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COMPARATIVE STUDY OF FINITE ELEMENT AND HAAR WAVELET COLLOCATION METHOD FOR THE NUMERICAL SOLUTION OF PARABOLIC TYPE PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we present the comparative study of Haar wavelet collocation method (HWCM) and Finite Element Method (FEM) for the numerical solution of parabolic type partial differential equations such as 1-D singularly perturbed convection-dominated diffusion equation and 2-D Transient heat conduction problems validated against exact solution. The distinguishing feature of HWCM is that it provides a fast converging series of easily computable components. Compared with FEM, this approach needs substantially shorter computational time, at the same time meeting accuracy requirements. It is found that higher accuracy can be attained by increasing the level of Haar wavelets. As Consequences, it avoids more computational costs, minimizes errors and speeds up the convergence, which has been justified in this paper through the error analysis.

Keywords: Haar wavelet collocation method, parabolic equation, Finite difference method, Finite element method, Heat conduction problems.

1. Introduction

Differential equations have several applications in many fields such as: physics, fluid dynamics and geophysics etc. Many reaction–diffusion problems in biology and chemistry are modeled by partial differential equations (PDEs). These problems have been extensively studied by many authors, like Singh and Sharma [1], Giuseppe and Filippo [2] in their literature and their approximate solutions have been accurately computed provided the diffusion coefficients, reaction excitations, initial and boundary data are given in a deterministic way. However, it is not always possible to get the solution in closed form and thus, many numerical methods come into the picture. These are Finite Difference, Spectral, Finite Element and Finite Volume Methods and so on to handle a variety of problems. Many researchers such as, Kadalbajoo and Awasti [3], F. de Monte [4] are involved in developing various numerical schemes for finding solutions of heat conduction problems appear in many areas of engineering and science. So, finding out flexible techniques for generating the solutions of such PDEs is quite meaningful. Researchers like Medvedskii and Sigunov [5] and Doss et.al [6] have used different techniques to solve the above problems and similar ones. Singularly perturbed problems appear in many branches of engineering, such as fluid mechanics, heat transfer, and problems in structural mechanics posed over thin domains. Theorems that list conditions for the existence and uniqueness of solutions of such problems are thoroughly discussed by Ross et.al [7] and Gamel [8].

The application of FEM to various heat conduction problems began through a paper by Zienkienicz and Cheung in 1965 [9]. Subsequently, Wilson and Nickel[10] have studied time dependent finite element with variational principle in their work on transient heat conduction problems with Gurtin's Variational principle [11]. Zienkienicz and Parekh [12] derived isoparametric finite element formulations for 2-D and 3-D transient heat conduction problems to approximate the solution in space with recursion process of the solution in time. Argyris et. al [13, 14] analyzed structural problems by using real time-space finite elements. A parabolic time-space element, an unconditionally stable in the solution of heat conduction problems through a quasivariational approach was used by Tham and Cheung [15]. Wood and Lewis [16] compared the heat equations for different time-marching schemes. However, it is necessary to choose very small time-steps in order to overcome unwanted numerically induced oscillations in the solution.

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From the past few years, wavelets have become very popular in the field of numerical approximations. Among the different wavelet families mathematically most simple are the Haar wavelets. Due to the simplicity the Haar wavelets are very effective for solving ordinary and partial differential equations. In the previous years, many researchers like Bujurke & Shiralashetti et al. [17, 18, and 19], Hariharan & Kannan [20] have worked with Haar wavelets and their applications. In order to take the advantages of the local property, Chen and Hsiao [21], Lepik [22, 23] researched the Haar wavelet to solve the differential and integral equations. Haar wavelet collocation method (HWCM) with far less degrees of freedom and with smaller CPU time provides better solutions than classical ones, see Islam et.al. [24]. In the present work, we use HWCM for solving typical heat conduction problems.

The present paper is organized as follows; Haar wavelets and its generalized operational matrix of integration are presented in section 2. In section 3and 4 Method of solution of FEM and HWCM are discussed respectively. Section 5 deals with the numerical findings with error analysis of the examples. Finally, conclusion of the proposed work is presented in section 6.

2. Haar wavelets and operational matrix of integration

The scaling function $h_1(x)$ for the family of the Haar wavelets is defined as

$$h_1(x) = \begin{cases} 1 & for \quad x \in [0,1) \\ 0 & Otherwise \end{cases} \tag{2.1}$$

The Haar wavelet family for $x \in [0,1)$ is defined as

$$h_i(x) = \begin{cases} 1 & for \quad x \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ -1 & for \quad x \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0 & Otherwise \end{cases} \tag{2.2}$$

In the above definition the integer $m = 2^l$, $l = 0,1,\ldots,J$, indicates the level of resolution of the wavelet and integer $k = 0,1,\ldots,m-1$ is the translation parameter.

Maximum level of resolution is J. The index i in Eq. (2.2) is calculated using $i = m+k+1$. In case of minimal values $m=1,\ k=0$ then $i=2$. The maximal value of i is $N = 2^{J+1}$.

