ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 10, No. 3, 2015, *pp*. 178-188

Numerical solution of two-dimensional nonlinear Volterra integral equations using Bernstein polynomials

Y. Ordokhani* , R. Javadi
Department of Applied Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran
(*Received December 30, 2014, accepted January23, 2015)*

Abstract. The purpose of this paper is to present a numerical method for finding an approximate solution of two-dimensional (2D) nonlinear volterra integral equations. First, we introduce two-dimensional Bernstein functions, then present their operational matrices of integration and product. Using this properties and collcation points, reduce integral equation to a system of nonlinear algebric equations. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique.

Keywords: Two-dimensional integral equations, Bernstein polynomials, Operational matrix of integration, Product operational matrix,Volterra, Nonlinear.

1. Introduction

Mathematical modeling of real-life problems usually results in functional equations, e.g. integral equations, integro-differential equations and others. Inparticular, integral and integro-differential equations arise in fluid dynamics, biological models and chemical kinetics. Finding the analytical solutions of mentioned equations is not possible, and thus numerical methods are required [1,2,3,4,5].

In recent years, researchers have allocated considerable effort to study of numerical solutions of the two-dimensional integral and integro-differential equations. Many powerful methods have been proposed. In [6] authors, have applied rationalized Haar functions to the solution of the two-dimensional nonlinear integral equations.While in [7] Legendre polynomials have been chosen. In [8] triangular functions have been used by the authors and Block-pulse functions have been chosen by the authors of [9, 10]. In [11] the authors, have applied the differential transform method to solve two-dimensional volterra integral equations. The Euler method have been used in [12] to approximate the solution of 2D Volterra integral equations numerically by authors.

Consider the second-kind Volterra integral equation [7]

$$\begin{aligned} U(x,t) &= \int_0^t\int_0^x k_1(x,t,y,z)\,H(y,z,U(y,z))\,dydz + \int_0^x k_2(x,t,y)\,G(y,t,U(y,t))\,dy \\ &\quad + \int_0^t k_3(x,t,z)\,F(x,z,U(x,z))\,dz + R(x,t), \qquad (x,t)\in[0,1]\times[0,1], \end{aligned} \tag{1}$$

where $U(x,t)$ is the unknown function in $\Omega\,(\Omega=[0,1]\times[0,1])$, the functions R, k_1, k_2 and k_3 are given smooth functions and the functions H, G and F are given continuos functions in $\Omega\times(-\infty,\infty)$, nonlinear in U.

As shown in [12], Eq. (1) arise from the transformation of certain Volterra integral equations of the first kind. In this paper, the numerical solution of Eq. (1) is computed by using 2D Bernstein polynomials.

The basis in the present method is the use operational matrices of the Bernstein polynomials. Using Bernstein polynomials expanded in terms of Legendre basis are given the mentioned operational matrices. The main reason to use this extension is reduce the computational, In particular operational matrices calculation.

This paper is organized as follows. In section 2, we introduce 2D Bernstein functions, their propertice

* Corresponding author. Tel: +98-21-88048931; fax: +98-21-88048931.
E-mail address: ordokhani@alzahra.ac.ir.

and present operational matrices of them. In section 3, how the collocation method can be used to reduce the problem to a system of nonlinear equations is explained. Numerical examples are given in section 4 to evaluation of our method and comparison with the numerical results obtained by other authors is provided. Finally, conclusions are given in section 5.

2. Properties of 2D Bernstein polynomials

2.1 Definition and function approximation

Two-dimensional Bernstein functions are defined on Ω as

$$B_{i,j,M,N}(x,t) = B_{i,M}(x)\, B_{j,N}(t), \qquad i = 0,\ldots,M, \quad j = 0,\ldots,N,$$

here $B_{i,M}$ and $B_{j,N}$ are the well-known Bernstein functions respectively of degree M and N, which are defined on the interval [0, 1] and can be determined with the following formula [13]:

$$B_{i,M}(x) = \binom{M}{i} x^i (1-x)^{M-i}, \qquad i = 0, \ldots, M,$$

where

$$\binom{M}{i} = \frac{M!}{i!(M-i)!} \, .$$

2D Bernstein polynomials form a single partition on Ω as:

$$\sum_{i=0}^{M}\sum_{j=0}^{N} B_{i,j,M,N}(x, t) = 1, \qquad (x,t) \in \Omega.$$

Suppose that $X = L^2(\Omega)$, the inner product in this space is defined by

$$\langle\, f(x,t),\, g(x,t) \rangle = \int_0^1\int_0^1 f(x,t)\, g(x,t)\, dxdt \,,$$

and the norm is as:

$$\| f(x,t) \|_2 = \langle\, f(x,t),\, f(x,t) \rangle^{\frac{1}{2}} = (\int_0^1\int_0^1 |\, f(x,t)\,|^2 \, dxdt)^{\frac{1}{2}}.$$

