ACADEMIC World Academic Union

ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 10, No. 3, 2015, *pp*. 199-208

The Method of Particular Solutions (MPS) for Solving One-Dimensional Hyperbolic Telegraph Equation

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(*Received January 19, 2015, accepted May 30, 2015)*

Abstract. In this paper, the method of particular solution (MPS) is employed for the numerical solution of the one-dimensional (1D) telegraph equation based on radical basis functions (RBFs). Coupled with the time discretization and MPS, the proposed method is a truly meshless method which requires neither domain or boundary discretization. The algorithm is very simple so it is very easy to implement. The results of numerical experiments are presented, and are compared with analytical solutions to confirm the good accuracy of the presented scheme, the obtained numerical results also have been compared with the results obtained by some existing methods to verify the accurate nature of our method.

Keywords: method of particular solution (MPS), radical basis function (RBF), numerical solution, hyperbolic telegraph equation.

1. Introduction

This paper is devoted to the numerical computation of the one-dimension (1D) hyperbolic telegraph equation:

$$\frac{\partial^2 u}{\partial^2 t}+2\alpha\frac{\partial u}{\partial t}+\beta^2 u=\frac{\partial^2 u}{\partial^2 x}+f(x,t),\ a\le x\le b,\ t>0, \tag{1.1}$$

with the initial conditions:

$$u(x,0)=h_0(x),\ u_t(x,0)=h_1(x),\ a\le x\le b, \tag{1.2}$$

and Dirichlet boundary conditions:

$$u(a,t)=g_0(t),\ u(b,t)=g_1(t),\ t>0 \tag{1.3}$$

where α and β are known constant coefficients, h_i and g_i ($i=0,1$) are known continuous functions. Both the electric voltage and the current in a double conductor, satisfy the telegraph equation, where x is distance and t is time. Note that, for $\alpha>0$, $\beta=0$, Eq. (1.1) represents a damped wave equation, and for $\alpha>\beta>0$, it is called telegraph equation [1].

The second-order telegraph equation with constant coefficients is commonly used in signal analysis for transmission and propagation of electrical signals [2] and also models mixture between diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to standard heat or mass transport equation [3]. In fact the telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion for such branches of sciences. Moreover, this equation also has applications in other fields (see [4] and the references therein).

Recently, much attention has been given to the development, analysis, and implementation of stable methods for the numerical solution of second-order hyperbolic equations (see [5] and the reference therein). Mohanty et al [6,7], developed new three-level implicit unconditionally stable alternating direction implicit schemes for the two and three-space dimensional linear hyperbolic equations. These schemes are second-order accurate both in space and time. Dehghan and Shokri [8] solved the one-dimensional telegraph equation

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using Kansa's method. Z. W. Jiang, et al [9] extend this problem considered in [8] to one kind of partial differential equations with variable coefficients. A numerical method based on the interpolating scaling functions were described by Lakestani and N. Saray [10]. Evans and Hasan [11] applied an Alternating Group Explicit (AGE) method to obtain numerical solution of the telegraph equation. Marzieh Dosti, Alireza Nazemi [12,13] and J. Rashidinia1, et al [14] developed a numerical method using quartic B-spline collocation and cubic B-spline quasi-interpolation.

In this article, we present a new numerical scheme to solve the second-order hyperbolic telegraph equation using the Method of Particular Solutions (MPS) with the Thin Plate Splines (TPS) Radial Basis Function (RBF). The results of numerical experiments are presented, and are compared with analytical solutions to confirm the good accuracy of the presented scheme, the obtained numerical results also have been compared with the results obtained by some existing methods to verify the accurate nature of our method.

In last 25 years, the radial basis functions (RBFs) method is known as a powerful tool for scattered data interpolation problem. The use of RBFs as a meshless procedure for numerical solution of partial differential equations is based on the collocation scheme. Because of the collection technique, this method does not need to evaluate any integral. The main advantage of numerical procedures which use RBFs over traditional techniques is meshless property of these methods. RBFs are used actively for solving partial differential equations. The examples see [15,16]. In the last decade, the development of the RBFs as a truly meshless method for approximating the solutions of PDEs has drawn the attention of many researchers in science and engineering [17–19]. Meshless method has became an important numerical computation method, and there are many academic monographs are published [20–22].