Let us define the collocation points $x_j = \frac{j-0.5}{N}$, $j = 1,2,\ldots,N$, discretize the Haar function $h_i(x)$, in this way, we get Haar coefficient matrix $\mathrm{H}(i,j) = \mathrm{h}_i(x_j)$, which has the dimension $N \times N$. For instance, $J = 3 \Rightarrow N = 16$, then we have

$$H(16,16)=\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1\\
1&1&1&1&-1&-1&-1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&-1&-1&-1&-1\\
1&1&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&-1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&-1&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&-1&-1\\
1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1
\end{pmatrix}$$

The operational matrix of integration via Haar wavelets is obtained by integrating (2.2) is as,

$$Ph_i=\int_0^x h_i(x)\,dx \tag{2.3}$$

and

$$Qh_i=\int_0^x Ph_i(x)\,dx \tag{2.4}$$

These integrals can be evaluated by using equation (2.2) and they are given by

$$Ph_i(x)=\begin{cases} x-\frac{k}{m} & for \quad x\in\left[\frac{k}{m},\frac{k+0.5}{m}\right) \\ \frac{k+1}{m}-x & for \quad x\in\left[\frac{k+0.5}{m},\frac{k+1}{m}\right) \\ 0 & Otherwise \end{cases} \tag{2.5}$$

and

$$Qh_i(x)=\begin{cases} \frac{1}{2}\left(x-\frac{k}{m}\right)^2 & for \quad x\in\left[\frac{k}{m},\frac{k+0.5}{m}\right) \\ \frac{1}{4m^2}-\frac{1}{2}\left(\frac{k+1}{m}-x\right)^2 & for \quad x\in\left[\frac{k+0.5}{m},\frac{k+1}{m}\right) \\ \frac{1}{4m^2} & for \quad x\in\left[\frac{k+1}{m},1\right) \\ 0 & Otherwise \end{cases} \tag{2.6}$$

For instance, $J=3 \Rightarrow N=16$, from (2.5) then we have

$$Ph(16,16)=\frac{1}{32}\begin{pmatrix}
1&3&5&7&9&11&13&15&17&19&21&23&25&27&29&31\\
1&3&5&7&9&11&13&15&15&13&11&9&7&5&3&1\\
1&3&5&7&7&5&3&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&3&5&7&7&5&3&1\\
1&3&3&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&3&3&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&3&3&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&3&3&1\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1
\end{pmatrix}$$

and from (2.6) we get

$$Qh(16,16)=\frac{1}{2048}\begin{pmatrix}
1&9&25&49&81&121&169&225&289&361&441&529&625&729&841&961\\
1&9&25&49&81&121&169&225&287&343&391&431&463&487&503&511\\
1&9&25&49&79&103&119&127&128&128&128&128&128&128&128&128\\
0&0&0&0&0&0&0&0&1&9&25&49&79&103&119&127\\
1&9&23&31&32&32&32&32&32&32&32&32&32&32&32&32\\
0&0&0&0&1&9&23&31&32&32&32&32&32&32&32&32\\
0&0&0&0&0&0&0&0&1&9&23&31&32&32&32&32\\
0&0&0&0&0&0&0&0&0&0&0&0&1&9&23&31\\
1&7&8&8&8&8&8&8&8&8&8&8&8&8&8&8\\
0&0&1&7&8&8&8&8&8&8&8&8&8&8&8&8\\
0&0&0&0&1&7&8&8&8&8&8&8&8&8&8&8\\
0&0&0&0&0&0&1&7&8&8&8&8&8&8&8&8\\
0&0&0&0&0&0&0&0&1&7&8&8&8&8&8&8\\
0&0&0&0&0&0&0&0&0&0&1&7&8&8&8&8\\
0&0&0&0&0&0&0&0&0&0&0&0&1&7&8&8\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&7
\end{pmatrix}$$

also

$Ch_i=\int_0^1 Ph_i(x)\,dx$ and for instance $J=3 \Rightarrow N=16$, then we have

$$Ch(16,16)=\frac{1}{256}\begin{pmatrix}
128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128\\
128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 64 & 64 & 64 & 64 & 64 & 64 & 64 & 64\\
128 & 128 & 128 & 128 & -16 & -16 & -16 & -16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 16\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 32 & 32 & 32 & 16 & 16 & 16 & 16\\
128 & 128 & -68 & -68 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4\\
0 & 0 & 0 & 0 & 72 & 72 & -28 & -28 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 32 & -4 & -4 & 4 & 4 & 4 & 4\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 4 & 4\\
128 & -97 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 98 & -71 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 72 & -49 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 50 & -31 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & -17 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18 & -7 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & -1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1
\end{pmatrix}$$

3. Finite Element Method for the Numerical Solution of Parabolic equations

Case (i) FEM in one dimension

The equation can be written with the given conditions

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+cu+c_1\frac{\partial u}{\partial t}=f(x,t)\quad in\,\Omega:0<x<1 \tag{3.1}$$