Let

$$\{ B_{00}(x,t),\ldots, B_{0N}(x,t), \ldots, B_{M0}(x,t),\ldots, B_{MN}(x,t) \} \subset X \,,$$

be the set of 2D Bernstein functions and

$$X_{M,N} = span\{ B_{00}(x,t), \ldots, B_{0N}(x,t), \ldots, B_{M0}(x,t), \ldots, B_{MN}(x,t) \},$$

and $f(x,t)$ be an arbitrary function in X. Since $X_{M,N}$ is a finite dimensional vector space, f has a unique best approximation $f_{M,N} \in X_{M,N}$ [7], such that

$$f(x,t) \cong f_{M,N}(x,t) = F^T B(x,t) = \sum_{i=0}^{M}\sum_{j=0}^{N} f_{ij}\, B_{i,j,M,N}(x,t) = \sum_{k=0}^{M}\sum_{h=0}^{N} c_{kh}\, L_{kh}(x,t), \qquad (2)$$

where $L_{kh}(x,t), k = 0, \ldots, M, h = 0, \ldots, N$ are 2D shifted Legendre functions on Ω [7] and coefficients c_{kh} are obtained by

$$c_{kh} = \frac{\langle\, f(x,t), L_{kh}(x,t) \rangle}{\| L_{kh}(\mathrm{x,t}) \|_2^2} \, .$$

2D Legendre polynomial $L_{mn}(x,t)$ can be expanded in terms of the Bernstein basis as follows:

$$L_{mn}(x,t)=\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j+m+n}\binom{m}{i}\binom{n}{j}B_{i,j,m,n}(x,t). \qquad (3)$$

We put:

$$B_{k,h,M,N}(x,t)=\sum_{i=0}^{M}\sum_{j=0}^{N}w_{k,h,i,j}\,L_{ij}(x,t), \qquad k=0,\ldots,M,\ h=0,\ldots,N. \qquad (4)$$

The coefficients $w_{k,h,i,j}$, $k,\ i=0,\ldots,M,\ \ j,\ h=0,\ldots,N$ form a $(M+1)(N+1)\times(M+1)(N+1)$ transformation matrix of base as W. We obtain the coefficients $w_{k,h,i,j}$ in the following manner. Multiplying both sides of Eq. (4) by $L_{mn}(x,t)$, $m=0,\ldots,M,\ \ n=0,\ldots,N$, and integrating the result yields

$$\int_0^1\int_0^1 B_{k,h,M,N}(x,t)\,L_{mn}(x,t)\,dx\,dt=\sum_{i=0}^{M}\sum_{j=0}^{N}w_{k,h,i,j}\int_0^1\int_0^1 L_{ij}(x,t)\,L_{mn}(x,t)\,dx\,dt,$$

2D shifted Legendre functions $L_{mn}(x,t)$ are orthogonal with each other as [7]:

$$\int_0^T\int_0^l L_{ij}(x,t)\,L_{mn}(x,t)\,dxdt=\begin{cases}\dfrac{lT}{(2m+1)(2n+1)}, & i=m \text{ and } j=n,\\ 0, & otherwise,\end{cases}$$

therefore

$$w_{k,h,m,n}=(2m+1)(2n+1)\int_0^1\int_0^1 B_{k,h,M,N}(x,t)\,L_{mn}(x,t)\,dxdt\,. \qquad (5)$$

Substituting Eq. (3) into Eq. (5) we have

$$w_{k,h,m,n}=(2m+1)(2n+1)\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j+m+n}\binom{m}{i}\binom{n}{j}\int_0^1\int_0^1 B_{k,h,M,N}(x,t)\,B_{i,j,m,n}(x,t)\,dx\,dt\,.$$