The layout of the article is as follows : In section 2, we introduce the MPS method and apply this method on the hyperbolic telegraph equation. The results of numerical experiments are presented in section 3. Section 4 is dedicated to a brief conclusion. Finally, some references are introduced at the end.

2. The Method of Particular Solutions (MPS)

2.1. Radial basis function approximation

The approximation of a distribution $u(\boldsymbol{x})$, using RBF, may be written as a linear combination of N radial basis functions, usually it takes the following form:

$$u(\boldsymbol{x}) \approx \sum_{j=1}^{N} \lambda_j \varphi(\boldsymbol{x}, \boldsymbol{x}_j) + \psi(\boldsymbol{x}), \text{ for } \boldsymbol{x} \in \Omega \subseteq R^d \tag{2.1.1}$$

where N is the number of data points, $\boldsymbol{x} = (x_1, x_2, \cdots, x_d)$, d is the dimension of the problem, the λ's are coefficients to be determined and φ is the radial basis function. Eq. (2.1.1) can be written without the polynomial ψ. In that case, φ must be unconditional positive definite to guarantee the solvability of the resulting system (e. g. Gaussian or Inverse Multiquadrics). However, ψ is usually required when φ is conditionally positive definite, i. e, when φ has a polynomial growth towards infinity. We will use the Thin Plate Splines (TPS), which defined as:

$$\text{TPS:} \quad \varphi(\boldsymbol{x}, \boldsymbol{x}_j) = \varphi(r_j) = r_j^{2m} \log(r_j), \quad m = 1,2,3,\cdots \tag{2.1.2}$$

where $r_j = \|\boldsymbol{x} - \boldsymbol{x}_j\|$ is the Euclidean norm.

If $\boldsymbol{P}_q^d$ denotes the space of d-variate polynomial of order not exceeding than q, and letting the polynomials $(P_1, P_2, \cdots, P_m)$ be the basis of $\boldsymbol{P}_q^d$ in R^d, then the polynomial $\psi(\boldsymbol{x})$ in Eq. (2.1.1) is usually written in the following form:

$$\psi(\boldsymbol{x}) = \sum_{i=1}^{m} \xi_i P_i(\boldsymbol{x}_j) \tag{2.1.3}$$

where $m = (q-1+d)!/(d!(q-1)!)$. To get the coefficients $(\lambda_1, \lambda_2, \cdots, \lambda_N)$ and $(\xi_1, \xi_2, \cdots, \xi_m)$, the collocation method is used. However, in addition to the N equations resulting from collecting Eq. (2.1.1) at

N points, an extra m equations are required. This is ensured by the m conditions for Eq. (2.1.1),

$$\sum_{j=1}^{N} \lambda_j P_i\left(\boldsymbol{x}_j\right) = 0, \quad i = 1,2,\cdots m. \tag{2.1.4}$$

In a similar representation as Eq. (2.1.1), for any linear partial differential operator ℓ, ℓu can be approximated by:

$$\ell u(\boldsymbol{x}) \approx \sum_{j=1}^{N} \lambda_j \ell \varphi\left(\boldsymbol{x}, \boldsymbol{x}_j\right) + \ell \psi(\boldsymbol{x}) \tag{2.1.5}$$

2.2. The hyperbolic telegraph equation

Let us consider the 1D hyperbolic telegraph equation Eq. (1.1), with the initial conditions Eq. (1.2) and the Dirichlet boundary conditions Eq. (1.3).

First, let us discretize Eq. (1.1) according to the following θ-weighted scheme:

$$\begin{aligned} &\frac{u(x,t+\tau) - 2u(x,t) + u(x,t-\tau)}{\tau^2} + 2\alpha \frac{u(x,t+\tau) - u(x,t-\tau)}{2\tau} = \\ &\theta\left(\Delta u(x,t+\tau) - \beta^2 u(x,t+\tau) + f(x,t+\tau)\right) + (1-\theta)\left(\Delta u(x,t) - \beta^2 u(x,t) + f(x,t)\right) \end{aligned} \tag{2.2.1}$$

where $0 \le \theta \le 1$, and τ is the time step size, and $\Delta u = \dfrac{\partial^2 u}{\partial^2 x}$, using the notation $u^n = u\left(x, t^n\right)$ where $t^n = t^{n-1} + \tau$, we get:

$$\Delta u^{n+1} = \frac{\left(1 + \alpha\tau + \tau^2\theta\beta^2\right)u^{n+1}}{\theta\tau^2} - \frac{\left(2 - \beta^2(1-\theta)\tau^2\right)u^n}{\theta\tau^2} - \frac{(\alpha\tau - 1)u^{n-1}}{\theta\tau^2} - \frac{(1-\theta)}{\theta}\Delta u^n - \frac{\left(\theta f^{n+1} + (1-\theta) f^n\right)}{\theta}$$