To formulate a FEM model of the governing differential equation, the domain $\Omega=(0,1)$ is divided into M (=2N) elements. A Typical element $\Omega=(x_a,x_b)$ where x_a,x_b are the global coordinates of the end nodes of the element. We begin with the weak formulation by multiplying the given equation with the test function w, we get

$$0=\int_{x_a}^{x_b} w\left[-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+c_0u+c_1\frac{\partial u}{dt}-f\right]dx \tag{3.2}$$

$$0=\int_{x_a}^{x_b}\left[a\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}+c_0wu+c_1w\frac{\partial u}{dt}-f\right]dx-w(x_a)-w(x_b) \tag{3.3}$$

We assume finite element solution in the form

$$u(x,t_s)=\sum_{j=1}^{n}u_j^sL_j(x) \tag{3.4}$$

Where t_s is the initial time and $\Delta t=t-t_s$ is the time interval and $L_j(x)$, for two linear elements

i.e. $n=2\Rightarrow j=1\,\&\,2$, $L_1(x)=1-\frac{x}{h}$ & $L_2(x)=\frac{x}{h}$.

The finite element solution which is continuous at space is obtained as

$$u(x,t)=\sum_{j=1}^{n}u_j(t_s)L_j(s)=\sum_{j=1}^{n}u_jL_j(x),$$

In matrix form, we get

$$[K]\{u\}+[M^1]\{\dot{u}\}=\{F\} \tag{3.5}$$

where $[K]=[K^1]+[M^0]$

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$$M_{ij}^{0} = \int_{x_a}^{x_b} c_0 L_i L_j dx, \quad M_{ij}^{1} = \int_{x_a}^{x_b} c_1 L_i L_j dx, \tag{3.6}$$

$$K_{ij}^{1} = \int_{x_a}^{x_b} a \frac{dL_i}{dx}\frac{dL_j}{dx} dx \, , F_i = \int_{x_a}^{x_b} L_i L_j dx + Q_i \tag{3.7}$$

The weak formulation is a variational statement of the given problem in which it is integrated against a test function, and hence after discretization, resulting matrices can be easily solved.

Discretization:

Rewritting the finite element model in the matrix form (3.5) can be rewritten as

(By taking $\left[M^1\right] = \left[M\right] = \left[M^0\right], \left[K^1\right] = \left[K\right]$)

$$\left[K\right]\left\{u\right\}_{t_s} + \left[M\right]\left\{\dot{u}\right\}_{t_s} = \left\{F\right\}_{t_s} \tag{3.8}$$

where

$$M_{ij} = \int_{x_a}^{x_b} L_i L_j dx, \quad K_{ij} = \int_{x_a}^{x_b} \frac{dL_i}{dx}\frac{dL_j}{dx} dx \tag{3.9}$$

The semidiscrete equations of a typical element for the choice of the linear interpolation functions are

$$\frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{3.10}$$

where h is the length of the element.

For different difference (i.e. forward, backward and Crank-Nicolson) schemes, general form of α -family of the approximation is given by

$$\left(\left[M\right] + \alpha\Delta t\left[K\right]\right)\left\{\dot{u}\right\}_{t_s+\Delta t} = \left(\left[M\right] - (1-\alpha)\Delta t\left[K\right]\right)\left\{u\right\}_{t_s} + \Delta t\left(\alpha\left\{F\right\}_{t_s+\Delta t} + (1-\alpha)\left\{F\right\}_{t_s}\right) \tag{3.11}$$

Where Δt is the time step and t_s is the initial time, and

$$\left[K\right]_{t_s+\Delta t} = \left[M\right] + b_1\left[K\right]_{t_s+\Delta t}, \quad \left[K\right]_{t_s} = \left[M\right] - b_2\left[K\right]_{t_s} \tag{3.12}$$

$$\left\{F\right\}_{t_s, t_s+\Delta t} = \Delta t\left[\alpha\left\{F\right\}_{t_s+\Delta t} + (1-\alpha)\left\{F\right\}_{t_s}\right], \; b_1 = \alpha\Delta t_{t_s+\Delta t}, \; b_2 = (1-\alpha)\Delta t_{t_s} \tag{3.13}$$

Here we used Backward difference scheme to approximate the solution with α =1 and which is stable and order of accuracy is $O(\Delta t)$.

For M = 2-Element model, the $\alpha-$family of time approximation schemes are put in the matrix form as

$$\begin{bmatrix} \frac{h}{3} + \alpha\Delta t\left(\frac{1}{h} - \frac{h}{3}\right) & \frac{h}{6} - \alpha\Delta t\left(\frac{1}{h} + \frac{h}{6}\right) & 0 \\ \frac{h}{6} - \alpha\Delta t\left(\frac{1}{h} + \frac{h}{6}\right) & 2\left(\frac{h}{3} + \alpha\Delta t\left(\frac{1}{h} - \frac{h}{3}\right)\right) & \frac{h}{6} - \alpha\Delta t\left(\frac{1}{h} + \frac{h}{6}\right) \\ 0 & \frac{h}{6} - \alpha\Delta t\left(\frac{1}{h} + \frac{h}{6}\right) & \frac{h}{3} + \alpha\Delta t\left(\frac{1}{h} - \frac{h}{3}\right)\frac{\Delta t}{h} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_{t_s+\Delta t} =$$