The integration of multiplication of Bernstein basis functions is as follows

$$\eta=\int_0^1\int_0^1 B_{k,h,M,N}(x,t)\,B_{i,j,m,n}(x,t)\,dx\,dt=\left(\int_0^1\binom{M}{k}\binom{m}{i}x^{i+k}(1-x)^{M+m-(i+k)}dx\right)$$
$$\times\left(\int_0^1\binom{N}{h}\binom{n}{j}t^{j+h}(1-t)^{N+n-(j+h)}dt\right),$$

using the following equation,

$$\int_0^1(1-t)^r t^i\,dt=\frac{1}{(r+i+1)\binom{r+i}{i}}, \qquad r,i\in N,$$

$$\eta=\frac{\binom{M}{k}\binom{m}{i}\binom{N}{h}\binom{n}{j}}{(M+m+1)(N+n+1)\binom{M+m}{k+i}\binom{N+n}{j+h}},$$

therefore

$$w_{k,h,m,n}=\frac{(2m+1)(2n+1)}{(M+m+1)(N+n+1)}\binom{M}{k}\binom{N}{h}\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(-1)^{i+j+m+n}}{\binom{M+m}{k+i}\binom{N+n}{j+h}}\binom{m}{i}\binom{m}{i}\binom{n}{j}\binom{n}{j}. \qquad (6)$$

We suppose

$$L_{mn}(x,t)=\sum_{i=0}^{M}\sum_{j=0}^{N}\Lambda_{m,n,i,j}\,B_{i,j,M,N}(x,t),\qquad m=0,\ldots,M,\ n=0,\ldots,N. \tag{7}$$

The coefficients $\Lambda_{m,n,i,j}$, $m,i=0,\ldots,M,\ j,n=0,\ldots,N$ form a $(M+1)(N+1)\times(M+1)(N+1)$ transformation matrix of base as Λ.

Substituting Eq. (7) into Eq. (2) we have

$$f_{ij}=\sum_{m=0}^{M}\sum_{n=0}^{N}c_{mn}\,\Lambda_{m,n,i,j},\qquad i=0,\ldots,M,\ j=0,\ldots,N. \tag{8}$$

Considering the following equation [13]

$$B_{i,m}(x)=\sum_{r=i}^{M-m+i}\frac{\binom{m}{i}\binom{M-m}{r-i}}{\binom{M}{r}}B_{i,M}(x),$$

thus

$$B_{i,j,m,n}(x,t)=\sum_{r=i}^{M-m+i}\sum_{s=j}^{N-n+j}\frac{\binom{m}{i}\binom{M-m}{r-i}\binom{n}{j}\binom{N-n}{s-j}}{\binom{M}{r}\binom{N}{s}}B_{i,j,M,N}(x,t). \tag{9}$$

Thus, substituting Eq. (9) into Eq. (3) , the elements of Λ are determined with the following formula :

$$\Lambda_{m,n,r,s}=\frac{1}{\binom{M}{r}\binom{N}{s}}\sum_{i=l}^{min\,\{r,m\}}\sum_{j=k}^{min\,\{s,n\}}(-1)^{i+j+m+n}\binom{m}{i}\binom{m}{i}\binom{n}{j}\binom{n}{j}\binom{M-m}{r-i}\binom{N-n}{s-j}, \tag{10}$$

$$l=max\{0,m+r-M\},\ k=max\{0,n+s-N\}.$$

Therefore we get

$$B(x,t)=W\,L(x,t), \tag{11}$$

$$L(x,t)=\Lambda\,B(x,t). \tag{12}$$

Each functions k_1 in $L_2(\Omega\times\Omega)$, k_2 and k_3 in $L^2(\Omega\times[0,1])$ can be expanded in terms of 2D Bernstein functions respectively as

$$k_1(x,t,y,z)\cong B^T(x,t)K_{1b}\,B(y,z),\qquad k_2(x,t,y)\cong B^T(x,t)K_{2b}\,B(y,t),$$

$$k_3(x,t,z)\cong B^T(x,t)K_{3b}\,B(x,z),\qquad R(x,t)\cong E_b\,B(x,t).$$

Using Eqs.(11)- (12) we have

$$K_{1b}=(W^{-1})^T K_{1l}\,(W^{-1}),\qquad K_{2b}=(W^{-1})^T K_{2l}\,(W^{-1}),$$