(2.2.2)

Assuming that $u^{n+1}(x)$ is the solution which we want to find, we can suppose the function $F(x)$ as the right section of Eq. (2.2.2), this means Eq. (2.2.2) is a Possion equation:

$$\Delta u^{n+1}(x) = F(x) \tag{2.2.3}$$

So, if the function $F(x)$ is known, the Eq. (2.2.2) is equivalent with Eq. (2.2.3) with the same boundary condition. Assuming that there are $N-2$ interpolation points, $F\left(x_k\right)$ can be approximated by:

$$F\left(x_k\right) \approx \sum_{j=1}^{N-2} \lambda_j^n \varphi\left(r_{kj}\right), \quad k, j = 1,2,\cdots, N-2. \tag{2.2.4}$$

So, when the time size is $l+1$, we have the following approximation:

$$u^{l+1} \approx \sum_{j=1}^{N-2} \lambda_j^{l+1} \Phi\left(r_j\right) \tag{2.2.5}$$

where $\Phi(x)$ should be support $\Delta\Phi(x) = \varphi(x)$. In this paper, we take the TPS radial basis function:

$$\varphi(r) = r^{2m} \log(r), \quad m = 1,2,3,\cdots$$

and we can easily get:

$$\Phi(r) = r^{2m+2} / \left((2m+1)(2m+2)\right) \cdot \left(\log(r) - 1/(2m+2) - 1/(2m+1)\right), \quad m = 1,2,3,\cdots.$$

To guarantee the positive definition, here we use the following approximation:

$$u^n\left(x_i\right) \approx \sum_{j=1}^{N-2} \lambda_j^n \Phi\left(r_{ij}\right) + \lambda_{N-1}^n x_i + \lambda_N^n \tag{2.2.6}$$

where r_{ij} is the Euclidean norm. The additional conditions due to Eq. (2.1.4) are written as:

$$\sum_{j=1}^{N-2} \lambda_j^n = \sum_{j=1}^{N-2} \lambda_j^n x_j = 0. \tag{2.2.7}$$

Writing Eq. (2.2.6) together with Eq.(2.2.7) in a matrix form we have:

$$[u]^n = A[\lambda]^n \tag{2.2.8}$$

where $[u]^n = [u_1^n \quad u_2^n \quad \cdots \quad u_{N-2}^n \quad 0 \quad 0]^T$, $[\lambda]^n = [\lambda_1^n \quad \lambda_2^n \quad \cdots \quad \lambda_N^n]$ and $A = [a_{ij}, \quad 1 \le i, j \le N]$ is given by:

$$A = \begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1(N-2)} & x_1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \Phi_{(N-2)1} & \cdots & \Phi_{(N-2)(N-2)} & x_{N-2} & 1 \\ x_1 & \cdots & x_{N-2} & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 \end{pmatrix}. \tag{2.2.9}$$

Assuming that there are $p < N-2$ internal points and $N-2-p$ boundary points, then the $N \times N$ matrix A can be split into: $A = A_d + A_b + A_e$, where

$$A_d = [a_{ij} \quad for \quad (1 \le i \le p, 1 \le j \le N) \quad and \quad 0 \quad elsewhere]$$
$$A_b = [a_{ij} \quad for \quad (p+1 \le i \le N-2, 1 \le j \le N) \quad and \quad 0 \quad elsewhere]$$
$$A_e = [a_{ij} \quad for \quad (N-1 \le i \le N, 1 \le j \le N) \quad and \quad 0 \quad elsewhere]$$