$$\begin{bmatrix} \frac{h}{3} - (1-\alpha)\Delta t\left(\frac{1}{h} - \frac{h}{3}\right) & \frac{h}{6} + (1-\alpha)\Delta t\left(\frac{1}{h} + \frac{h}{6}\right) & 0 \\ \frac{h}{6} + (1-\alpha)\Delta t\left(\frac{1}{h} + \frac{h}{6}\right) & 2\left(\frac{h}{3} - (1-\alpha)\Delta t\left(\frac{1}{h} - \frac{h}{3}\right)\right) & \frac{h}{6} + (1-\alpha)\Delta t\left(\frac{1}{h} + \frac{h}{6}\right) \\ 0 & \frac{h}{6} + (1-\alpha)\Delta t\left(\frac{1}{h} + \frac{h}{6}\right) & \frac{h}{3} - (1-\alpha)\Delta t\left(\frac{1}{h} - \frac{h}{3}\right) \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_{t_s} + \Delta t\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_{t_s} \tag{3.14}$$

FEM consistency, accuracy and stability:

The equation (3.11) represents an α -family of approximation, error is in the solution $\{u\}_{t_s+\Delta t}$ at each time step. If the error is bounded, the solution scheme is said to be stable. The numerical scheme is said to be consistent, if the round off and truncation error tends to zero as $\Delta t \to 0$. The size of the time step will control both accuracy and stability. The numerical solution converges to the exact solution when the numbers of elements are increased and time step Δt is decreased. If the numerical scheme is stable and consistent, it is also convergent.

Case (i) FEM in Two dimensions

The governing equation for transient heat conduction problems with a distributed source $F(x, y,t)$ may be given by

$$K\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+F=\rho c\frac{\partial u}{\partial t} \qquad (in\ \Omega) \tag{3.15}$$

Subjected to

$$u(x, y,0) = u_0(x, y) \quad \left(in\ \overline{\Omega}\right) \tag{3.16}$$

Where $u(x, y,t)$ is the temperature function, u_0 is initial temperature field, F the specified thermal conductivity, ρ the density, c the specific heat, $\overline{\Omega} = \Omega \cup \partial\Omega$, Ω is a bounded domain with a boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with the following conditions

$$u = u_s \ \ (\text{On}\,\Gamma_1) \tag{3.17}$$

$$-K\frac{\partial u}{\partial n} = q \ \ (\text{On}\,\Gamma_2) \tag{3.18}$$

$$-K\frac{\partial u}{\partial n} = h\left(u-u_a\right) \ \ (\text{On}\,\Gamma_3) \tag{3.19}$$

Where u_s is boundary surface temperature, q is the intensity of heat input, h the heat transfer coefficient, u_s, q and h known functions, n is the outward normal vector of the boundary surface, and u_a the environmental temperature.

Time-domain discretization:

Integrating the field equation (3.15) w r t 't' and using condition (3.16), we obtain

$$\rho u(x, y,t) = \rho c u_0(x, y) + \int_0^t \left[K\nabla^2 u(x, y.\tau)+F\right]d\tau \tag{3.20}$$

The Integral equation cannot be calculated analytically, so to approximate the temperature $u(x, y,t)$ by given functions, divide the time domain $\left[0,T\right]$ into M equal intervals $\left[t_m, t_{m+1}\right]$, where T is a given time. We can approximate $u(x, y,t)$ as a linear function of time variables as

$$u(x, y,t) \approx u_m(x, y) + \left[u_{m+1}\left(x, y\right) - u_m\left(x, y\right)\right]\left(t-t_m\right)/\nabla t \tag{3.21}$$

Where $\nabla t = \dfrac{T}{M}$

Putting (3.21) into (3.20), we get

$$\rho c u(x, y,t) = \rho c u_m(x, y) + \int_0^t \left[K\nabla^2\{u_m(x, y)+\left[u_{m+1}(x, y)-u_m(x, y)\right](\tau - t_m)/\nabla t\} + F(x, y,\tau)\right]d\tau \tag{3.22}$$

Let $t = t_{m+1}$ then equation (3.22) becomes

$$\left(k_0 K\nabla^2 - \rho c\right)u_{m+1} = -F_m(x, y) \qquad \left(m = 0,1,2,..., M-1\right) \tag{3.23}$$

Where $k_0 = \Delta t/2$ and

$$F_m(x,y)=\int_{t_m}^{t_{m+1}} F(x,y,t)dt+\nabla tK\nabla^2 u_m(x,y) \tag{3.24}$$

Hence, the related boundary conditions become;

$$u_{m+1}(x,y)=u_s(x,y,t_{m+1}) \qquad (\text{On}\,\Gamma_1) \tag{3.25}$$

$$-K\frac{\partial u_{m+1}}{\partial n}=q(x,y,t_{m+1}) \qquad (\text{On}\,\Gamma_2) \tag{3.26}$$

$$-K\frac{\partial u_{m+1}}{\partial n}=h\left[u_{m+1}-u_a(x,y,t_{m-1})\right] \quad (\text{On}\,\Gamma_3) \tag{3.27}$$

