)$ and $b_1 = b_1(a_0, a_1)$.

Step 2: Let $m \to m+1$ and $k \to k+1$ and find $\alpha_k = \inf_{Q_{m,n}} (j)$

Step 3: If $\left|\alpha_k - \alpha_{k-1}\right| < \varepsilon_1$ then go to step 4, otherwise go to step 2.

Step 4: Let $n \to n+1$ and $k \to k+1$ and go to step 4 otherwise go to step 2.

Step 5: If $\left|\alpha_k - \alpha_{k-1}\right| < \varepsilon_2$ then go to Step 4.

4. Numerical examples

Example 1 For the first example consider the optimal control problem of minimizing:

$$J(x,u) = \int_0^1 \left(x(t) - 1 + t\right)^2 + \left(u(t) - 1 - t\right)^2 dt, \tag{22}$$

subject to following nonlinear Fredholm integral equation:

$$x(t) = \frac{1}{3} - \frac{7}{6}t + \int_0^1 \left(st + u(s)\right) x(s) ds. \tag{23}$$

The exact optimal solution of Eqs. (22) and (23) are: $x^*(t) = 1 - t$ and $u^*(t) = 1 + t$, with optimal criteria $J\left(x^*(t), u^*(t)\right) = 0$.

Table 1: The estimated values of J in Example 2.

Iteration	*n*	*m*	*J in presented method*	*J in method of [18]*
1	1	1	0.00105437	0.0444
2	1	2	1.17808×10^{-5}	0.0107
3	1	3	2.15566×10^{-7}	1.5226×10^{-4}
4	2	2	1.16891×10^{-7}	7.8920×10^{-5}
5	2	3	1.3105×10^{-8}	2.5166×10^{-5}
6	2	4	1.3462×10^{-10}	1.0922×10^{-7}

For this example we have $\lambda_2 = 1, \lambda_1 = 0$; Eq. (23) can be written as

$$x(t) = \frac{1}{3} - \frac{7}{6}t + \left(\int_0^1 s x(s) ds\right) t + \int_0^1 u(s) x(s) ds, \tag{24}$$

substituting Eqs. (16), (17) and (18) into Eq. (24), we obtain

$$B^T = Y^T + F^T DBF^T + A^T DBE_1^T, \tag{25}$$

where

$$E_1^T = [0,1,\cdots,0], \ t = F^T L(t).$$

For $m = n = 1$, by solving Eq. (25), we have

$$b_0 = -\frac{-10-7a_1}{3(-8+11a_0+a_1)}, \ b_1 = -\frac{7}{11} - \frac{-10-7a_1}{11(-8+11a_0+a_1)}, \quad (26)$$

by replacing b_0 and b_1 into Eq. (17) we get

$$x_1(t) = -\frac{-10-7a_1}{3(-8+11a_0+a_1)} L_0(t) - \frac{7}{11} - \frac{-10-7a_1}{11(-8+11a_0+a_1)} L_1(t), \quad (27)$$

now substituting Eqs. (16) and (27) into Eq. (22) to optimization J we get

$$\frac{\partial J(a_0,a_1)}{\partial a_0} = -3 + 2a_0 - \frac{682(4-3a_0+a_1)^2}{9(-8+11a_0+a_1)^3} - \frac{62(4-3a_0+a_1)}{3(-8+11a_0+a_1)^2} = 0, \quad (28)$$

$$\frac{\partial J(a_0,a_1)}{\partial a_1} = -\frac{1}{3} + \frac{2a_1}{3} - \frac{62(4-3a_0+a_1)^2}{9(-8+11a_0+a_1)^3} + \frac{62(4-3a_0+a_1)}{9(-8+11a_0+a_1)^2} = 0, \quad (29)$$

Solving Eqs. (28) and (29) we have $a_0 = 1.5, \ a_1 = 0.5.$

Substituting the values a0 and a1 into Eq. (26), we obtain: $b_0 = 0.5, \ b_1 = -0.5.$, by applying the method of section (3) on this example leads to exact solutions. Therefore $x_1(t) = 1-t, \ u_1(t) = 1+t$ and $J = 0.$

Example 2 For the second example consider the following optimal control problem [18]

$$J(x,u) = \int_0^1 \left(x(t)-1+t\right)^2 + \left(u(t)-1-t\right)^2 dt, \quad (30)$$

subject to the following nonlinear Volterra integral equation:

$$x(t) = y(t) + \int_0^t u^2(s)\left(x(s)+ts\right) ds, \quad (31)$$

where

$$y(t) = (1-2t)\cos(t) - \left(t^2-2\right)\sin(t) - \frac{t^5}{4},$$

Table 2: The exact and approximate values of control function for Example 2.