Using the notation ℓA to designate the matrix of the same dimension as A and containing the elements $\hat{a}_{ij}$ where $\hat{a}_{ij} = \ell a_{ij}$, $1 \le i, j \le N$, then Eq. (2.2.1) together with the boundary conditions Eq. (1.3) can be written in matrix form as:

$$B\lambda^{n+1} = C\lambda^n + (\alpha\tau - 1)[u_d]^{n-1} + \tau^2\left(\theta[f]^{n+1} + (1-\theta)[f]^n\right) + [G]^{n+1} \tag{2.2.10}$$

where

$$B = (1 + \alpha\tau + \beta^2\theta\tau^2)A_d - \theta\tau^2\Delta A_d + A_b + A_e,$$
$$C = (2 - \beta^2(1-\theta)\tau^2)A_d + (1-\theta)\tau^2\Delta A_d,$$
$$[u_d]^{n-1} = [u_1^{n-1} \quad \cdots \quad u_p^{n-1} \quad 0 \quad \cdots \quad 0]^T,$$
$$[f]^{n+1} = [f_1^{n+1} \quad \cdots \quad f_p^{n+1} \quad 0 \quad \cdots \quad 0]^T,$$
$$[f]^n = [f_1^n \quad \cdots \quad f_p^n \quad 0 \quad \cdots \quad 0]^T,$$

and $[G]^{n+1} = [0 \quad \cdots \quad 0 \quad g_{p+1}^{n+1} \quad \cdots \quad g_{N-2}^{n+1} \quad 0 \quad 0]^T$. Eq. (2.2.10) is obtained by combining Eq. (2.2.1), which applies to the domain points, while Eq. (1.3) applies to the boundary points.

At $n = 0$, the Eq. (2.2.10) has the following form:

$$B\lambda^1 = C\lambda^0 + (\alpha\tau - 1)[u_d]^{-1} + \tau^2\left(\theta[f]^1 + (1-\theta)[f]^0\right) + [G]^1 . \tag{2.2.11}$$

To approximate u^{-1}, the initial condition $u_t(x,0) = h_1(x)$ can be used. For this purpose, we discretize this initial condition as:

$$\frac{u^1(x) - u^{-1}(x)}{2\tau} = h_1(x), \quad x \in \Omega \tag{2.2.12}$$

Writing Eq. (2.2.11) together with Eq. (2.2.12) we have:

$$(B + (1+\alpha\tau)A_d)[\lambda]^1 = C[\lambda]^0 + 2\tau(1-\alpha\tau)[H] + \tau^2\left(\theta[f]^1 + (1-\theta)[f]^0\right) + [G]^1, \tag{2.2.13}$$

where $[H] = [(h_1)_1 \quad \cdots \quad (h_1)_p \quad 0 \quad \cdots \quad 0]^T$. Together with the initial condition $u(x,0) = h_0(x)$ and Eq. (2.2.10), we can get all λ s, thus we can get the numerical solutions.

Since the coefficient matrix is unchanged in time steps, we use the LU factorization to the coefficient matrix only once and use this factorization in our algorithm.

Remark: Although Eq. (2.2.10) is valid for any value of $\theta \in [0,1]$, we will use $\theta = \frac{1}{2}$ (The famous Crank-Nicolson scheme) in our computation.

3. Numerical Examples

In this section, we present some numerical results to confirm the efficiency of our algorithm for solving the 1D hyperbolic telegraph equations.

3.1. Example 1

In this example, we consider the hyperbolic telegraph Eq. (1.1) in $[0,\pi]$, with the initial conditions:

$$\begin{cases} u(x,0)=\sin(x) \\ u_t(x,0)=-\sin(x) \end{cases}$$

and the boundary condition $u(0,t)=u(\pi,t)=0$. The exact solution is $u(x,t)=\exp(-t)\sin(x)$. The function $f(x,t)$ has the form

$$f(x,t)=(2-2\alpha+\beta^2)\exp(-t)\sin(x).$$

We use TPS radical basis function with $m=1$ for the computation, the L_∞ and L_2 errors are obtained in **Table. 1** for T = 0.4, 0.8, 1.2, 1.6 and 2 with time steps $\tau=0.0001$ and $dx=0.02$. The results are also compared with the results in papers [12,14].