Finite element formulation:

The finite element formulation related to (3.23) to (3.27) is based on an extended variational principle. It can be stated as

$$\pi(u_m)=\left(\frac{1}{2}\right)\left\{\iint_\Omega\left[k_0K\left(\left(\frac{\partial u_m}{\partial x^2}\right)^2+\left(\frac{\partial u_m}{\partial y^2}\right)^2+\rho c u_m^2\right)dx\,dy+k_0\int_{\Gamma_1} hu^2{}_m ds\right]\right\}$$
$$+k_0\int_{\Gamma_2} q(x,y,t_m)u_m ds-k_0\int_{\Gamma_3} hu_m(x,y,t_m)u_m ds-\int_\Omega \bar{F}_{m-1}u_m\,dx\,dy=stationary \tag{3.28}$$

The Finite Element method is applied to obtain the numerical solution of (3.28). For this the domain Ω is divided into a number of elements. For each element, the unknown function u_m may be obtained by

$$u_m=\sum_{i=1}^{N} N_i(x,y)u_m^i \tag{3.29}$$

Where N_i is the shapes function, u_m^i the nodal value of $u_m(x,y)$ in the element, N is the number of nodes in an element. In this work, a 4-node quadrilateral element is used and N_i is a linear function of x and y.

Substituting (3.29) in (3.28), we get

$$\pi(u_m)=\sum_e\left[(1/2)\{u_m\}_e^T K^e\{u_m\}_e-\{u_m\}_e^T G^e\right]$$

Where e is the element number, K^e is the stiffness matrix and G^e the equivalent nodal force vector, which gives to

$$K^e=\iint_{\Omega_e}\left[k_0K\left\{\left(\frac{\partial N}{\partial x}\right)^T\left(\frac{\partial N}{\partial x}\right)+\left(\frac{\partial N}{\partial y}\right)^T\left(\frac{\partial N}{\partial y}\right)\right\}+\rho c N^T N\right]dx\,dy+\int_{\Gamma_3^e} hN^T ds$$

$$G^e=-k_0\int_{\Gamma_2^e} N^T q\,ds+k_0\int_{\Gamma_3^e} hN^T u_a\,ds-\iint_{\Omega_e} N^T\bar{Q}_{m-1}\,dx\,dy$$

Where $\Gamma_1^e=\Gamma^e\cap\Gamma_1,\quad \Gamma_2^e=\Gamma^e\cap\Gamma_2,\quad \Gamma_3^e=\Gamma^e\cap\Gamma_3,$

Here $\Gamma^e=\partial\Omega_e$ denotes the entire boundary of element e.

4. Haar wavelet collocation method for the numerical solution of parabolic equations

Consider the parabolic equation of the form (3.1) with the given conditions, let us assume that

$$\dot{u}''(x,t)=\sum_{i=1}^{N} a_i h_i(x) \tag{4.1}$$

where a_i's, $i=1,2,\ldots,N$ are Haar coefficients to be determined and $\cdot$ & $'$ are differentiations with respect to t & x respectively.

Integrating the equation (4.1) w. r. t. t from t_s to t, we get

$$u''(x, t) = (t - t_s)\sum_{i=1}^{N} a_i h_i(x) + u''(x, t_s) \tag{4.2}$$

Where t_s is the initial time and $\Delta t = t - t_s$ is the time interval

Integrating the equation (4.2) twice w. r. t. x from 0 to x, we get

$$u'(x, t) = \Delta t\sum_{i=1}^{N} a_i Ph_i(x) + u'(x, t_s) - u'(0, t_s) + u'(0, t) \tag{4.3}$$

$$u(x, t) = \Delta t\sum_{i=1}^{N} a_i Qh_i(x) + u(x, t_s) - u(0, t_s) - xu'(0, t_s) + xu'(0, t) + u(0, t) \tag{4.4}$$

Put $x = 1$ in (4.4) and by using given conditions we get

$$u'(0, t) - u'(0, t_s) = g_2(t) - \Delta t\sum_{i=1}^{N} a_i Ch_i(x) - g_2(t_s) + g_1(t_s) - g_1(t)$$

Then equation (4.4) becomes

$$\begin{aligned} u(x, t) = {} & \Delta t\sum_{i=1}^{N} a_i Qh_i(x) + u(x, t_s) - g_1(t_s) + g_1(t) + \\ & x\left(g_2(t) - \Delta t\sum_{i=1}^{N} a_i Ch_i(x) - g_2(t_s) + g_1(t_s) - g_1(t)\right) \end{aligned} \tag{4.5}$$

Differentiating equation (4.5) w. r. t. t then we have

$$\begin{aligned} \dot{u}(x, t) = {} & \sum_{i=1}^{N} a_i Qh_i(x) + \dot{u}(x, t_s) - \dot{g}_1(t_s) + \dot{g}_1(t) + \\ & x\left(\dot{g}_2(t) - \sum_{i=1}^{N} a_i Ch_i(x) - \dot{g}_2(t_s) + \dot{g}_1(t_s) - \dot{g}_1(t)\right) \end{aligned} \tag{4.6}$$

Substituting the expressions of (4.2)-(4.6) in (1.1) and by solving we get the Haar wavelet coefficients a_i's using Inexact Newton's method [21]. Substituting the values of a_i's in (4.5), to obtain the Haar wavelet collocation method (HWCM) based numerical solution of the problem (3.1).