$$K_{3b}=(W^{-1})^T K_{3l}\,(W^{-1}),\qquad E_b=E^T\Lambda,$$

where K_{1l},K_{2l} and K_{3l} are block matrices of the form

$$K_{ql}=[k_{ql}^{(i,m)}]_{i,m=0}^{M},\qquad q=1,2,3,$$

in which

$$k_{ql}^{(i,m)}=[k_{ijmn}^{ql}]_{j,n=0}^{N},\quad i,m=0,\ldots,M,\ q=1,2,3,$$

and 2D shifted Legendre coefficients k_{ijmn}^{ql}, $q = 1,2,3$ are given by [7]

$$k_{ijmn}^{1l} = \frac{\langle\langle k_1(x,t,y,z), L_{mn}(y,z)\rangle, L_{ij}(x,t)\rangle}{\|L_{ij}(x,t)\|_2^2 \|L_{mn}(y,z)\|_2^2}, \qquad i,m = 0,\dots,M, \quad j,n = 0,\dots,N,$$

$$k_{ijmn}^{2l} = \frac{\langle\langle k_2(x,t,y), L_{mn}(y,t)\rangle, L_{ij}(x,t)\rangle}{\|L_{ij}(x,t)\|_2^2 \|L_{mn}(y,t)\|_2^2}, \qquad i,m = 0,\dots,M, \quad j,n = 0,\dots,N,$$

$$k_{ijmn}^{3l} = \frac{\langle\langle k_3(x,t,z), L_{mn}(x,z)\rangle, L_{ij}(x,t)\rangle}{\|L_{ij}(x,t)\|_2^2 \|L_{mn}(x,z)\|_2^2}, \qquad i,m = 0,\dots,M, \quad j,n = 0,\dots,N,$$

$$E_{mn} = \frac{\langle R(x,t), L_{mn}(x,t)\rangle}{\|L_{mn}(x,t)\|_2^2}, \qquad m = 0,\dots,M, \quad n = 0,\dots,N.$$

2.2 Operational matrices of integration

The integration of the vector $B(x, t)$ can be approximating obtained as:

$$\int_0^t \int_0^x B(x', t')\, dx' dt' \cong Q_{1b}\, B(x,t) \qquad (x,t) \in \Omega,$$

where Q_{1b} is the $(M+1)(N+1) \times (M+1)(N+1)$ operational matrix of integration. Using Eqs. (11)- (12)

$$Q_{1b} = W\, Q_{1l}\, \Lambda, \tag{13}$$

here Q_{1l} is the operational matrix of 2D shifted Legendre polynomials as follows [7]:

$$Q_{1l} = P_1 \otimes P_2,$$

where P_1 and P_2 are the operational matrices of 1D shifted Legendre polynomials as

$$P_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-1}{2M-1} & 0 & \frac{1}{2M-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2M+1} & 0 \end{bmatrix},$$

$$P_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-1}{2N-1} & 0 & \frac{1}{2N-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2N+1} & 0 \end{bmatrix},$$

and $\otimes$ denotes the Kronecher product defind for two arbitrary matrices A and B as [14]

$$A \otimes B = (a_{ij} B).$$

Analogously, using Eqs.(11)- (12), we write

$$\int_0^x B(x', t)\, dx' \cong Q_{2b}\, B(x,t), \tag{14}$$

$$\int_0^t B(x, t')\, dt' \cong Q_{3b}\, B(x,t), \tag{15}$$

where Q_{2b} and Q_{3b} are $(M+1)(N+1)\times(M+1)(N+1)$ matrices of the form

$$Q_{2b} = W Q_{2l}\Lambda, \qquad Q_{3b} = W Q_{3l}\Lambda,$$

such that Q_{2l} and Q_{3l} are $(M+1)(N+1)\times(M+1)(N+1)$ matrices of the form [7]

$$Q_{2l} = \frac{1}{2}\begin{bmatrix} I & I & 0 & \cdots & 0 & 0 & 0 \\ \frac{-I}{3} & 0 & \frac{I}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-I}{2M-1} & 0 & \frac{I}{2M-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-I}{2M+1} & 0 \end{bmatrix}, \qquad Q_{3l} = \begin{bmatrix} P_2 & 0 & 0 & \cdots & 0 \\ 0 & P_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_2 \end{bmatrix},$$

where I and 0 are identity and zero matrix of order N+1, respctively.