T	*Errors*	Present Method	Cubic B-spline [14]	Quartic B-spline [12]
0.4	$L_\infty-errors$	9.776×10^{-6}	8.748×10^{-6}	2.423×10^{-3}
	$L_2-errors$	1.293×10^{-5}	1.010×10^{-6}	2.900×10^{-3}
0.8	$L_\infty-errors$	1.773×10^{-5}	1.578×10^{-5}	3.192×10^{-3}
	$L_2-errors$	2.296×10^{-5}	1.822×10^{-6}	3.200×10^{-3}
1.2	$L_\infty-errors$	5.072×10^{-5}	1.829×10^{-5}	3.059×10^{-3}
	$L_2-errors$	6.427×10^{-5}	2.112×10^{-6}	2.800×10^{-3}
1.6	$L_\infty-errors$	2.048×10^{-5}	1.814×10^{-5}	2.627×10^{-3}
	$L_2-errors$	2.629×10^{-5}	2.095×10^{-6}	2.300×10^{-3}
2	$L_\infty-errors$	1.886×10^{-5}	1.664×10^{-5}	2.140×10^{-3}
	$L_2-errors$	2.409×10^{-5}	1.921×10^{-6}	1.800×10^{-3}

Table. 1: $\alpha=4$, $\beta=2$ Numerical Errors with $\tau=0.0001$ and $dx=0.02$ for Example 1

The space-time graph of analytical and numerical solutions for T=2 are given in **Fig. 1**. The results obtained show the very good accuracy and efficiency of the new approximate scheme. Note that we can not distinguish the exact solution from the estimated solution in **Fig. 1**.

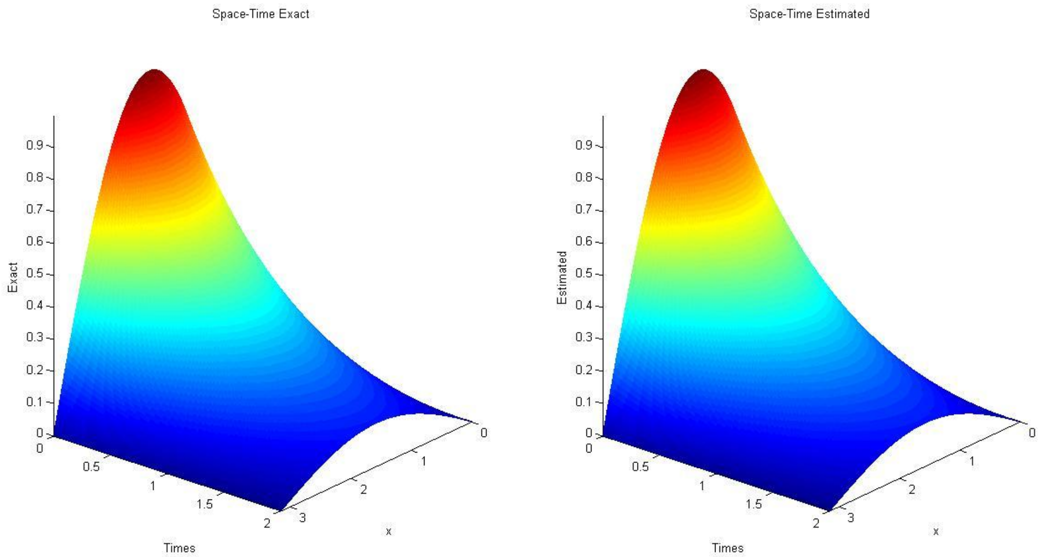

Fig. 1: $\alpha = 4$, $\beta = 2$ Space-time graph of the exact and estimated solutions with T=2 for Example 1.

3.2. Example 2

In this example, we consider Eq. (1.1) with $\alpha = 6$, $\beta = 2$ and the initial conditions:

$$\begin{cases} u(x,0) = \sin(x) \\ u_t(x,0) = 0 \end{cases} \qquad 0 \le x \le 1$$

the exact solution is $u(x,t) = \cos(t)\sin(x)$, we get the boundary conditions from the exact solution. The right side function $f(x,t)$ has the form:

$$f(x,t) = -2\alpha\sin(t)\sin(x) + \beta^2\cos(t)\sin(x).$$

The $L_\infty - errors$ compared with the results in papers[13] are given in **Table. 2** for T = 0.2, 0.4, 0.6, 0.8 and 1 with $\tau = 0.0005$ and $dx = 0.002$.