Convergence analysis of the Haar wavelets:

Lemma*:* Assume that $u(x,t) \in L_2(\mathbb{R})$ with the bounded first derivative on (0, 1), then the error norm at j^{th} level satisfies the following inequality

$$\left\| e_j(x,t) \right\| \le \sqrt{\frac{K}{7}} C 2^{-(3/2)N/2}$$

From the above equation, it is obvious that the error bound is inversely proportional to the level of resolution of the Haar wavelet. This ensures the convergence of the Haar wavelet approximation when *N* is increased.

5. Test Problems

In this section, Implementing the FEM and HWCM as discussed in section 3 and 4 to find the numerical solution of some of the parabolic type problems.

Test Problem 1. First consider the equation of the form

$$u_t = u_{xx}, \quad 0 < x < 1,\ t > 0 \tag{5.1}$$

Subject to the conditions $u(x,0) = \sin \pi x$, $u(0,t) = 0$ and $u(1,t) = 0$

FEM Solution:

Comparing the equation (5.1) with (3.1), we get $a = 1,\ c = 0,\ f = 0$, then from equation (3.2)

By putting $h=\frac{1}{M}$, $M=4, \Delta t=\frac{2}{M}$ and by assembling the matrix elements, using $f_i=0,\ u_0=[0, \sin\pi/4, \sin 2\pi/4, \sin 3\pi/4, 0]$ and omitting the first and last row and columns (due to the boundary conditions), we get,

$$\begin{bmatrix} 0.1667 & 0.0417 & 0 \\ 0.0417 & 0.1667 & 0.0417 \\ 0 & 0.0417 & 0.1667 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 4.1667 & -1.9583 & 0 \\ 0 & 4.166 & -1.9583 \\ 0 & -1.9583 & 4.166 \end{bmatrix} \begin{Bmatrix} 0.7071 \\ 1.000 \\ 0.7071 \end{Bmatrix}$$

Hence the solutions are $u_2=0.1142, u_3=0.1615, u_4=0.1142$. For higher values of N, FEM based numerical solutions are presented in the Table 1 & 2 and in the Fig. 1.

HWCM Solution:

Assume that

$$\dot{u}''(x,t)=\sum_{i=1}^{N} a_i h_i(x) \tag{5.2}$$

Integrating the equation (5.2) w. r. t. t from t_s to t , we get

$$u''(x,\ t)=(t-t_s)\sum_{i=1}^{N} a_i h_i(x)+u''(x,\ t_s) \tag{5.3}$$

Where t_s is the initial time and $\Delta t = t - t_s$ is the time interval

Integrating the equation (3.16) twice w. r. t. x from 0 to x, we get

$$u'(x,\ t)=\Delta t\sum_{i=1}^{N} a_i Ph_i(x)+u'(x,\ t_s)-u'(0,\ t_s)+u'(0,\ t) \tag{5.4}$$

$$u(x,\ t)=\Delta t\sum_{i=1}^{N} a_i Qh_i(x)+u(x,\ t_s)-u(0,\ t_s)-xu'(0,\ t_s)+xu'(0,\ t)+u(0,\ t) \tag{5.5}$$

Put $x=1$ in (5.5) and by using given conditions we get

$$u'(0,\ t)-u'(0,\ t_s)=0-\Delta t\sum_{i=1}^{N} a_i Ch_i(x)-0+0-0$$

Then equation (5.5) becomes

$$u(x,\ t)=\Delta t\sum_{i=1}^{N} a_i Qh_i(x)+\sin\pi x+x\left(-\Delta t\sum_{i=1}^{N} a_i Ch_i(x)\right) \tag{5.6}$$

Differentiating equation (5.6) w. r. t. then we have

$$\dot{u}(x,\ t)=\sum_{i=1}^{N} a_i Qh_i(x)+0+x\left(-\sum_{i=1}^{N} a_i Ch_i(x)\right) \tag{5.7}$$

Substituting the expressions of (5.3) & (5.7) in (5.1) we have

$$\sum_{i=1}^{N} a_i Qh_i(x)+x\left(-\sum_{i=1}^{N} a_i Ch_i(x)\right)=\Delta t\sum_{i=1}^{N} a_i h_i(x)+u''(x,\ t_s) \tag{5.8}$$

By solving (5.8) using Inexact Newton's method [25], we get the Haar wavelet coefficients a_i 's = [38.74, 2.36, -11.01, 13.74, -10.09, -2.15, 3.34, 10.44, -7.54, -3.28, -1.70, -0.46, 0.89, 2.50, 4.37 & 5.96]. Substituting the values of a_i 's in (5.6), to obtain the numerical solution of the problem (5.1) and is presented with Finite element method (FEM) and Finite difference method (FDM) solutions in comparison with the exact solution $u(x,t)=e^{-\pi^2 t}\sin\pi x$ in the Table 1 for N=16 and Fig. 1 for N=32. The error analysis for higher values of N is given in Table 2 with $\Delta t = 1/N$.