2.3 The product operational matrix

Let

$$B(x,t)\,B^T(x,t)F \cong \tilde{F}\,B(x,t), \qquad F = [\,f_{00},\,\ldots,\,f_{0N},\,\ldots,f_{M0},\,\ldots,\,f_{MN}\,]^T,$$

where $\tilde{F}$ is an $(M+1)(N+1)\times(M+1)(N+1)$ product operational matrix and can be obtained with using Eqs. (11)- (12) as:

$$\tilde{F} = W\tilde{C}\Lambda, \tag{16}$$

here $C = W^T F$ and $\tilde{C}$ is product operational matrix of 2D shifted legendre functions [7].

Finally, for an $(M+1)(N+1)\times(M+1)(N+1)$ matrix $H = [H^{(i,j)}],\ i,\ j = 0,\,\ldots,\,M$ in which

$$H^{(i,j)} = [h_{im\,jn}]_{m,n=0}^{N}, \qquad i,\ j = 0,\,\ldots,\,M,$$

we have

$$B^T(x,t)\,H\,B(x,t) \cong \hat{H}\,B(x,t), \tag{17}$$

where $\hat{H}$ is a $1\times(M+1)(N+1)$ vector and using Eqs. (11)- (12) is obtained as follows:

$$\hat{H} = \hat{K}\Lambda,$$

here $K = W^T H W$ and $\hat{K}$ is a $1\times(M+1)(N+1)$ vector defind by [7]

$$\hat{K} = [K_{00},\,\ldots,\,K_{0N},\,\ldots,\,K_{M0},\,\ldots,K_{MN}],$$

and

$$K_{mn} = (2m+1)(2n+1)\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{r=0}^{M}\sum_{s=0}^{N}\omega_{i,r,m}\,\omega'_{j,s,n}\,k_{ijrs}, \qquad m = 0,\,\ldots,\,M,\ n = 0,\,\ldots,\,N.$$

3. Numerical solution

In this section, we introduce a numerical method for the solution of nonlinear 2D Volterra integral equations of the form (1).

For this purpose, assume that

$$H_1(x,t) = H(x,t,U(x,t)), \tag{18}$$

$$G_1(x,t) = G(x,t,U(x,t)), \tag{19}$$

$$F_1(x,t) = F(x,t,U(x,t)). \tag{20}$$

By using Eqs. (18)- (20), Eq. (1) can be written as:

$$U(x,t)=\int_0^t\int_0^x k_1(x,t,y,z)H_1(y,z)\,dydz+\int_0^x k_2(x,t,y)G_1(y,t)\,dy$$
$$+\int_0^t k_3(x,t,z)F_1(x,z)\,dz+R(x,t). \tag{21}$$

Using the methods described in the previous section, we approximate the functions in Eq. (21) as:

$$U(x,t)\cong D^T B(x,t), \tag{22}$$

$$H_1(x,t)\cong A^T B(x,t), \tag{23}$$

$$G_1(x,t)\cong B^T B(x,t), \tag{24}$$

$$F_1(x,t)\cong C^T B(x,t), \tag{25}$$

$$R(x,t)\cong E_b\, B(x,t), \tag{26}$$

$$k_1(x,t,y,z)\cong B^T(x,t)K_{1b}\,B(y,z), \tag{27}$$

$$k_2(x,t,y)\cong B^T(x,t)K_{2b}\,B(y,t), \tag{28}$$

$$k_3(x,t,z)\cong B^T(x,t)K_{3b}\,B(x,z), \tag{29}$$

where A, B, C and D are $(M+1)(N+1)\times 1$ unknown vectors.

Substituting Eqs. (22)- (29) into Eq. (21), we obtain

$$D^T B(x,t)=B^T(x,t)K_{1b}\int_0^t\int_0^x B(y,z)B^T(y,z)A\,dydz+B^T(x,t)K_{2b}\int_0^x B(y,t)B^T(y,t)B\,dy$$
$$+B^T(x,t)K_{3b}\int_0^t B(x,z)B^T(x,z)C\,dz+E_b\,B(x,t). \tag{30}$$

Using the operational matrices of integration (13)- (15) and the product operational matrix (16), Eq. (30) can be written as:

$$D^T B(x,t)=B^T(x,t)\;K_{1b}\,\tilde{A}Q_{1b}\,B(x,t)+B^T(x,t)K_{2b}\,\tilde{B}Q_{2b}\,B(x,t)$$
$$+B^T(x,t)K_{3b}\tilde{C}Q_{3b}\,B(x,t)+E_b B(x,t). \tag{31}$$