t	Approximate with $n=1, m=1$	Approximate with $n=1, m=2$	Approximate with $n=1, m=3$	Exact values
0	-9.627183×10^{-4}	-9.27183×10^{-4}	1.20528×10^{-6}	0.0
0.1	0.0987431	0.0995164	0.100018	0.1
0.2	0.198456	0.19966	0.200024	0.2
0.3	0.298169	0.299804	0.300029	0.3
0.4	0.397882	0.39947	0.400035	0.4
0.5	0.497595	0.500091	0.500041	0.5
0.6	0.597308	0.600234	0.600047	0.6
0.7	0.697021	0.700234	0.700053	0.7
0.8	0.796734	0.800522	0.800058	0.8
0.9	0.896446	0.900665	0.900064	0.9
1	0.996159	1.00081	1.000070	1

the exact optimal solution of Eqs. (30) and (31) are: $x^*(t) = \cos(t)$ and $u^*(t) = t$, with optimal criteria $J\left(x^*, u^*\right) = 0.$

Eq. (31) can be expressed approximately as:

$$B^T = Y^T + A^T \tilde{A}\tilde{B}P + A^T \tilde{A}\tilde{F}P\tilde{F}. \tag{32}$$

where $\tilde{A}$ calculated from Eq. (14).

In Table 1, we list the approximate values of the cost function J in any iteration and comparison between proposed method and the method given in [18]. Tables 2 and 3 show the comparison between exact and approximate solutions in any iteration.

Table 3: The exact and approximate values of state function for Example 2.

t	Approximate with $n=1, m=1$	Approximate with $n=1, m=2$	Approximate with $n=1, m=3$	Exact values
0	1.07296	0.995791	1.00057	1
0.1	1.02715	0.991653	0.995155	0.995004
0.2	0.981348	0.978099	0.979767	0.980067
0.3	0.935544	0.955129	0.954748	0.955336
0.4	0.889739	0.922741	0.920433	0.921061
0.5	0.843935	0.880937	0.877194	0.877583
0.6	0.798131	0.829715	0.825346	0.825336
0.7	0.759326	0.769077	0.765242	0.764842
0.8	0.706522	0.699023	0.697225	0.696707
0.9	0.660717	0.619551	0.621636	0.62161
1	0.614913	0.530663	0.538828	0.540302

Table 4: The estimated values of J for Example 3

Iteration	*n*	*m*	*J*	*CPU Time*
1	1	1	4.5574×10^{-10}	4.556
2	1	2	2.666362×10^{-18}	4.867

Table 5: The absolute difference between exact and approximate solutions, for Example 3 with n=1, m=2.

t	$\lvert x_{Exact}(t) - x_{approximate}(t)\rvert$	$\lvert u_{Exact}(t) - u_{approximate}(t)\rvert$
0	4.30244×10^{-3}	3.00704×10^{-12}
0.1	2.2157×10^{-3}	4.23794×10^{-12}
0.2	5.55634×10^{-4}	5.4689×10^{-12}
0.3	6.77745×10^{-4}	6.69981×10^{-12}
0.4	1.48444×10^{-3}	7.93074×10^{-12}
0.5	1.86446×10^{-3}	9.16167×10^{-12}
0.6	1.81779×10^{-3}	1.03926×10^{-11}
0.7	1.34444×10^{-3}	1.6235×10^{-11}
0.8	4.4441×10^{-4}	1.28544×10^{-11}
0.9	8.82303×10^{-4}	1.40854×10^{-11}
1	2.6357×10^{-3}	1.53163×10^{-11}

Example 3 Consider the optimal control problem of minimizing:

$$J(x,u) = \int_0^1 \left(x(t)-t\right)^2 + \left(u(t)-t\right)^2 dt, \tag{33}$$

subject to following nonlinear Volterra-Fredholm integral equation:

$$x(t) = \frac{3t}{4} + \frac{t^5}{4} + \int_0^t stu(s)x(s)ds + \int_0^1 stu(s)x(s)ds, \tag{34}$$

the exact optimal solutions of Eqs. (33) and (34) are $x^*(t) = t$ and $u^*(t) = t$, with optimal criteria $J\left(x^*, u^*\right) = 0.$

The approximate solutions is obtained by using the method in section (3). In Table 4 the approximate values of the cost function J in any iteration is listed. Table 5 shows the absolute difference between exact and approximate solutions, as we consider from Table 5, the maximum error of state function and control function, for the presented method are 10^{-3} and 10^{-11} respectively, with $n=1$ and $m=2$. Figure 1, displays the comparison between exact and approximate solutions with $n=1$ and $m=2$.