T	0.2	0.4	0.6	0.8	1
Present Method	2.1953×10^{-5}	4.573×10^{-5}	6.0887×10^{-5}	6.892×10^{-5}	7.0940×10^{-5}
cubic B-spline [13]	3.5005×10^{-5}	5.576×10^{-5}	6.9334×10^{-4}	7.686×10^{-5}	7.8908×10^{-5}

Table. 2: The $L_\infty - errors$ compared with the cubic B-spline method with $\tau = 0.0005$ and $dx = 0.002$.

The space-time graph of numerical solutions for T=1 are given in **Fig. 2**. Absolute errors between the numerical and analytical solutions are also depicted at different time in **Fig. 3**.

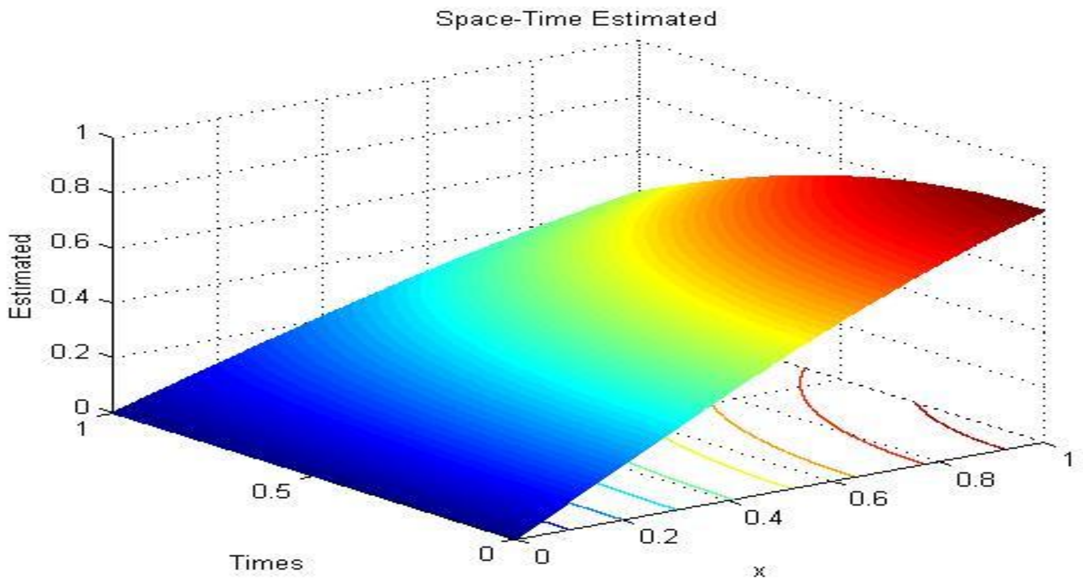

Fig. 2: The space-time graph of numerical solutions with T=1 for Example 2.

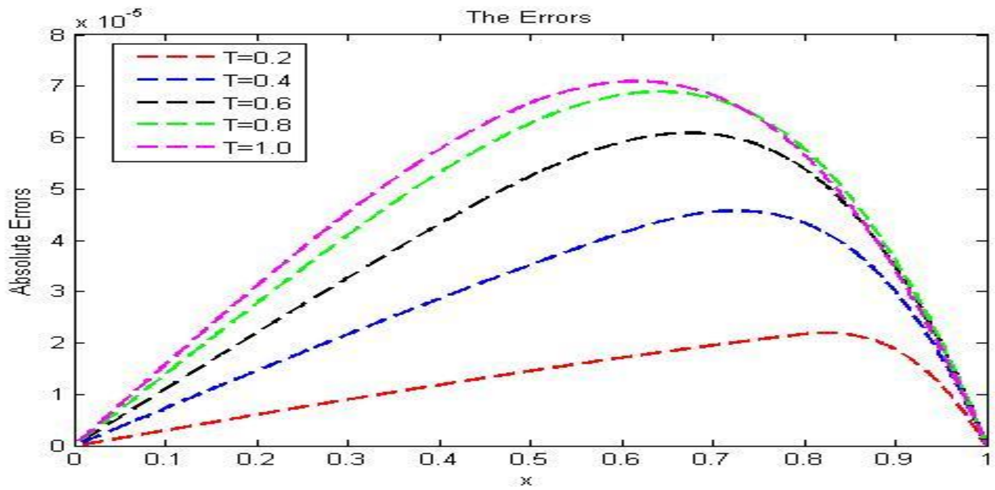

Fig. 3: Absolute errors between the numerical and analytical solution at different time for Example 2.