Let

$$\Lambda_1=K_{1b}\,\tilde{A}Q_{1b},\;\Lambda_2=K_{2b}\,\tilde{B}Q_{2b},\;\Lambda_3=K_{3b}\,\tilde{C}Q_{3b}. \tag{32}$$

By using (17) to define the vectors $\hat{\Lambda}_1,\hat{\Lambda}_2,\hat{\Lambda}_3$ and using this approximation in (31) we obtain the following system:

$$D^T-\hat{\Lambda}_1-\hat{\Lambda}_2-\hat{\Lambda}_3-E_b=0. \tag{33}$$

Using Eqs. (18)- (20) and (22), Eqs. (23)- (25) for $(x,t)\in\Omega$ can be rewritten as:

$$H(x,t,D^T B(x,t))=A^T\,B(x,t), \tag{34}$$

$$G(x,t,D^T B(x,t))=B^T\,B(x,t), \tag{35}$$

$$F(x,t,D^T B(x,t))=C^T\,B(x,t). \tag{36}$$

Collocating of Eqs. (34)- (36) at $(M+1)(N+1)$ point $(x_i,t_j),\,(i=0,\ldots,M,\,j=0,\ldots,N)$, we have

$$H(x_i,t_j,D^T B(x_i,t_j))-A^T B(x_i,t_j)=0, \tag{37}$$

$$G(x_i,t_j,D^T B(x_i,t_j))-B^T B(x_i,t_j)=0, \tag{38}$$

$$F(x_i,t_j,D^T B(x_i,t_j))-C^T B(x_i,t_j)=0, \tag{39}$$

where x_i and t_j are zeros of $L_{M+1}(x)$ and $L_{N+1}(t)$ respectively.

Eqs. (33) and (37)- (39) form a system of $4(M+1)(N+1)$ nonlinear equations with the same number of unknowns.

In the Linear case, we have $H(y, z, U(y, z)) = U(y, z)$, $G(y, t, U(y, t)) = U(y, t)$, $F(x, z, U(x, z)) = U(x, z)$, in this case the matrices Λ_1, Λ_2 and Λ_3 in (32), depend linearly on the unknown vector D.

In the general case, the dependence of Λ_1, Λ_2 and Λ_3 from D is nonlinear. Our numerical expriments have shown that for a large variety of cases the syetem (33), (37)- (39) is solvable and its solution can be efficiently approximated by classical iterative methods.

In our implementation, we have solved this system using the Matlab function fsolve, which uses Newton's method as the default method. This function has succeeded to obtain an accurate approximate solution of the system, even starting with a zero initial approximation.

4. Numerical examples

In this section, some examples are presented to evaluation of our method. In order to show the error of the method we introduce the notation:

$$e_{M,N}(x, t) = |U(x, t) - U_{M,N}(x, t)|, \qquad (x, t) \in \Omega,$$

where $U(x,t), U_{M,N}(x,t)$ are the exact solution and the computed solution by the present method, respectively. The absolute values of the errors $e_{M,N}(x, t)$ are reported at some points for the examples.

Example 1. Consider the following nonlinear two-dimensional VIE [6, 15]

$$U(x, t) = \int_0^t \int_0^x (x + t - y - z) U^2(y, z)\, dydz + x + t - \frac{1}{12} xt\,(x^3 + 4x^2 t + 4xt^2 + t^3), \quad (x, t) \in \Omega.$$

The exact solution is $U(x, t) = x + t$. Table 1 shows the numerical results obtained here and the numerical results of [6] and [15] for this example. The exact solution of this equation is obtained with M=N=3.

Table1: Numerical results for Example 1.

$(x, t) = (\frac{1}{2^i}, \frac{1}{2^i})$ i	Present method for M=N=1	Present method for M=N=3	Method of [6] for m=32	Method of [15] by Legendre , N=4
1	8.0×10^{-4}	0	3.1×10^{-2}	9.2×10^{-10}
2	9.6×10^{-3}	0	3.1×10^{-2}	8.0×10^{-10}
3	5.0×10^{-3}	0	3.1×10^{-2}	7.0×10^{-10}
4	3.7×10^{-4}	0	3.1×10^{-2}	5.3×10^{-10}
5	2.6×10^{-3}	0	3.1×10^{-2}	8.0×10^{-10}

Example 2. Consider the following linear 2D Volterra integral equation

$$U(x, t) = f(x, t) + \int_0^t \int_0^x U(y, z)\, dydz, \qquad (x, t) \in \Omega,$$

where $f(x, t) = x^2 e^t - \frac{1}{3} x^3 (e^t - 1)$.