Test Problem 2. Now consider the equation of the form

$$u_t = u_{xx}, \quad 0<x<1,\ t>0 \tag{5.9}$$

with the given conditions $u(x,0)=0$, $u(0,t)=0$ and $u(1,t)=t$

Due to the initial condition, the FEM gives the trivial solution as discussed in section 3.
The solution of (5.9) is obtained using the methods presented in section 4, Haar coefficients a_i 's = [4.37, -2.73, -0.62, -2.89, -0.37, -0.32, -0.73, -2.43, -0.25, -0.14, -0.14, -0.18, -0.28, -0.47, -0.87 & -1.61] and the corresponding HWCM solution is presented in comparison with the FDM and exact solution $u(x,t)=\frac{1}{6}\left(x^3-x+6xt\right)+\frac{2}{\pi^3}\sum_{n=1}^{N}\frac{(-1)^n}{n^3}e^{-n^2\pi^2 t}\sin n\pi x$ in the Table 3 for N=16 and Fig. 2 for N=32. The error analysis for higher values of N is given in Table 4 with $\Delta t = 1/N$.

Test Problem 3. Next consider the equation of the form

$$u_t = u_{xx} + u, \quad 0<x<1,\ t>0 \tag{5.10}$$

With the given conditions $u(x,0)=\cos\pi x$, $u(0,t)=e^{(1-\pi^2)t}$ and $u(1,t)=-e^{(1-\pi^2)t}$

FEM Solution:

Comparing the equation (5.10) with (3.1), By putting $h=\frac{1}{M},\ M=4, \Delta t=\frac{2}{M}$ and by assembling the matrix elements, using $f_i=0,\ u_0=[0,\cos\pi/4,\cos 2\pi/4,\cos 3\pi/4,0]$ and omitting the first and last row and columns (due to the boundary conditions),we get

$$\begin{bmatrix} 2.1667 & -0.9583 & 0 \\ -0.9583 & 2.1667 & -0.9583 \\ 0 & -0.9583 & 2.1667 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix}=\begin{bmatrix} 0.3333 & 0.0833 & 0 \\ 0.0833 & 0.3333 & 0.0833 \\ 0 & 0.0833 & 0.3333 \end{bmatrix}\begin{Bmatrix} 0 \\ 1.000 \\ 0 \end{Bmatrix}$$

Hence the solution as $u_2=-0.1749, u_3=-0.3087, u_4=-0.1749$. For higher values of N, FEM based numerical solutions are presented in the Table 5 & 6 and in the Fig. 3.

HWCM Solution:

With the given conditions $u(x,0)=\cos\pi x$, $u(0,t)=e^{(1-\pi^2)t}$ and $u(1,t)=-e^{(1-\pi^2)t}$

As in previous examples the solution of (5.10) is obtained with the Haar coefficients a_i 's =
[-7.00, 32.61, 16.55, 15.80, 9.69, 7.85, 8.51, 6.74, 6.17, 4.03, 3.84, 4.02, 4.22, 4.25, 3.85 & 2.80] and the corresponding HWCM solution is presented in comparison with the FEM, FDM and exact solution $u(x,t)=e^{(1-\pi^2)t}\cos\pi x$ in the Table 5 for N=16 and Fig. 3 for N=32. The error analysis for higher values of N is given in Table 6 with $\Delta t=1/N$.

Test Problem 4. Consider singularly perturbed convection-dominated diffusion equation

$$u_t=\varepsilon u_{xx}-u_x+\eta(x,t), \quad 0<x<1,\ t>0 \text{ and } \varepsilon>0 \tag{5.11}$$

Where

$$\eta(x,t)=\frac{\left(t^2-t+1-x\right)}{\varepsilon\left(1-e^{-1/\varepsilon}\right)}e^{-(1-x)t/\varepsilon}$$

with the given conditions $u(x,0)=1,\ u(0,t)=1+\frac{1-e^{-t/\varepsilon}}{1+e^{-1/\varepsilon}}$ and $u(1,t)=1$.

As in previous Test Problems, the solution of (5.11) is obtained with the Haar coefficients a_i 's = [30.40, -58.06, -15.30, -89.37, -19.72, -2.38, -4.32, -144.76, -23.26, -3.17, -1.41, -1.07, -1.40, -3.25, -14.10 & -190.47] and the corresponding HWCM solution is presented in comparison with the FDM and exact solution $u(x,t)=1+\frac{1-e^{-(1-x)t/\varepsilon}}{1+e^{-1/\varepsilon}}$ in the Table 7 for N=16 and Fig. 4 for N=32 for $\varepsilon=0.08$. The error analysis for higher values of N is given in Table 9 with $\Delta t=1/N$ for different ε.