Example 4 Consider the optimal control problem of minimizing:

$$J(x,u)=\int_0^1 \left|x(t)+u(t)-t^3-t^2-2\right| dt, \tag{35}$$

subject to following nonlinear Fredholm integral equation:

$$x(t)=\frac{7}{4}+\frac{8}{15}-t^2+\int_0^1 ((s^2-1)t-u(s))x(s)ds, \tag{36}$$

the exact optimal solutions of Eqs. (35) and (36) are $x^*(t)=1-t^2$ and $u^*(t)=1+t^3$, with optimal criteria $J\left(x^*,u^*\right)=0.$

By using the method in section 3, in Table 6 we listed the approximate values of the cost function J in any iteration. Figure 2, displays the maximum error $x(t)$ and $u(t)$ for example 4 with $n=3$ and $m=3$. As you see from this figure, the maximum error of state function and control function, for the presented method are 10^{-16}.

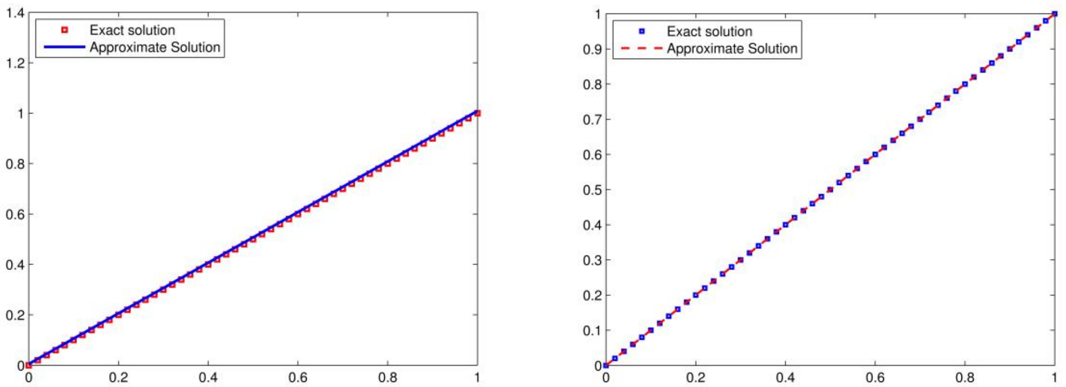

Fig. 1: The exact and approximate state function (left) and control function (right) for Example 3 with $n=1$, $m=2$.

Table 6: The estimated values of J for Example 4

Iteration	*n*	*m*	*J*	*CPU Time*
1	1	1	0.302025	1.997
2	1	2	1.28674×10^{-2}	2.02
3	2	2	3.57143×10^{-4}	2.94
4	3	3	6.93889×10^{-17}	3.59

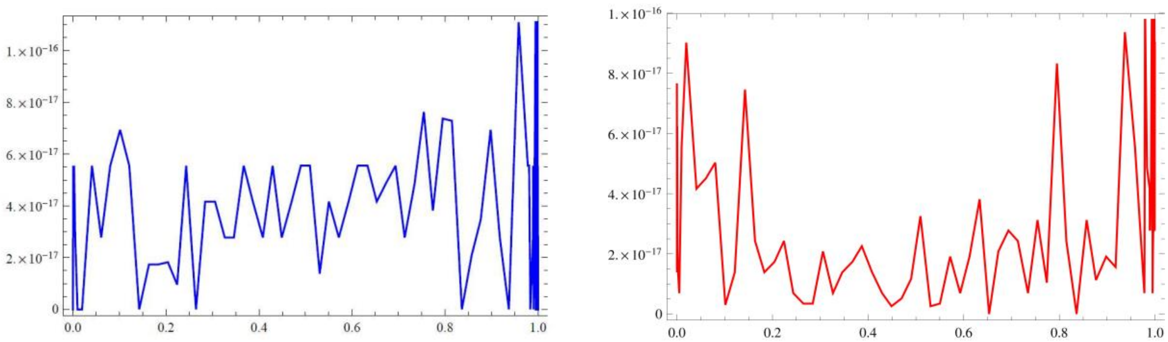

Fig. 2: The absolute difference between exact and approximate solutions of $x(t)$ (left) and $u(t)$ (right) for Example 4.