The exact solution of this problem is $U(x, t) = x^2 e^t$. We have solved this equation numerically with present method and the method of [7] using Legendre basis. Numericall results are shown in Table 2. The absolute error function for M =N= 4 is plotted in Fig. 1.

Table 2: Numerical results for Example 2.

(x, t)	Present method for $M=N=4$	Method of [7] for $M=N=5$
$(\frac{1}{2}, \frac{1}{2})$	2.39×10^{-7}	1.96×10^{-5}
$(\frac{1}{4}, \frac{1}{4})$	4.20×10^{-6}	6.07×10^{-5}
$(\frac{1}{8}, \frac{1}{8})$	1.23×10^{-6}	9.42×10^{-5}
$(\frac{1}{16}, \frac{1}{16})$	4.42×10^{-7}	1.28×10^{-4}
$(\frac{1}{32}, \frac{1}{32})$	1.44×10^{-7}	1.50×10^{-4}
(0.2, 0.3)	3.36×10^{-6}	5.29×10^{-5}
(0.5, 0.6)	9.72×10^{-6}	2.27×10^{-5}
(0.5, 0.7)	1.53×10^{-5}	3.42×10^{-5}
(0,001, 0,001)	1.98×10^{-10}	1.98×10^{-4}
(0,002, 0.002)	7.48×10^{-10}	1.96×10^{-4}

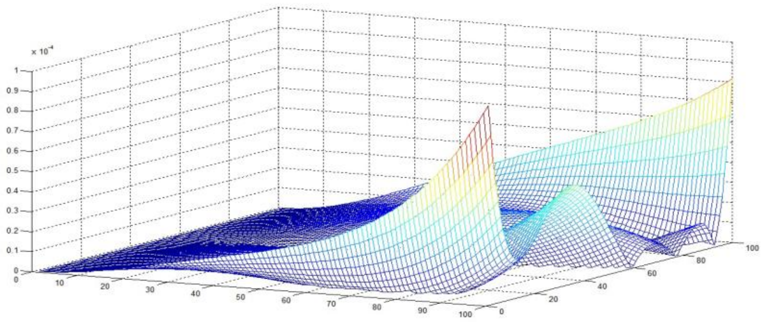

Fig 1: Plot of the function $e_{M,N}(x, t)$ with $M = N = 4$ for Example 2.

Example 3. Consider the following nonlinear 2D Volterra integral equation

$$U(x, t) = f(x, t) + \int_0^t \int_0^x U^2(y, z)\,dydz + \int_0^x (t - y)U(y, t)\,dy, \qquad (x, t) \in \Omega ,$$

where $f(x, t) = xt^2 - \frac{1}{15}x^3t^5 - \frac{1}{2}x^2t^3 + \frac{1}{3}x^3t^2$.

The exact solution of this problem is $U(x,t) = xt^2$. Table 3 shows the numerical results obtained for this example with M=N=1, M=N=2. The absolute error function for M = N= 2 is plotted in Fig. 2.

Table 3: Numerical results for Example 3.

(x, t)	Present method with M=N=1	Present method with M=N=2
(0.1, 0.1)	8.40×10^{-3}	1.79×10^{-5}
(0.1, 0.2)	2.18×10^{-4}	8.92×10^{-6}
(0.2, 0.2)	3.76×10^{-4}	9.28×10^{-6}
(0.2, 0.4)	1.72×10^{-2}	6.72×10^{-6}
(0.2, 0.6)	1.88×10^{-2}	3.22×10^{-7}
(0.3, 0.4)	2.43×10^{-2}	1.50×10^{-5}
(0.25, 0.25)	6.70×10^{-3}	5.47×10^{-6}
(0.5, 0.5)	4.29×10^{-2}	1.62×10^{-4}

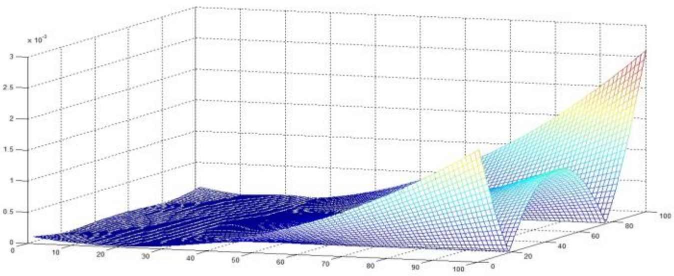

Fig 2: Plot of the function $e_{M,N}(x,t)$ with $M = N = 2$ for Example 3.