3.3. Example 3

We consider the hyperbolic telegraph equation Eq. (1.1) with $\alpha = \frac{1}{2}$, $\beta = 1$, in the interval $[0,4]$, the exact solution is given in [23] as $u(x,t) = \exp(x-t)$. The initial conditions are:

$$\begin{cases} u(x,0) = \exp(x), \\ u_t(x,0) = -\exp(x), \end{cases} \quad 0 \le x \le 4.$$

In this case, the right side function $f(x,t) = 0$, and we extract the boundary conditions from the exact solution. These results are obtained with $dx = 0.01$, $\tau = 0.001$. The $L_\infty - errors$, $L_2 - errors$ and RMS errors for T=1, 2, 3, 4 and 5 are obtained in **Table. 3**. The space-time graph of analytical and numerical solution for T=5 is given in **Fig. 4**. We also give the numerical solutions for different times in **Fig. 5**.

T	1	2	3	4	5
$L_\infty - errors$	1.856×10^{-5}	1.178×10^{-5}	5.875×10^{-6}	3.047×10^{-6}	2.464×10^{-6}
$L_2 - errors$	1.526×10^{-5}	7.881×10^{-6}	3.798×10^{-6}	1.824×10^{-6}	1.534×10^{-6}
$RMS - errors$	7.619×10^{-6}	3.936×10^{-6}	1.897×10^{-6}	9.108×10^{-7}	7.609×10^{-7}

Table. 3: The $L_\infty - errors$, $L_2 - errors$ and RMS errors for T=1, 2, 3, 4 and 5 for Example 3.

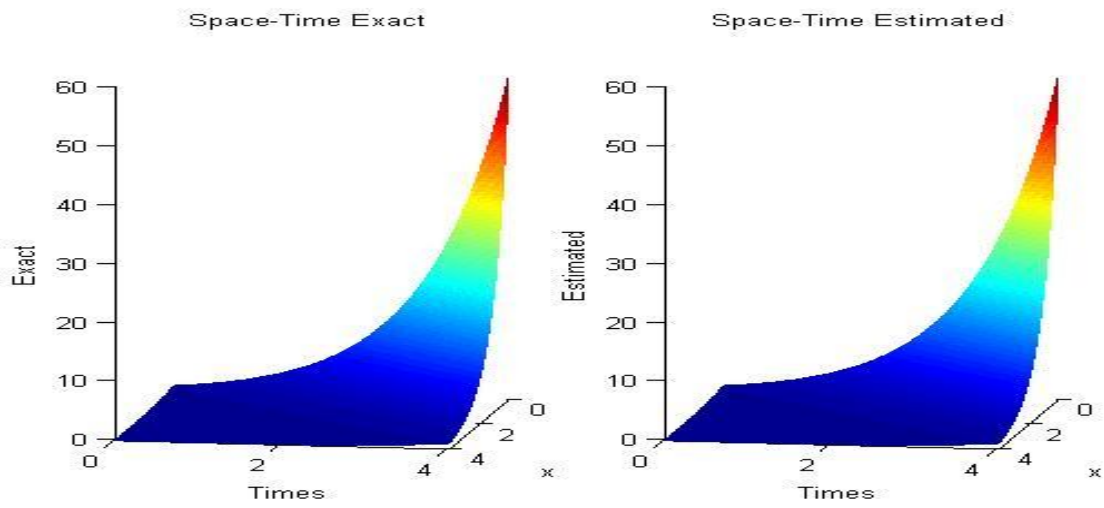

Fig. 4: Space-time graph of the exact and estimated solutions with T=4 for Example 3.