Test Problem 5. Now consider the two dimensional problem as,

$$K\nabla^2 u + F = \rho c \frac{\partial u}{\partial t} \quad \text{, (In } \Omega = \{x, y\}\text{, } 0 < x < 3, 0 < y < 3) \tag{5.12}$$

Where $\rho c = 1, K = 1.25, L = 3, F = 0$, subject to the boundary and initial conditions as

$$u(0, y, t) = u(x, 0, t) = u(L, y, t) = u(x, L, t) = 0; \quad u_0(x, y) = 30. \tag{5.13}$$

The analytical solution for the above problem is,

$$u(x, y, t) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} A_n \sin(n\pi x / 3)\sin(j\pi y / 3)\exp[-K\pi^2 \left(n^2 + j^2\right)t / 3^2]$$

Where $A_n = 4 \times 30 \times [(-1)^n - 1]\left[(-1)^j - 1\right] / nj\pi^2$.

Due to the symmetry of the problem, only one quadrant of the solution domain is formed by $N \times N$ elements in the problem. Some results obtained by the proposed method are shown in Tables 11 and 12, where Table 11 gives the distribution of temperature with analytical solution, Table 12 gives the variation of temperature at $(x, y, t) = (1.5, 1.5, 1.2h)$ with N and Time step Δt .The Results of HWCM are based on Section 4. The distributions of temperature with analytical solution for Test Problem 5 are given in Table 11.

Test Problem 6. Consider the transient heat conduction problem

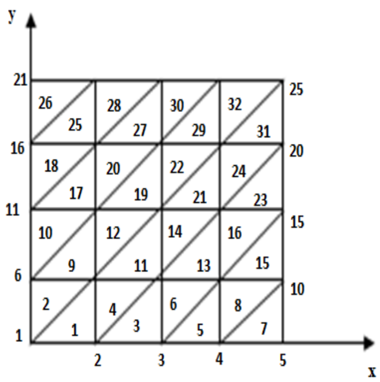
$$\frac{\partial T}{\partial t} - \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = 1, \tag{5.14}$$

Subject to the boundary conditions, for $t \geq 0$,

$$\frac{\partial T}{\partial x}(0, y, t) = 0, \quad \frac{\partial T}{\partial y}(x, 0, t) = 0, T(1, y, t) = 0, \quad T(x, 1, t) = 0 \tag{5.15}$$

And the initial conditions $T(x, y, 0) = 0 \ \forall (x, y) \in \Omega$. Analytical solution is $\lambda_{mn} = \frac{1}{4}\pi^2\left(m^2 + n^2\right) \ (m, n = 1, 3, 5, ...)$.

We check for 4×4 mesh of linear triangular elements to model the domain, and analyze the Stability and accuracy of the Crank-Nicolson method for 0.5which is unconditionally stable. For the higher values of Δt, we take $\Delta t_{cri} = \frac{2}{\lambda_{\max}} = \frac{2}{386.4} = 0.005176$.



The boundary conditions of the problem are given by $U_5 = U_{10} = U_{15} = U_{20} = U_{25} = 0.0$.

Haar wavelet collocation method and Finite element based method numerical solutions are obtained for the different values of N of the Test Problem 6, Temperature against mesh N and Time step Δt . $([x, y, t] = 1.5, 1.5, 1.2)$ are shown in Table 12. Results of Crank-Nicolson are given in Table 13.

Test Problem 7; Lastly**,** Consider the 2-D Parabolic problem,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad 0 < x, \ y < 2 \tag{5.16}$$

Subjected to the conditions, $u(x,y,0)=\sin\left(\frac{\pi y}{2}\right),\ 0\le x,\ y\le 2;\ u\left(x,y,0\right)=0,\ \forall\left(x,y\right)\in\delta\Omega,\ \forall t>0.$

(5.17)

With the analytical solution,

$$u(x,y,t)=\sin\frac{\pi}{2}y\sum_{n=1}^{\infty}\left[1-\left(-1\right)^{n}\right]\frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\exp\left(\frac{-\pi^{2}(n^{2}+1)t}{4}\right)$$

Errors of the Test Problem 7 with $t=1$ are given in Table 14.

6. Conclusion

In this paper, we applied the Haar wavelet collocation method (HWCM) for the numerical solution of parabolic set of differential equations. It has been well demonstrated that while applying the nice properties of Haar wavelets, the parabolic type partial differential equations can be solved conveniently and accurately by using HWCM systematically. In the first Test Problem FEM & FDM gives better results than the HWCM. While in the second Test Problem, FDM results closer to HWCM where as FEM gives the trivial solution due to the initial condition. Third Test Problem shows that the FEM & FDM gives the poor performance as compared to HWCM. In the fourth Test Problem due to the value of the results are varied, as the value of is less than 1, the FDM results are better than HWCM. The HWCM results closer to the FDM as the value of is closer to 1. For the higher value of , the HWCM results are better than the FDM. The last four i.e. 2-D Test Problem shows the robustness of the HWCM over FEM when compared with exact solution. The main advantages of the HWCM are its simplicity and small computation costs: it is due to the sparcity of the transform matrices and to the small number of significant wavelet coefficients. Hence the Haar wavelet collocation method is competitive in comparison with the classical methods.

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