5. Conclusion

We have introduced a new method for the numerical solution of the form (1), based on expanding the solution in terms of 2D Bernstain polynomials. Since the Bernstein polynomials are not orthogonal, we use their expanded in terms of Legendre basis. The main reason to use this extention is reduce the computational. As the numerical results have shown, in the case of a sufficiently smooth solution, a small number of basis functions is enough to obtain high accuracy. We leave the numerical analysis of the present method as future work.

Acknowledgements

The work of the first author was supported by Alzahra University.

6. References

[1] F. Bloom, Asymptotic bounds for solutions to a system of damped integro-differential equations of electromagnetic theory, Journal of Mathematical Analysis and Applications, 73.2 (1980), pp. 524-542.

[2] R. Y. Chang, and M. L. Wang, *Shifted Legendre direct method for variational problems*, Journal of Optimization Theory and Applications, 39.2 (1983), pp. 299-307.

[3] A. Jerri, Introduction to integral equations with applications. John Wiley Sons, (1999).

[4] D. M. Doddrell, L. K. Forbes, and S. Crozier, *Caluculating current densities and fields produced by shielded magnetic resonance imaging probes*, SIAM Journal on Applied Mathematics, 57.2 (1997), pp. 401-425.

[5] Y. Mahmoudi, *Wavelet Galerkin method for numerical solution of nonlinear integral equation*, Applied

Mathematics and Computation, 167.2 (2005), pp. 1119-1129.

[6] E. Babolian, S. Bazm, and P. Lima, *Numerical solution of nonlinear two-dimensional integral equations using rationalized Haar functions*, Communications in Nonlinear Science and Numerical Simulation, 16.3 (2011), pp. 1164-1175.

[7] S. Nemati, PM. Lima, and Y. Ordokhani, *Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials,* Journal of Computational and Applied Mathematics, 242 (2013), pp. 53-69.

[8] E. Babolian, K. Maleknejad, M. Roodaki, and H. Almasieh, *Two-dimensional triangular functions and their applications to nonlinear 2D Volterra-Fredholm integral equations,* Computers Mathematics with Applications, 60.6 (2010), pp. 1711-1722.

[9] K. Maleknejad, S. Sohrabi, and B. Baranji, *Application of 2D-BPFs to nonlinear integral equations*, Communications in Nonlinear Science and Numerical Simulation, 15.3 (2010), pp. 527-535.

[10] K. Maleknejad, and K. Mahdiani, *Solving nonlinear mixed Volterra Fredholm integral equations with two dimensional Block-pulse functions using direct method*, Communications in Nonlinear Science and Numerical Simulation, 16.9 (2011), pp. 3512-3519.

[11] A. Tari, M. Y. Rahimi, S. Shahmorad, and F. Talati, *Solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method*, Journal of Computational and Applied Mathematics, 228.1 (2009), pp. 70-76.

[12] S. McKee, T. Tang, and T. Diogo, *An Euler-type method for two-dimensional Volterra integral equations of the first kind*, IMA Journal of Numerical Analysis, 20.3 (2000), pp. 423-440.

[13] R. T. Farouki, *Legendre Bernstein basis transformations*, Journal of Computational and Applied Mathematics, 119.1 (2000), pp. 145-160.

[14] P. Lancaster, and M. Tismenetsky, *The theory of matrices: with applications*, Academic press, 1985.

[15] J. Saberi Nadjafi, O. Navid Samadi, and E. Tohidi, *Numerical solution of two-dimensional Volterra integral equations by spectral Galerkin method*, Journal of Applied Mathematics and Bioinformatics, 1.2 (2011), pp. 159-174.

[16] E. Babolian, and N. Dastani, homotopy perturbation method: An effective tool for solving a nonlinear system of two-dimensional Volterra Fredholm integral equations. Mathematical and Computer Modelling, 55.3 (2012), pp. 1233-1244.