Table 7: The estimated values of J in Example 5.

Iteration	*n*	*m*	*J in presented method*	*J in method of [18]*
1	1	1	2.34138×10^{-4}	2.1070×10^{-4}
2	1	2	1.6288×10^{-7}	2.385×10^{-7}
3	1	3	3.2183×10^{-8}	1.9795×10^{-8}
4	2	2	5.96275×10^{-10}	1.4383×10^{-8}
5	2	3	3.4351×10^{-12}	4.2201×10^{-10}

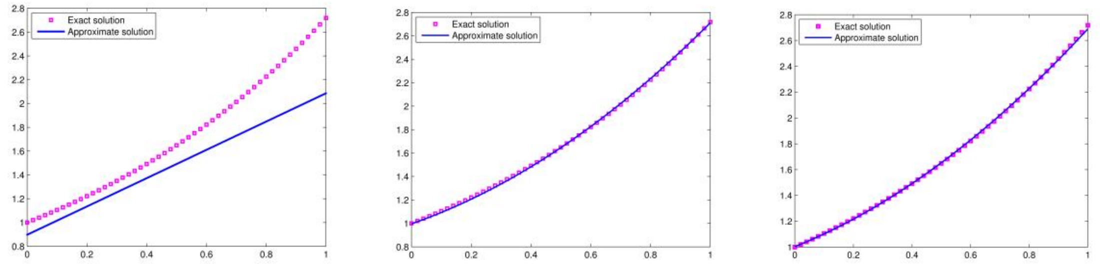

Fig. 3: The exact and approximate of $x(t)$ for Example 5 with $(n=1, m=1)$, $(n=2, m=1)$, $(n=2, m=2)$, respectively.

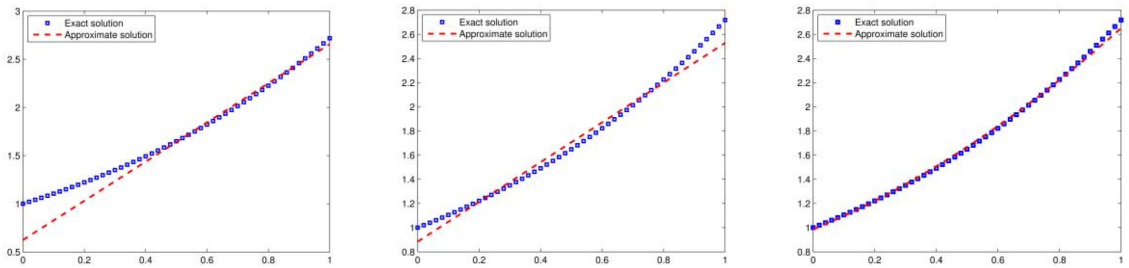

Fig. 4: The exact and approximate of $u(t)$ for Example 5 with $(n=1, m=1)$, $(n=2, m=1)$, $(n=2, m=2)$, respectively.

Example 5 Consider the optimal control problem of minimizing [18]

$$J(x,u)=\int_0^1 \left(x(t)-e^t\right)^2 \left(u(t)-e^t\right)^2 dt, \tag{37}$$

subject to following nonlinear Volterra integral equation:

$$x(t)=y(t)+\int_0^t u(s)\left(x(s)+t\right)ds, \tag{38}$$

where $y(t)=e^t(1-t-\frac{1}{2}e^t)+t+\frac{1}{2}$, the exact optimal solution of Eqs. (37) and (38) are: $x^*(t)=e^t$ and $u^*(t)=e^t$, with optimal criteria $J\left(x^*,u^*\right)=0.$

The approximate solutions is obtained by using the method in section 3, with $n=2$ and $m=2$. Table 7 shows the comparison between results for J the proposed method and the method given in [18]. Figures 3 and 4 show the comparison between exact and approximate solutions in any iteration for this example.

5. Conclusion

In this article, we have proposed a numerical scheme for finding approximate solution of optimal control problems. The method is based upon reducing the solution of integral equation into a set of algebraic equations by expanding shifted Legendre polynomials with unknown coefficients in any iteration. Then by using the parametrization the coefficients of state function was obtained to correspond with coefficients of control function. Comparing with expanding Taylor, the results of numerical examples demonstrated that this method by using shifted Legendre polynomials is more accurate than Taylor polynomials. Also we suggested the procedure which is simple and effective and some numerical results showed that the given scheme can produce the approximate solutions with high precision able to compare the results with their exact solutions.

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