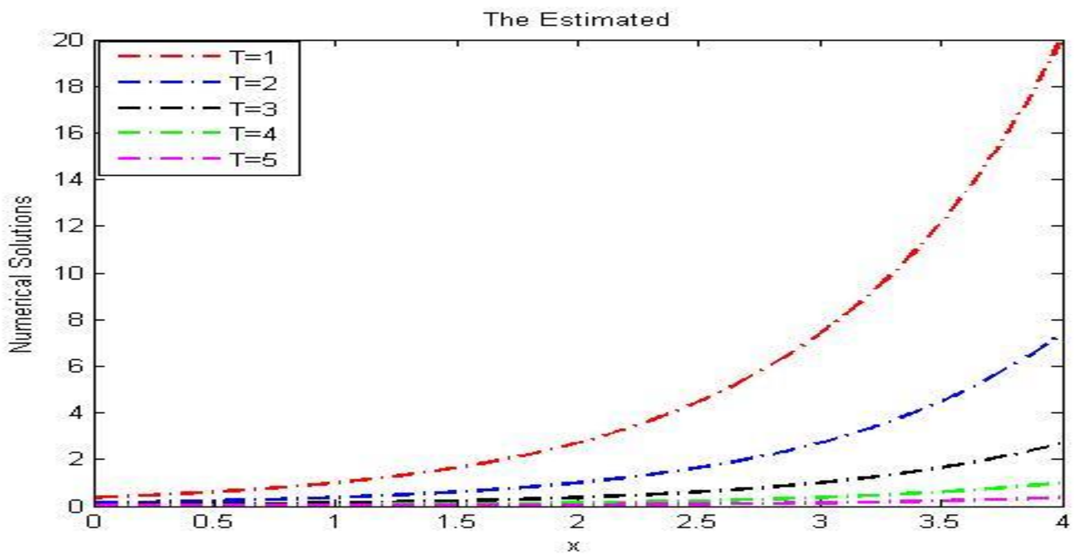

Fig. 5: The numerical solutions at different times for Example 3.

In this example, we also consider the telegraph equation in $[0,1]$ for $\alpha = 2$, $\beta = 2$ with the same exact solution [24] and initial conditions and boundary condition. We results are obtained with $\tau = 0.0001$, $dx = 0.01$. **Table. 4** gives the absolute error for T=0.5, 0.75 and compared with the results in papers [25].

DOI https://doi.org/2024-JICS-22545 | Generated on 2025-04-09 07:43:15

x	Present Method	Haar-Sinc Collocation [25]		Present Method	Haar-Sinc Collocation [25]	
	$T=0.5$	$k=16$	$k=32$	$T=0.75$	$k=16$	$k=32$
0.1	$9.34e-6$	$4.16e-4$	$2.12e-5$	$5.42e-6$	$3.83e-4$	$1.32e-4$
0.2	$4.62e-6$	$1.30e-3$	$5.44e-4$	$5.33e-7$	$1.38e-3$	$3.98e-4$
0.3	$7.98e-7$	$2.08e-3$	$1.03e-3$	$5.34e-7$	$2.60e-3$	$1.23e-3$
0.4	$2.85e-6$	$2.52e-3$	$1.28e-3$	$2.36e-6$	$3.24e-3$	$1.62e-3$
0.5	$3.81e-6$	$2.65e-3$	$1.32e-3$	$3.66e-6$	$3.39e-3$	$1.65e-3$
0.6	$6.16e-6$	$2.55e-3$	$1.24e-3$	$3.40e-6$	$3.17e-3$	$1.46e-3$
0.7	$3.79e-6$	$2.22e-3$	$1.04e-4$	$1.41e-6$	$2.64e-3$	$1.09e-3$
0.8	$2.01e-7$	$1.59e-3$	$6.72e-4$	$2.41e-6$	$1.69e-3$	$4.83e-4$
0.9	$5.50e-6$	$6.06e-4$	$8.98e-5$	$2.81e-6$	$3.76e-4$	$3.02e-4$

Table. 4: The absolute error compared with the results from Haar-Sinc Collocation Method [25]

4. Conclusion

In this paper, the method of particular solution (MPS) is employed for the numerical solution of second –order hyperbolic telegraph equation based on radical basis functions (RBFs). Coupled with the time discretization and MPS, the proposed method is a truly meshless method which requires neither domain or boundary discretization. The results of numerical experiments are presented, and are compared with analytical solutions confirmed the good accuracy of the presented scheme, the obtained numerical results also compared with the results obtained by some existing methods verified the accurate nature of our method..

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